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RINGS WITHOUT NILPOTENT ELEMENTS

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In what follows R stands for an associative (but not necessarily commutative) ring without nonzero nilpotent elements.

In this paper, without the use of the axiom of Choice, we prove Theorem 1 which states that if a product $r_1r_2 \ldots r_n$ of (not necessarily distinct) elements (factors) r_i of R is equal to zero then every product (of elements of R) whose factors include (in any order whatsoever) at least once every *distinct* factor of $r_1r_2 \ldots r_n$ is also equal to zero.

Based on the axiom of Choice and Theorem 1 we easily derive Theorem 2 which in turn implies that R is isomorphic to a subdirect product of rings without divisors of zero (cf. [1], Thm. 2).

Let a and b be elements of R such that $ab = 0$. But then $(ba)(ba) = b(ab)a = 0$ and since R has no nonzero nilpotent elements we see that $ba = 0$. Thus, for every element a and b of R we have:

$$(1) \qquad ab = 0 \quad \textit{if and only if} \quad ba = 0$$

Next, let a, b, r be elements of R such that $ab = 0$. Then from (1) it follows that $r(ba) = (rb)a = arb = 0$. Thus, for every element a, b, r of R we have:

$$(2) \qquad ab = 0 \quad \textit{implies} \quad arb = 0$$

Let us call a product $s_1s_2 \ldots s_n$ of elements s_i of R a *supersequent* of a product $r_1r_2 \ldots r_m$ of elements r_i of R if and only if $r_1, r_2, \ldots, r_m$ is a subsequence of $s_1, s_2, \ldots, s_n$. Thus, the product *acbabcas* is a supersequent of the product *cbc*. However, the product *abc* is not a supersequent of the product *cab*.

From (2) it readily follows that for every element $r_1, r_2 \ldots, r_m$ of R we have:

(3) *if* $r_1r_2 \ldots r_m = 0$ *then every supersequent of* $r_1r_2 \ldots r_m$ *is also equal to zero.*

Based on (3) we prove.

Theorem 1. *Let $r_1r_2 \ldots r_m$ be a product of (not necessarily distinct) elements r_i of R. Let $s_1s_2 \ldots s_n$ be a product of elements s_i of R which includes (in any order whatsoever) at least once every distinct factor of $r_1r_2 \ldots r_m$. Then*

(4) $$r_1r_2\ldots r_m = 0 \quad \textit{implies} \quad s_1s_2\ldots s_n = 0$$

Proof. Clearly $(s_1s_2\ldots s_n)^m$ is a supersequent of $r_1r_2\ldots r_m$ and therefore from (3) and the hypothesis of (4) it follows that $(s_1s_2\ldots s_n)^m = 0$. But then $s_1s_2\ldots s_n = 0$ since R has no nonzero nilpotent elements. Thus, (4) is established.

Accordingly, if in R we have $aabac = 0$ then

$$cba = pcbca = qbbca = paaacccbcabbq = 0.$$

Lemma 1. *Let M be a multiplicative system (i.e., $u \in M$ and $v \in M$ imply $uv \in M$) of R such that M is maximal with respect to the property of not containing* 0 *as an element. Then $R - M$ is a completely prime ideal (i.e., $xy \in (R - M)$ implies $x \in (R - M)$ or $y \in (R - M)$ of R.*

Proof. First we show that $R - M$ is closed under subtraction. Assume on the contrary that for some elements a and b of R it is the case that

(5) $$a \in (R - M) \quad \textit{and} \quad b \in (R - M) \quad \textit{and} \quad (a - b) \in M.$$

From the maximality of M it follows that there are elements $x_1, \ldots, x_m$ of R with $x_1\ldots x_m = 0$ such that, for every $i \in \{1, \ldots, m\}$, either $x_i \in M$ or $x_i = a$. But then from Theorem 1 it follows that

$$m_1\ldots m_ka = 0 \qquad \textit{with} \qquad m_i \in M$$

Since M is a multiplicative system, from the above equality we obtain

(6) $$m_1'a = 0 \qquad \textit{with} \qquad m_1' \in M.$$

Similarly, we obtain

(7) $$m_2'b = 0 \qquad \textit{with} \qquad m_2' \in M.$$

But then from (6), (7) and Theorem 1 it follows that

$$m_1'm_2'a = m_1'm_2'b = m_1'm_2'(a - b) = 0$$

which in view of (5) and the fact that $m_1'm_2' \in M$ implies $0 \in M$, contradicting $0 \notin M$.

Thus, our assumption is false and $R - M$ is closed under subtraction.

Next we show that $R - M$ is closed under outside (left and right) multiplication. Assume on the contrary that for some elements a and b of R it is the case that

(8) $$a \in (R - M) \qquad \textit{and} \qquad ab \in M \quad (\text{or } ba \in M)$$

Let $M(a)$ be the smallest multiplicative system of R described above. But then again we see that (8) implies (6). Therefore $m_3m_4ab = 0$ (with $m_3 \in M$

and $m_4 \in M$), and also $m_3 m_4 ba = 0$, by Theorem 1. Hence, from (8) in view of the fact that $m_3 m_4 \in M$ it follows (under either assumption) that $0 \in M$, contradicting $0 \notin M$.

Thus, $R - M$ is closed under outside (left and right) multiplication.

From the above it follows that $R - M$ is an ideal of R. Moreover, $R - M$ is a completely prime ideal of R since M is a multiplicative system.

Theorem 2. *Let a be a nonzero element of R. Then there exists a completely prime ideal P of R such that $a \notin P$.*

Proof. Clearly, $A = \{a, a^2, a^3, \ldots\}$ is a multiplicative system of R such that $0 \notin A$. But then from Zorn's lemma it follows that there exists a multiplicative system M of R such that $A \subseteq M$ and M is maximal with respect to the property of not containing 0 as an element. Hence, the conclusion of Theorem 2 follows from Lemma 1 by choosing $R - M$ for P.

From Theorem 2 we see that the family (P_i) of all completely prime ideals P_i of R has zero intersection. Moreover, it is clear that R/P_i is a ring without divisors of zero. Furthermore, it is obvious that a subdirect product of rings without divisors of zero is a ring without nonzero nilpotent elements. Thus, we have:

Corollary (cf. [1], Thm. 2). *A ring is without nonzero nilpotent elements if and only if it is isomorphic to a subdirect product of rings without divisors of zero.*

REFERENCE

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