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INTERSECTION GRAPHS OF SET FAMILIES CLOSED UNDER INTERSECTION AND SATISFYING THE DESCENDING CHAIN CONDITION

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Let $\mathscr{F}$ be some family of sets. The intersection graph of $\mathscr{F}$ is the undirected graph $G(\mathscr{F})$ without loops and multiple edges whose vertex set is $\mathscr{F}$ and in which two distinct vertices are joined by an edge if and only if they have a non-empty intersection. Instead of "intersection graph of $\mathscr{F}$" we shall say sometimes only "graph of $\mathscr{F}$". It is wellknown that each undirected graph without loops and multiple edges is isomorphic to the graph $G(\mathscr{F})$ for some set family $\mathscr{F}$. Therefore it is not interesting to study intersection graphs in general, but some special cases are of importance. Some authors studied the case when $\mathscr{F}$ is the family of all proper subalgebras of a given algebra. This study was begun by J. Bosák [1]. Such a family is either closed under intersection, or obtains this property after adding the empty set as a new element. (We say that a family $\mathscr{F}$ of sets is closed under intersection, if and only if for any non-empty subfamily $\mathscr{F}'$ of $\mathscr{F}$ the intersection of all sets of $\mathscr{F}'$ belongs to $\mathscr{F}$.)

In this paper we shall characterize the graphs $G(\mathscr{F})$ for all set families closed under intersection and satisfying the Descending Chain Condition.

The Descending Chain Condition: Let S_i for $i = 1, 2, \ldots$ be an infinite sequence of sets of $\mathscr{F}$ (of the ordinal ω_0) such that $S_{i+1} \subset S_i$ for any positive integer i. Then there exists such a positive integer N that for each $n \geqq N$ the set $S_n = S_N$.

Now we shall define the concept of an atom. An atom of the set family $\mathscr{F}$ is such a non-empty set A of $\mathscr{F}$ that no non-empty proper subset of A is in $\mathscr{F}$.

If $\mathscr{F}$ satisfies the Descending Chain Condition, then any non-empty set of $\mathscr{F}$ contains at least one atom of $\mathscr{F}$. If $\mathscr{F}$ is closed under intersection, then $A \in \mathscr{F}$ is an atom of $\mathscr{F}$, if and only if A is non-empty and $A \cap B \neq \emptyset$ implies $A \subset B$ for any $B \in \mathscr{F}$.

In the first item we shall study such families in general, the second item will describe graphs of finite abstract algebras.

1. General theory

Consider a family $\mathscr{F}$ closed under intersection and satisfying the Descending Chain Condition. The family $\mathscr{F} - \{\emptyset\}$ can be partitioned into classes such that two sets belong to the same class if and only if they contain the same atoms as subsets.

Theorem 1. *The intersection graph $G(\mathscr{F})$ of a set family $\mathscr{F}$ closed under inter section and not containing the empty set is a complete graph. Every complete graph is isomorphic to the graph $G(\mathscr{F})$ for some set family $\mathscr{F}$ closed under intersection and not containing the empty set.*

Proof. Let $\mathscr{F}$ satisfy the assumption. As $\mathscr{F}$ is closed under intersection, the intersection of any two sets of $\mathscr{F}$ must be in $\mathscr{F}$. As $\emptyset \notin \mathscr{F}$, this intersection must be always non-empty and $G(\mathscr{F})$ is complete. Now let a complete graph G with the vertex set of the cardinality n be given. Let us choose a set M of the cardinality n and an element $a \in M$. Let $\mathscr{F}$ be the family of all sets $\{a, b\}$ for $b \in M$, $b \neq a$ with the set $\{a\}$ added. The graph $G(\mathscr{F})$ is evidently isomorphic to G.

Theorem 2. *The intersection graph $G(\mathscr{F})$ of a set family $\mathscr{F}$ closed under intersection, satisfying the Descending Chain Condition and containing the empty set has the following structure:*

(i) *The vertex set V of $G(\mathscr{F})$ is the union of pairwise disjoint subsets V_i for $i \in I$, where I is some subscript set; each of these subsets induces a complete subgraph of G and for $i \in I$, $j \in I$, $i \neq j$ either each vertex of V_i is joined with each vertex of V_j, or no vertex of V_i is joined with a vertex of V_j.*

(ii) *The graph $G(\mathscr{F})$ contains at least one isolated vertex.*

(iii) *If G' is the graph obtained from $G(\mathscr{F})$ by contracting all vertices of V_i for each $i \in I$, then there exists in G' a kernel K of the maximal cardinality which has the following properties:*

(a) *any two vertices of G' not belonging to K and both joined to the same vertex of K are joined together and vice versa;*

(b) *if $K' \subset K$, then there exists at most one vertex of G' joined to all the vertices of K' and with no vertex of $K - K'$;*

(c) *if some family $\mathscr{K}$ of subsets of K has the property that to each $K' \in \mathscr{K}$ there exists a vertex of G' joined to all the vertices of K' and to no vertex of $K - K'$ and the intersection $P(\mathscr{K})$ of all sets of $\mathscr{K}$ has at least two elements, then there exists a vertex of G' joined to all the vertices of $P(\mathscr{K})$ and to no other vertex of K.*

Any graph satisfying the above described conditions is isomorphic to $G(\mathscr{F})$ for some set family $\mathscr{F}$ closed under intersection, satisfying the Descending Chain Condition and containing the empty set.

Proof. Let $\mathscr{A}$ be the family of atoms of $\mathscr{F}$. Let $\mathscr{B} \subset \mathscr{A}$. The family of sets of $\mathscr{F}$, each of which contains all atoms from $\mathscr{B}$ as subsets and no other atom, will be denoted by $\mathscr{E}(\mathscr{B})$. If $\mathscr{E}(\mathscr{B}) \neq \emptyset$, then any two sets from $\mathscr{E}(\mathscr{B})$ have a non-empty intersection, because this intersection contains all atoms from $\mathscr{B}$. Therefore the subgraph of $G(\mathscr{F})$ induced by the set of vertices corresponding to sets from $\mathscr{E}(\mathscr{B})$ is complete. Any non-empty set of $\mathscr{F}$ belongs to exactly one $\mathscr{E}(\mathscr{B})$, therefore for $\mathscr{B}_1 \neq \mathscr{B}_2$ we have $\mathscr{E}(\mathscr{B}_1) \cap \mathscr{E}(\mathscr{B}_2) = \emptyset$. Let $\mathscr{B}_1 \subset \mathscr{A}$, $\mathscr{B}_2 \subset \mathscr{A}$ and consider the families $\mathscr{E}(\mathscr{B}_1)$, $\mathscr{E}(\mathscr{B}_2)$. If $\mathscr{B}_1 \cap \mathscr{B}_2 \neq \emptyset$, then the intersection of $M_1 \in \mathscr{E}(\mathscr{B}_1)$, $M_2 \in \mathscr{E}(\mathscr{B}_2)$ is always non-empty, because it contains all atoms from $\mathscr{B}_1 \cap \mathscr{B}_2$. If $\mathscr{B}_1 \cap \mathscr{B}_2 = \emptyset$, then the set $M_1 \cap M_2$ for $M_1 \in \mathscr{E}(\mathscr{B}_1)$, $M_2 \in \mathscr{E}(\mathscr{B}_2)$ cannot contain any atom of $\mathscr{F}$, because such an atom would be contained in both $\mathscr{B}_1$ and $\mathscr{B}_2$, therefore in $\mathscr{B}_1 \cap \mathscr{B}_2$, which is a contradiction with the assumption that $\mathscr{B}_1 \cap \mathscr{B}_2 = \emptyset$. But $M_1 \cap M_2 \in \mathscr{F}$, because $\mathscr{F}$ is closed under intersection; any non-empty set from $\mathscr{F}$ must contain at least one atom of $\mathscr{F}$, therefore $M_1 \cap M_2 = \emptyset$. Now let G' be the graph whose vertices are all families $\mathscr{B} \subset \mathscr{A}$ for which $\mathscr{E}(\mathscr{B}) \neq \emptyset$ and in which two vertices are joined by an edge if and only if the corresponding families have a non-empty intersection. Evidently G' can be obtained from $G(\mathscr{F})$ by contracting all the vertices corresponding to sets of $\mathscr{E}(\mathscr{B})$ for each $\mathscr{E}(\mathscr{B}) \neq \emptyset$ and by omitting the vertex corresponding to the empty set. Let K be the subset of the vertex set of G' consisting of all one-element subfamilies of $\mathscr{A}$. For any of such subfamilies $\mathscr{B}$ the family $\mathscr{E}(\mathscr{B}) \neq \emptyset$, because $\mathscr{B} = \{B\}$ for some $B \in \mathscr{A}$ and $B \in \mathscr{E}(\mathscr{B})$. The set K is evidently a kernel of G' and no kernel of G' can have a greater cardinality than K (see for example [4]). Let $\mathscr{B} \in K$. Then $\mathscr{B} = \{B\}$ for some $B \in \mathscr{A}$ and for any other set $\mathscr{C} \subset \mathscr{A}$ we have $\mathscr{B} \cap \mathscr{C} \neq \emptyset$ if and only if $B \in \mathscr{C}$, therefore $\mathscr{B} \subset \mathscr{C}$. For any two such sets $\mathscr{C}_1$, $\mathscr{C}_2$ we have $\mathscr{C}_1 \cap \mathscr{C}_2 \supset \mathscr{B}$ and therefore $\mathscr{C}_1 \cap \mathscr{C}_2 \neq \emptyset$ and $\mathscr{C}_1$ and $\mathscr{C}_2$ are joined by an edge in G'. If $K' \subset K$, then the vertex of G' joined with all vertices of K' and with no vertex of $K - K'$ is the set $\{B \in \mathscr{A} \mid \{B\} \in K'\} = \mathscr{D}$ if $\mathscr{E}(\mathscr{D}) \neq \emptyset$; therefore it is at most one. Thus if V_i for $i \in I$ are the families $\mathscr{E}(\mathscr{B})$ for $\mathscr{B} \subset \mathscr{A}$, then the assertion holds. The vertex of $G(\mathscr{F})$ corresponding to the empty set is always isolated in $G(\mathscr{F})$, because the empty set cannot have a non-empty intersection with another set. The property (c) follows from the fact that $\mathscr{F}$ is closed under intersection.

Now let G be a graph with the structure described in the assertion of the theorem. Let us construct the graph G' and find the described kernel K. To any vertex u of K we assign the set $\{u\}$. If $K' \subset K$ and the vertex v of G' is joined with all the vertices of K' and with no vertex of $K - K'$ (according to (b) such a vertex is at most one), then to v we assign the set K'. Thus to each vertex w of G' we have assigned some set $K(w)$ so that two vertices are joined by an edge if and only if the intersection of the assigned sets is non-

-empty (according to (a)). Now return to G. In any set V_i for $i \in I$ choose an arbitrary vertex v_i. If w_i is the vertex of G' obtained by contracting V_i and if $K_i = K(w_i)$ is the set assigned to w_i in the above described way, then assign K_i also to v_i in G. Now for any $i \in I$ denote the vertices of $V_i - \{v_i\}$ by x_{ij} for $j \in J_i$, where J_i is some subscript set. Then consider the set Y consisting of pairwise different elements y_{ij} for $j \in J_i$, $i \in I$. To any vertex x_{ij} assign the set $K_i \cup \{y_{ij}\}$. To one isolated vertex of G the empty set is assigned. The graph thus obtained is the intersection graph of the set family $\mathscr{F}$ whose elements were assigned to the vertices of G.

2. Intersection graphs and lattices of subalgebras of finite abstract algebras

If the family $\mathscr{F}$ from the first item is the family of all proper subalgebras of a given abstract algebra, we call $G(\mathscr{F})$ the intersection graph of this algebra or simply the graph of this algebra. Here we shall study the interrelations between this concept and the concept of the lattice of subalgebras.

In certain cases we define the modified intersection graph of an algebra. Before defining this concept we shall define a minimal subalgebra of a given algebra.

A minimal subalgebra of an algebra $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$ which does not contain any proper subalgebra.

If the algebra $\mathfrak{A}$ is finite, it certainly contains minimal subalgebras. But it may happen that some algebra contains only one minimal subalgebra. For example, the only minimal subgroup of a group is the subgroup consisting of the unit element, the only minimal subfield of a field is the subfield generated by the unit element and the zero element. In this case we define the quasi-minimal subalgebra and the modified intersection graph of an algebra.

If an algebra $\mathfrak{A}$ contains only one minimal subalgebra, then a quasi-minimal subalgebra of $\mathfrak{A}$ is a subalgebra of $\mathfrak{A}$ which is not minimal and does not contain any proper subalgebra except for the minimal one.

If an algebra $\mathfrak{A}$ contains only one minimal subalgebra, then the modified intersection graph (or shortly modified graph) of $\mathfrak{A}$ is the undirected graph $G^*(\mathfrak{A})$ without loops and multiple edges whose vertex set is the set of all proper subalgebras of $\mathfrak{A}$ which are not minimal and two distinct vertices of $G^*(\mathfrak{A})$ are joined by an edge if and only if the corresponding subalgebras have an intersection different from the minimal subalgebra of $\mathfrak{A}$.

Such a graph for groups was introduced by B. Csákány and G. Pollák [2].

Evidently if a finite algebra $\mathfrak{A}$ contains only one minimal subalgebra, this minimal subalgebra is contained in each subalgebra of $\mathfrak{A}$ and therefore also the intersection of any two subalgebras of $\mathfrak{A}$ contains it. Here we see the reason of defining the modified intersection graph. In this case the (unmodified) intersection graph of $\mathfrak{A}$ is a complete graph and does not say much about the structure of $\mathfrak{A}$; the modified intersection graph tells us much more about it.

Now let us study the lattice $\mathscr{L}(\mathfrak{A})$ of subalgebras of a finite algebra $\mathfrak{A}$ (if the intersection of some two subalgebras of $\mathfrak{A}$ is empty, we join the empty set as a new element to this lattice) and its lower semilattice $\mathscr{L}_{\wedge}(\mathfrak{A})$. Let $\mathscr{M}(\mathfrak{A})$ be the set of minimal subalgebras of $\mathfrak{A}$. To any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ a subset $\mathscr{M}(\mathfrak{B})$ of $\mathscr{M}(\mathfrak{A})$ is uniquely assigned, consisting of all elements of $M(\mathfrak{A})$ which are subalgebras of $\mathfrak{B}$. Now let us define a relation ϱ on $\mathscr{L}(\mathfrak{A})$ such that for $\mathfrak{B} \in \mathscr{L}(\mathfrak{A})$, $\mathfrak{C} \in \mathscr{L}(\mathfrak{A})$ we have $(\mathfrak{B}, \mathfrak{C}) \in \varrho$ if and only if $\mathscr{M}(\mathfrak{B}) = \mathscr{M}(\mathfrak{C})$.

Theorem 3. *Let ϱ be the relation on the lattice $\mathscr{L}(\mathfrak{A})$ of subalgebras of a finite algebra $\mathfrak{A}$ defined so that for two subalgebras $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{A}$ we have $(\mathfrak{B}, \mathfrak{C}) \in \varrho$ if and only if the subalgebras $\mathfrak{B}$, $\mathfrak{C}$ contain exactly the same minimal subalgebras of $\mathfrak{A}$. Then the relation ϱ is a congruence on the lower semilattice $\mathscr{L}_{\wedge}(\mathfrak{A})$, but in general not on the lattice $\mathscr{L}(\mathfrak{A})$.*

Proof. From the definition of ϱ it is clear that ϱ is an equivalence on $\mathscr{L}_{\wedge}(\mathfrak{A})$. Now let $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{C}_1, \mathfrak{C}_2$ be subalgebras of $\mathfrak{A}$ and let $(\mathfrak{B}_1, \mathfrak{B}_2) \in \varrho$, $(\mathfrak{C}_1, \mathfrak{C}_2) \in \varrho$. This means $\mathscr{M}(\mathfrak{B}_1) = \mathscr{M}(\mathfrak{B}_2)$, $\mathscr{M}(\mathfrak{C}_1) = \mathscr{M}(\mathfrak{C}_2)$. The meets $\mathfrak{B}_1 \wedge \mathfrak{C}_1$, $\mathfrak{B}_2 \wedge \mathfrak{C}_2$ in the lattice $\mathscr{L}(\mathfrak{A})$ are the set intersections $\mathfrak{B}_1 \cap \mathfrak{C}_1$, $\mathfrak{B}_2 \cap \mathfrak{C}_2$. The set $\mathscr{M}(\mathfrak{B}_1 \wedge \mathfrak{C}_1)$ contains evidently all minimal subalgebras of $\mathfrak{A}$ which are contained in both $\mathfrak{B}_1$ and $\mathfrak{C}_1$ and cannot contain any other minimal subalgebra of $\mathfrak{A}$. Therefore $\mathscr{M}(\mathfrak{B}_1 \wedge \mathfrak{C}_1) = \mathscr{M}(\mathfrak{B}_1) \cap \mathscr{M}(\mathfrak{C}_1)$ and also $\mathscr{M}(\mathfrak{B}_2 \wedge \mathfrak{C}_2) = \mathscr{M}(\mathfrak{B}_2) \cap \mathscr{M}(\mathfrak{C}_2) = \mathscr{M}(\mathfrak{B}_1) \cap \mathscr{M}(\mathfrak{C}_1) = \mathscr{M}(\mathfrak{B}_1 \wedge \mathfrak{C}_1)$. Thus we have $(\mathfrak{B}_1 \wedge \mathfrak{C}_1, \mathfrak{B}_2 \wedge \mathfrak{C}_2) \in \varrho$. We have proved that ϱ is a congruence on $\mathscr{L}_{\wedge}(\mathfrak{A})$.

Now we shall give a counterexample showing that ϱ need not be a congruence on $\mathscr{L}(\mathfrak{A})$. In a finite semigroup the minimal subalgebras are one-element subsemigroups consisting of idempotents. Let $\mathfrak{A}$ be a finite semigroup with the elements a, b, c, d, e given by the following Cayley table:

	a	b	c	d	e
a	b	b	d	e	e
b	b	b	e	e	e
c	c	c	c	c	c
d	d	d	d	d	d
e	e	e	e	e	e

The minimal subalgebras of the semigroup $\mathfrak{A}$ are the subsemigroups $\{b\}$, $\{c\}$, $\{d\}$, $\{e\}$. Let $\mathfrak{B}_1 = \{a, b\}$, $\mathfrak{B}_2 = \{b\}$, $\mathfrak{C}_1 = \mathfrak{C}_2 = \{c\}$. We have $\mathscr{M}(\mathfrak{B}_1) = \mathscr{M}(\mathfrak{B}_2) = \{\{b\}\}$, $\mathscr{M}(\mathfrak{C}_1) = \mathscr{M}(\mathfrak{C}_2) = \{\{c\}\}$, therefore $(\mathfrak{B}_1, \mathfrak{B}_2) \in \varrho$, $(\mathfrak{C}_1, \mathfrak{C}_2) \in \varrho$. The join $\mathfrak{B}_1 \vee \mathfrak{C}_1$ in the lattice $\mathscr{L}(\mathfrak{A})$ is the subalgebra (subsemigroup) of $\mathfrak{A}$ generated by the union $\mathfrak{B}_1 \cup \mathfrak{C}_1$. Thus it is the subsemigroup of $\mathfrak{A}$ generated by the set $\{a, b, c\}$; this subsemigroup is the whole $\mathfrak{A}$. The join $\mathfrak{B}_2 \vee \mathfrak{C}_2$ is the subsemigroup of $\mathfrak{A}$ generated by the set $\{b, c\}$; this is the subsemigroup of $\mathfrak{A}$ consisting of the elements b, c, e. We have $\mathscr{M}(\mathfrak{B}_1 \vee \mathfrak{C}_1) = \{\{b\}, \{c\}, \{d\}, \{e\}\}$, $\mathscr{M}(\mathfrak{B}_2 \vee \mathfrak{C}_2) = \{\{b\}, \{c\}, \{e\}\}$ and $(\mathfrak{B}_1 \vee \mathfrak{C}_1, \mathfrak{B}_2 \vee \mathfrak{C}_2) \notin \varrho$.

By $\mathscr{L}_\wedge(\mathfrak{A})/\varrho$ we shall denote the factor-semilattice of $\mathscr{L}_\wedge(\mathfrak{A})$ according to the congruence ϱ, i. e. the semilattice whose elements are the classes of ϱ and in which the meet of two elements is the class of ϱ which contains the meet of representatives of these classes.

Now we shall define an auxiliary concept, the concept of an outerly free set in a graph, according to W. Dörfler [3].

A subset X of the vertex set V of an undirected graph G is called outerly free, if and only if any vertex $v \in V - X$ is joined by edges either to all the vertices of X, or to no vertex of X.

An outerly free set in a graph G with the property that the subgraph of G generated by it is a clique (a complete graph) will be called a CF-set.

We shall prove a lemma.

Lemma 1. *Let X, Y be two CF-sets in a graph G. If $X \cap Y \neq \emptyset$, then $X \cup Y$ is a CF-set in G.*

Proof. First we shall prove that $X \cup Y$ is outerly free. If $X \cup Y = V$, this holds trivially. If not, let $v \in V - (X \cup Y)$, $x \in X \cap Y$. According to the definition v is joined either to all the vertices of X, or to no vertex of X. If it is joined to all the vertices of X, it is joined also to x. But $x \in Y$ and Y is also an outerly free set. As v is joined to one vertex of Y, it must be joined to all of them. Therefore v is joined to all the vertices of $X \cup Y$. Now assume that v is joined to no vertex of X. Then it is not joined to x and as $x \in Y$, it is joined to no vertex of Y. Thus it is joined to no vertex of $X \cup Y$. It remains to prove that the subgraph generated by $X \cup Y$ is a clique. If $X \subset Y$ or $Y \subset X$, this is trivial. Thus let $X - Y \neq \emptyset$, $Y - X \neq \emptyset$. Let $a \in X - Y$. As X generates a clique and both a and x are in X, a is joined by an edge to x. But $x \in Y$ and Y is outerly free, therefore a is joined to all the vertices of Y. As it is joined also to all vertices of $X - \{a\}$, it is joined to ail the vertices of $X \cup Y - \{a\}$. Analogously if $a \in Y - X$. For $a \in X \cap Y$ this follows from the fact that X, Y generate cliques. We have proved that $X \cup Y$ is a CF-set.

Lemma 2. *Any CF-set in a finite graph G is contained in a uniquely determined maximal CF-set, i. e. in a CF-set which is a proper subset of no CF-set.*

Proof. Let X be a CF-set in G. Let $\bar{X}$ be the union of all CF-sets in G which contain X. These CF-sets are pairwise non-disjoint, because any of them contains X, and they are finitely many, because G is finite. According to

Lemma 1 the union of any two non-disjoint CF-sets is a CF-set; this can be easily generalized for any finite number of these sets. Thus $\bar{X}$ is a CF-set. If $\bar{X}$ were a proper subset of some CF-set Y in G, there would also be $X \subset Y$ and therefore Y would be one of the sets whose union is X, which would mean $Y \subset \bar{X}$; this is a contradiction. Therefore $\bar{X}$ is a maximal CF-set.

Now we return to abstract algebras.

Lemma 3. *For any two subalgebras* $\mathfrak{B}$, $\mathfrak{C}$ *of a finite algebra* $\mathfrak{A}$ *the following equivalence holds:*

$$\mathfrak{B} \cap \mathfrak{C} = \emptyset \Leftrightarrow \mathscr{M}(\mathfrak{B}) \cap \mathscr{M}(\mathfrak{C}) = \emptyset$$

Proof. The implication $\mathfrak{B} \cap \mathfrak{C} = \emptyset \Rightarrow \mathscr{M}(\mathfrak{B}) \cap \mathscr{M}(\mathfrak{C}) = \emptyset$ is evident. Now let $\mathfrak{B} \cap \mathfrak{C} \neq \emptyset$. The intersection $\mathfrak{B} \cap \mathfrak{C}$ is a subalgebra of $\mathfrak{A}$, thus it contains some minimal subalgebra $\mathfrak{X}$ of $\mathfrak{A}$. But then $\mathfrak{X} \in \mathscr{M}(\mathfrak{A})$, $\mathfrak{X} \subset \mathfrak{B}$, $\mathfrak{X} \subset \mathfrak{C}$, therefore $\mathfrak{X} \in \mathscr{M}(\mathfrak{B})$, $\mathfrak{X} \in \mathscr{M}(\mathfrak{C})$, which means $\mathfrak{X} \in \mathscr{M}(\mathfrak{B}) \cap \mathscr{M}(\mathfrak{C})$ and $\mathscr{M}(\mathfrak{B}) \cap$ $\cap \mathscr{M}(\mathfrak{C}) \neq \emptyset$.

Theorem 4. *Let* $G(\mathfrak{A})$ *be the intersection graph of a finite algebra* $\mathfrak{A}$. *Let* ϱ *be the congruence on the lower semilattice* $\mathscr{L}_{\wedge}(\mathfrak{A})$ *of the subalgebras of* $\mathfrak{A}$ *defined in Theorem* 3. *Let* $\mathscr{X}$ *be a class of the congruence* ϱ. *Then* $\mathscr{X} - \{\mathfrak{A}\}$ *is a maximal CF-set in* $G(\mathfrak{A})$.

Remark. We write $\mathscr{X} - \{\mathfrak{A}\}$, because in $G(\mathfrak{A})$ there does not exist a vertex corresponding to the whole $\mathfrak{A}$. If we speak about the graph whose vertices are all subalgebras of $\mathfrak{A}$ (not only proper ones) and in which the adjacency is the same as in $G(\mathfrak{A})$, then we can assert the same with $\mathscr{X}$ instead of $\mathscr{X} - \{\mathfrak{A}\}$; we shall prove this assertion, which evidently implies the assertion of this theorem.

Proof. Put $\mathscr{M}(\mathfrak{B}) = \mathscr{M}_{\mathfrak{X}}$ for any $\mathfrak{B} \in \mathscr{X}$. Let $\mathfrak{C}$ be any subalgebra of $\mathfrak{A}$. If $\mathfrak{B}_0 \cap \mathfrak{C} = \emptyset$ for some $\mathfrak{B}_0 \in \mathscr{X}$, then according to Lemma 3 we have $\mathscr{M}(\mathfrak{B}_0) \cap$ $\cap \mathscr{M}(\mathfrak{C}) = \mathscr{M}_{\mathfrak{X}} \cap \mathscr{M}(\mathfrak{C}) = \emptyset$. As $\mathscr{M}_{\mathfrak{X}} = \mathscr{M}(\mathfrak{B})$ for any $\mathfrak{B} \in \mathscr{X}$, we have $\mathscr{M}(\mathfrak{B}) \cap \mathscr{M}(\mathfrak{C}) = \emptyset$ for all $\mathfrak{B} \in \mathscr{X}$ and therefore $\mathfrak{C}$ is joined to no vertex of $\mathscr{X}$. If $\mathfrak{B}_0 \cap \mathfrak{C} \neq \emptyset$ for some $\mathfrak{B}_0 \in \mathscr{X}$, we have $\mathscr{M}(\mathfrak{B}_0) \cap \mathscr{M}(\mathfrak{C}) \neq \emptyset$ and $\mathscr{M}(\mathfrak{B}) \cap$ $\cap \mathscr{M}(\mathfrak{C}) \neq \emptyset$ for all $\mathfrak{B} \in \mathscr{X}$. Therefore $\mathfrak{C}$ is joined to all the vertices of $\mathscr{X}$ (possibly $\mathfrak{C}$ itself). Any two vertices corresponding to subalgebras of $\mathscr{X}$ are joined by an edge, because these subalgebras contain the same minimal subalgebras, i. e. they have a non-empty intersection. We have proved that $\mathscr{X}$ is a CF-set in the graph $G(\mathfrak{A})$ extended according to the Remark. Now assume that there exists a CF-set $\mathscr{Y}$ in this graph containing $\mathscr{X}$ as a proper subset. Let $\mathfrak{D} \in \mathscr{Y} - \mathscr{X}$; we have $\mathscr{M}(\mathfrak{D}) \neq \mathscr{M}_{\mathfrak{X}}$. This means either $\mathscr{M}(\mathfrak{D}) - \mathscr{M}_{\mathfrak{X}} \neq \emptyset$, or $\mathscr{M}_{\mathfrak{X}} - \mathscr{M}(\mathfrak{D}) \neq \emptyset$. If $\mathscr{M}(\mathfrak{D}) - \mathscr{M}_{\mathfrak{X}} \neq \emptyset$, let $\mathfrak{E} \in \mathscr{M}(\mathfrak{D}) - \mathscr{M}_{\mathfrak{X}}$. We have $\mathfrak{E} \subset \mathfrak{D}$ and $\mathfrak{E} \not\subset \mathfrak{B}$ for all $\mathfrak{B} \in \mathscr{X}$. Then $\mathfrak{D} \cap \mathfrak{E} = \mathfrak{E} \neq \emptyset$, $\mathfrak{B} \cap \mathfrak{E} = \emptyset$ for all $\mathfrak{B} \in \mathscr{X}$ and $\mathfrak{E}$ is joined to $\mathfrak{D}$, but to no element of $\mathscr{X}$; this means that $\mathscr{Y}$

is not a CF-set. If $\mathscr{M}_{\mathfrak{X}} - \mathscr{M}(\mathfrak{D}) \neq \emptyset$, let $\mathfrak{F} \in \mathscr{M}_{\mathfrak{X}} - \mathscr{M}(\mathfrak{D})$. Then $\mathfrak{F} \subset \mathfrak{B}$ for all $\mathfrak{B} \in \mathscr{X}$, $\mathfrak{F} \not\subset \mathfrak{D}$ and $\mathfrak{F}$ is joined to all the elements of $\mathscr{X}$ (possibly $\mathfrak{F}$ itself), but not with $\mathfrak{D}$; we obtain again a contradiction. Thus $\mathscr{X}$ is a maximal CF-set.

Lemma 4. *The vertex set V of a finite undirected graph G is uniquely partitioned into maximal CF-sets.*

Proof. According to Lemma 3 any CF-set in G is contained in a unique maximal CF-set. From the definition of the CF-set it trivially follows that any one-element subset of the vertex set of G is a CF-set. Therefore the maximal CF-sets of G form a partition of the vertex set of G.

Theorem 5. *Let $\mathscr{L}_{\vee}(\mathfrak{A})$ be the lower semilattice of subalgebras of a finite algebra $\mathfrak{A}$, let ϱ be the congruence on $\mathscr{L}_{\wedge}(\mathfrak{A})$ defined in Theorem 3. Then the factor-semilattice $\mathscr{L}_{\wedge}(\mathfrak{A})/\varrho$ is uniquely determined by the intersection graph $G(\mathfrak{A})$ of $\mathfrak{A}$.*

Proof. We find the partition of the vertex set of $G(\mathfrak{A})$ into maximal CF-sets. According to Theorem 4 any of these sets corresponds to a class of the congruence ϱ (eventually after omitting the subalgebra equal to the whole $\mathfrak{A}$), i. e. to an element of $\mathscr{L}_{\wedge}(\mathfrak{A})/\varrho$. Evidently for two elements $\mathscr{X}$, $\mathscr{Y}$ of $\mathscr{L}_{\wedge}(\mathfrak{A})\,\varrho$ we have $\mathscr{X} \leqq \mathscr{Y}$ if and only if $\mathscr{M}_{\mathfrak{X}} \subset \mathscr{M}_{\mathfrak{Y}}$, where $\mathscr{M}_{\mathfrak{X}} = \mathscr{M}(\mathfrak{B})$ for all $\mathfrak{B} \in \mathscr{X}$ and analogously for $\mathscr{M}_{\mathfrak{Y}}$. We shall prove that $\mathscr{X} \leqq \mathscr{Y}$ if and only if any vertex of $G(\mathfrak{A})$ joined to a vertex of $\mathscr{X}$ is joined to a vertex of $\mathscr{Y}$. (or eventually of $(\mathscr{Y}) - \{\mathfrak{A}\}$). For any subalgebra $\mathfrak{D}$ of $\mathfrak{A}$ which has a non--empty intersection with a vertex of $\mathscr{X}$ we have $\mathscr{M}(\mathfrak{D}) \cap \mathscr{M}_{\mathfrak{X}} \neq \emptyset$ (Lemma 3). If $\mathscr{X} \leqq \mathscr{Y}$, then $\mathscr{M}_{\mathfrak{X}} \subset \mathscr{M}_{\mathfrak{Y}}$ and $\mathscr{M}(\mathfrak{D}) \cap \mathscr{M}_{\mathfrak{X}} \subset \mathscr{M}(\mathfrak{D}) \cap \mathscr{M}_{\mathfrak{Y}}$ and therefore $\mathscr{M}(\mathfrak{D}) \cap \mathscr{M}_{\mathfrak{Y}} \neq \emptyset$ and $\mathfrak{D}$ has a non-empty intersection with some element of $\mathscr{Y}$. If $\mathscr{X} \leqq \mathscr{Y}$ does not hold, we have $\mathscr{M}_{\mathfrak{X}} - \mathscr{M}_{\mathfrak{Y}} \neq \emptyset$. Let $\mathfrak{C} \in \mathscr{M}_{\mathfrak{X}} - \mathscr{M}_{\mathfrak{Y}}$. We have $\mathfrak{C} \subset \mathfrak{B}$, therefore $\mathfrak{B} \cap \mathfrak{C} \neq \emptyset$ for any $\mathfrak{B} \in \mathscr{X}$, but $\mathfrak{C} \cap \mathfrak{C} = \emptyset$ for all $\mathfrak{C} \in \mathscr{Y}$. Therefore $\mathfrak{C}$ is joined to the element $\mathfrak{B} \in \mathscr{X}$, but to no element of $\mathscr{Y}$. According to this assertion we can obtain the ordering of $\mathscr{L}_{\wedge}(\mathfrak{A})/\varrho$.

Analogously to ϱ we may define the relation ϱ^*. If an algebra $\mathfrak{A}$ has only one minimal subalgebra, then for two subalgebras $\mathfrak{B}, \mathfrak{C}$ of $\mathfrak{A}$ we have $(\mathfrak{A}, \mathfrak{B}) \in \varrho^*$ if and only if they contain exactly the same quasi-minimal subalgebras. Now the following three theorems hold.

Theorem 6. *Let ϱ^* be the relation on the lattice $\mathscr{L}(\mathfrak{A})$ of subalgebras of a finite algebra $\mathfrak{A}$ with only one minimal subalgebra defined so that for two subalgebras $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{A}$ we have $(\mathfrak{B}, \mathfrak{C}) \in \varrho^*$ if and only if the subalgebras $\mathfrak{B}$, $\mathfrak{C}$ contain exactly the same quasi-minimal subalgebras of $\mathfrak{A}$. Then the relation ϱ^* is a congruence on the lower semilattice $\mathscr{L}_{\wedge}(\mathfrak{A})$, but in general not on the lattice $\mathscr{L}(\mathfrak{A})$.*

Theorem 7. *Let $G^*(\mathfrak{A})$ be the modified intersection graph of a finite algebra $\mathfrak{A}$ with only one minimal subalgebra. Let ϱ^* be the congruence on the lower semi-*

lattice $\mathscr{L}_\wedge(\mathfrak{A})$ of subalgebras of $\mathfrak{A}$ defined in Theorem 6. Let $\mathscr{X}$ be a class of the congruence ϱ^*. Then $\mathscr{X} - \{\mathfrak{A}\}$ is a maximal CF-set in $G^*(\mathfrak{A})$.

Theorem 8. *Let $\mathscr{L}_\wedge(\mathfrak{A})$ be the lower semilattice of subalgebras of a finite algebra $\mathfrak{A}$ with only one minimal subalgebra, let ϱ^* be the congruence on $\mathscr{L}_\wedge(\mathfrak{A})$ defined in Theorem 6. Then the factor-semilattice $\mathscr{L}_\wedge(\mathfrak{A})/\varrho^*$ is uniquely determined by the modified intersection graph $G^*(\mathfrak{A})$ of $\mathfrak{A}$.*

Proofs of these theorems are analogous to the proofs of Theorems 3, 4, 5. Here we shall give only a counterexample showing that ϱ^* need not be a congruence on the lattice $\mathscr{L}(\mathfrak{A})$. By $\mathscr{M}^*(\mathfrak{D})$, where $\mathfrak{D}$ is a subalgebra of $\mathfrak{A}$ (not minimal), we shall denote the set of quasi-minimal subalgebras of $\mathfrak{D}$.

Let $\mathfrak{A}$ be the semigroup given by the following Cayley table:

	a	b	b^2	c	d	e
a	a	a	a	a	a	a
b	a	b^2	a	d	e	a
b^2	a	a	a	e	a	a
c	a	c	c	a	a	a
d	a	d	d	a	a	a
e	a	e	e	a	a	a

The semigroup $\mathfrak{A}$ contains only one idempotent a and therefore only one minimal subsemigroup $\{a\}$. The quasi-minimal subalgebras of $\mathfrak{A}$ are the subsemigroups $\{a, b^2\}$, $\{a, c\}$, $\{a, d\}$, $\{a, e\}$. Now let $\mathfrak{B}_1 = \{a, b, b^2\}$, $\mathfrak{B}_2 = \{a, b^2\}$, $\mathfrak{C}_1 = \mathfrak{C}_2 = \{a, c\}$. We have $\mathscr{M}^*(\mathfrak{B}_1) = \mathscr{M}^*(\mathfrak{B}_2) = \{\{a, b^2\}\}$, $\mathscr{M}^*(\mathfrak{C}_1) = \mathscr{M}^*(\mathfrak{C}_2) = \{\{a, c\}\}$, therefore $(\mathfrak{B}_1, \mathfrak{B}_2) \in \varrho^*$, $(\mathfrak{C}_1, \mathfrak{C}_2) \in \varrho^*$. The semigroup $\mathfrak{B}_1 \vee \mathfrak{C}_1$ is the whole semigroup $\mathfrak{A}$, therefore $\mathscr{M}^*(\mathfrak{B}_1 \vee \mathfrak{C}_1) = \{\{a, b^2\}, \{a, c\}, \{a, d\}, \{a, e\}\}$. The semigroup $\mathfrak{B}_2 \vee \mathfrak{C}_2 = \{a, b^2, c, e\}$ and $\mathscr{M}^*(\mathfrak{B}_2 \vee \mathfrak{C}_2) = \{\{a, b^2\}, \{a, c\}, \{a, e\}\} \neq \mathscr{M}^*(\mathfrak{B}_1 \vee \mathfrak{C}_1)$.

Obviously, in some cases ϱ may be a congruence even on $\mathscr{L}(\mathfrak{A})$. In this case for all the subalgebras $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{A}$ necessarily $\mathscr{M}(\mathfrak{B} \vee \mathfrak{C}) = \mathscr{M}(\mathfrak{D})$, where $\mathfrak{D}$ is the subalgebra of $\mathfrak{A}$ generated by the union of all subalgebras from $\mathscr{M}(\mathfrak{B}) \cup \mathscr{M}(\mathfrak{C})$. One of these cases is the case of a commutative semigroup.

Theorem 9. *Let $\mathfrak{A}$ be a finite commutative semigroup. Let ϱ be the relation defined in Theorem 3. Then ϱ is a congruence on the lattice $\mathscr{L}(\mathfrak{A})$ of subsemigroups of $\mathfrak{A}$.*

Proof. According to Theorem 3 the relation ϱ is a congruence on $\mathscr{L}_\wedge(\mathfrak{A})$. Therefore it is sufficient to prove that $(\mathfrak{B}_1, \mathfrak{B}_2) \in \varrho$, $(\mathfrak{C}_1, \mathfrak{C}_2) \in \varrho$ implies $(\mathfrak{B}_1 \vee \mathfrak{C}_1, \mathfrak{B}_2 \vee \mathfrak{C}_2) \in \varrho$. The minimal subalgebras of a finite semigroup are one-element subsemigroups consisting of idempotents. Therefore $\mathfrak{B}_1$ and $\mathfrak{B}_2$ have equal sets of idempotents and so have $\mathfrak{C}_1$ and $\mathfrak{C}_2$. The subsemigroup

$\mathfrak{B}_1 \vee \mathfrak{C}_1$ contains all the idempotents which are contained in $\mathfrak{B}_1$, all the idempotents which are contained in $\mathfrak{C}_1$ and all the idempotents which are products of an idempotent of $\mathfrak{B}_1$ and an idempotent of $\mathfrak{C}_1$. Now assume that $\mathfrak{B}_1 \vee \mathfrak{C}_1$ contains an idempotent a such that $a \notin \mathfrak{B}_1$, $a \notin \mathfrak{C}_1$ and a is not a product of an idempotent of $\mathfrak{B}_1$ with an idempotent of $\mathfrak{C}_1$. As $\mathfrak{B}_1 \vee \mathfrak{C}_1$ is generated by the set $\mathfrak{B}_1 \cup \mathfrak{C}_1$ and $\mathfrak{B}_1, \mathfrak{C}_1$ are subsemigroups, the element a must be a product of an element $b \in \mathfrak{B}_1$ and an element $c \in \mathfrak{C}_1$. According to the assumption at least one of the elements b, c is not an idempotent; without the loss of generality let it be b. As $\mathfrak{A}$ is a finite semigroup, there exist positive integers k, l such that b^k, c^l are idempotents (any periodical semigroup, therefore also a cyclic one, contains an idempotent). Now as a is an idempotent, we have $a^{kl} = a$, this means $(bc)^{kl} = b^{kl}\ c^{kl} = a$. But as b^k, c^l are idempotents, we have $b^{kl} = (b^k)^l = b^k$, $c^{kl} = (c^l)^k = c^l$, so $b^k c^l = a$ and a is the product of an idempotent $b^k \in \mathfrak{B}_1$ and an idempotent $c^l \in \mathfrak{C}_1$, which is a contradiction. Thus we have proved that $\mathscr{M}(\mathfrak{B}_1 \vee \mathfrak{C}_1)$ contains all the idempotents of $\mathfrak{B}_1$, all the idempotents of $\mathfrak{C}_1$, all the products of the idempotents of $\mathfrak{B}_1$ with the idempotents of $\mathfrak{C}_1$ and no other idempotents. Analogously $\mathscr{M}(\mathfrak{B}_2 \vee \mathfrak{C}_2)$ contains all the idempotents of $\mathfrak{B}_2$, all the idempotents of $\mathfrak{C}_2$, all the products of the idempotents of $\mathfrak{B}_2$ with the idempotents of $\mathfrak{C}_2$ and no other idempotents. But as $\mathscr{M}(\mathfrak{B}_1) = \mathscr{M}(\mathfrak{B}_2)$, $\mathscr{M}(\mathfrak{C}_1) = \mathscr{M}(\mathfrak{C}_2)$, we see that $\mathscr{M}(\mathfrak{B}_1 \vee \mathfrak{C}_1) = \mathscr{M}(\mathfrak{B}_2 \vee \mathfrak{C}_2)$.

Analogous considerations can be made also for ϱ^*. If ϱ^* is a congruence on $\mathscr{L}(\mathfrak{A})$, then for all subalgebras $\mathfrak{B}$, $\mathfrak{C}$ of $\mathfrak{A}$ necessarily $\mathscr{M}^*(\mathfrak{B} \vee \mathfrak{C}) = \mathscr{M}^*(\mathfrak{D})$, where $\mathfrak{D}$ is the subalgebra of $\mathfrak{A}$ generated by the union of all subalgebras from $\mathscr{M}^*(\mathfrak{B}) \cup \mathscr{M}^*(\mathfrak{C})$.

Theorem 10. *Let $\mathfrak{A}$ be a finite Abelian group. Let ϱ^* be the relation defined in Theorem 6. Then ϱ^* is a congruence on the lattice $\mathscr{L}(\mathfrak{A})$ of the subgroups of $\mathfrak{A}$.*

Proof. The quasi-minimal subalgebras of a finite Abelian group are cyclic subgroups of prime orders. Let $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{C}_1, \mathfrak{C}_2$ be the subgroups of $\mathfrak{A}$. Consider the subgroup $\mathfrak{B}_1 \vee \mathfrak{C}_1$ generated by the set $\mathfrak{B}_1 \cup \mathfrak{C}_1$. Among the elements of prime order in $\mathfrak{B}_1 \vee \mathfrak{C}_1$ there are all the elements of prime order from $\mathfrak{B}_1$ and from $\mathfrak{C}_1$ and all the elements which are products of an element of prime order from $\mathfrak{B}_1$ with an element of the same order from $\mathfrak{C}_1$. We shall prove that $\mathfrak{B}_1 \vee \mathfrak{C}_1$ contains only these elements of prime order. As it is well known, in an Abelian group the order of a product is the least common multiple of the orders of the factors. Therefore the product of two non-unit elements has a prime order if and only if these elements have both the same prime order. Thus we see that $\mathscr{M}^*(\mathfrak{B}_1 \vee \mathfrak{C}_1)$ is uniquely determined by $\mathscr{M}^*(\mathfrak{B}_1)$ and $\mathscr{M}^*(\mathfrak{C}_1)$. As $\mathscr{M}^*(\mathfrak{B}_2) = \mathscr{M}^*(\mathfrak{B}_1)$, $\mathscr{M}^*(\mathfrak{C}_2) = \mathscr{M}^*(\mathfrak{C}_1)$, we have $\mathscr{M}^*(\mathfrak{B}_1 \vee \mathfrak{C}_1) = \mathscr{M}^*(\mathfrak{B}_2 \vee \mathfrak{C}_2)$ and $(\mathfrak{B}_1 \vee \mathfrak{C}_1, \mathfrak{B}_2 \vee \mathfrak{C}_2) \in \varrho^*$.

Finally we shall pay attention to the factor-semilattices $\mathscr{L}_\wedge(\mathfrak{A})/\varrho$ and $\mathscr{L}_\wedge(\mathfrak{A})/\varrho^*$. The semilattice $\mathscr{L}_\wedge(\mathfrak{A})/\varrho$ is a one-element semilattice if and only if $\mathfrak{A}$ contains only one minimal subalgebra. In this case all the subalgebras of $\mathfrak{A}$ contain this minimal subalgebra and no other, therefore the relation ϱ is the universal relation on $\mathscr{L}_\wedge(\mathfrak{A})$. Analogously $\mathscr{L}_\wedge(\mathfrak{A})/\varrho^*$ is a one-element semilattice if and only if $\mathfrak{A}$ contains only one quasi-minimal subalgebra.

Now let us study the reverse case: when $\mathscr{L}_\wedge(\mathfrak{A})/\varrho \cong \mathscr{L}_\wedge(\mathfrak{A})$, i. e. when ϱ is the relation of equality. In this case we have $\mathscr{M}(\mathfrak{B}) = \mathscr{M}(\mathfrak{C})$ if and only if $\mathfrak{B} = \mathfrak{C}$.

Theorem 11. *Let $\mathfrak{A}$ be a finite algebra, let ϱ be the relation defined in Theorem* 3. *Then the following three assertions are equivalent:*

(i) *ϱ is the relation of equality on $\mathscr{L}(\mathfrak{A})$.*

(ii) *Any subalgebra of $\mathfrak{A}$ is either minimal, or generated by a union of minimal subalgebras.*

(iii) *Any monogeneous subalgebra of $\mathfrak{A}$ is either minimal, or generated by a union of minimal subalgebras.*

Remark. A monogeneous subalgebra of an algebra is a subalgebra generated by a single element.

Proof. (i) $\Rightarrow$ (ii). Let ϱ be the relation of equality on $\mathscr{L}(\mathfrak{A})$. This means that $(\mathfrak{B}, \mathfrak{C}) \in \varrho$ implies $\mathfrak{B} = \mathfrak{C}$. As $(\mathfrak{B}, \mathfrak{C}) \in \varrho$ means $\mathscr{M}(\mathfrak{B}) = \mathscr{M}(\mathfrak{C})$, any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is uniquely determined by the set $\mathscr{M}(\mathfrak{B})$ of minimal subalgebras contained in it. Let $\mathfrak{B}'$ be the subalgebra of $\mathfrak{A}$ generated by the union of all subalgebras from $\mathscr{M}(\mathfrak{B})$; this algebra must be contained in $\mathfrak{B}$, because all the subalgebras from $\mathscr{M}(\mathfrak{B})$ are subalgebras of $\mathfrak{B}$. Thus $\mathfrak{B}' \subset \mathfrak{B}$ and therefore also $\mathscr{M}(\mathfrak{B}') \subset \mathscr{M}(\mathfrak{B})$. But $\mathfrak{B}'$ must contain all the elements of $\mathscr{M}(\mathfrak{B})$ as its subalgebras, therefore $\mathscr{M}(\mathfrak{B}) \subset \mathscr{M}(\mathfrak{B}')$. This implies $\mathscr{M}(\mathfrak{B}) = \mathscr{M}(\mathfrak{B}')$, which means $(\mathfrak{B}, \mathfrak{B}') \in \varrho$. As ϱ is the relation of equality, we have $\mathfrak{B} = \mathfrak{B}'$ and thus $\mathfrak{B}$ is generated by a union of minimal subalgebras.

(ii) $\Rightarrow$ (iii) trivially.

(iii) $\Rightarrow$ (ii). Let $\mathfrak{B}$ be a non-monogeneous subalgebra of $\mathfrak{A}$. The subalgebra $\mathfrak{B}$ is generated by all monogeneous subalgebras of $\mathfrak{A}$ generated by the elements of $\mathfrak{B}$. As any of these monogeneous subalgebras is either minimal, or generated by a union of minimal subalgebras, the subalgebra $\mathfrak{B}$ is generated by the union of all of these minimal subalgebras and unions of minimal subalgebras, which is again a union of minimal subalgebras of $\mathfrak{A}$.

(ii) $\Rightarrow$ (i). If (ii) holds, we shall prove that any subalgebra $\mathfrak{B}$ of $\mathfrak{A}$ is generated by the union of elements of $\mathscr{M}(\mathfrak{B})$. Let $\mathscr{N}(\mathfrak{B})$ be some set of minimal subalgebras of $\mathfrak{A}$ whose union generates $\mathfrak{B}$ (if $\mathfrak{B}$ is minimal, then $\mathscr{N}(\mathfrak{B})$ is a one-element set). Any element of $\mathscr{N}(\mathfrak{B})$ must be a subalgebra of $\mathfrak{B}$, therefore $\mathscr{N}(\mathfrak{B}) \subset \mathscr{M}(\mathfrak{B})$ and thus $\mathfrak{B}$, as the subalgebra generated by the union of elements of $\mathscr{N}(\mathfrak{B})$, must be contained in the subalgebra generated by the union of elements of $\mathscr{M}(\mathfrak{B})$. On the other hand, the subalgebra of $\mathfrak{A}$ generated by the union of elements of $\mathscr{M}(\mathfrak{B})$ must be contained in $\mathfrak{B}$, because this union is in $\mathfrak{B}$. We have proved that $\mathfrak{B}$ is generated by the union of the elements of $\mathscr{M}(\mathfrak{B})$, therefore it is uniquely determined by $\mathscr{M}(\mathfrak{B})$. Thus $\mathscr{M}(\mathfrak{B}) = \mathscr{M}(\mathfrak{C})$ implies $\mathfrak{B} = \mathfrak{C}$ and ϱ is the relation of equality.

With the help of Theorem 5 we obtain a corollary.

Corollary 1. *Let the assertions of Theorem* 11 *hold. Then the lattice* $\mathscr{L}(\mathfrak{A})$ *of the subalgebras of* $\mathfrak{A}$ *is uniquely determined by the intersection graph* $G(\mathfrak{A})$ *of the algebra* $\mathfrak{A}$.

Now we shall express a theorem analogous to Theorem 10 and concerning the relation ϱ^*.

Theorem 12. *Let* $\mathfrak{A}$ *be a finite algebra with exactly one minimal subalgebra, let* ϱ^* *be the relation defined in Theorem* 6. *Then the following three assertions are equivalent*:

(i) ϱ^* *is the relation of equality on* $\mathscr{L}(\mathfrak{A})$.

(ii) *Any non-minimal subalgebra of* $\mathfrak{A}$ *is either quasi-minimal, or generated by a union of quasi-minimal subalgebras.*

(iii) *Any monogeneous non-minimal subalgebra of* $\mathfrak{A}$ *is either quasi-minimal, or generated by a union of quasi-minimal subalgebras.*

The proof is analogous to the proof of Theorem 11.

Corollary 2. *Let the assertions of Theorem* 12 *hold. Then the lattice* $\mathscr{L}(\mathfrak{A})$ *of the subalgebras of* $\mathfrak{A}$ *is uniquely determined by the modified intersection graph* $G^*(\mathfrak{A})$ *of the algebra* $\mathfrak{A}$.

Now we shall apply Theorems 11 and 12 to semigroups, groups and lattices.

Theorem 13. *Let* $\mathfrak{A}$ *be a finite semigroup, let* ϱ *be the relation defined in Theorem* 3. *Then* ϱ *is the relation of equality on* $\mathscr{L}(\mathfrak{A})$ *if and only if all the elements of* $\mathfrak{A}$ *are idempotents.*

Proof. If all the elements of $\mathfrak{A}$ are idempotents, then any monogeneous subsemigroup of $\mathfrak{A}$ is a one-element subalgebra and therefore it is minimal. Thus (ii) from Theorem 11 holds, which implies (i). If some element b of $\mathfrak{A}$ is not an idempotent, the monogeneous subsemigroup $\mathfrak{B}$ of $\mathfrak{A}$ generated by b contains more than one element and contains exactly one idempotent c. Then $\mathscr{M}(\mathfrak{B}) = \mathscr{M}(\{c\}) = \{\{c\}\}$, but $\mathfrak{B} \neq \{c\}$, thus ϱ is not the relation of equality.

Theorem 14. *Let* $\mathfrak{A}$ *be a finite lattice, let* ϱ *be the relation defined in Theorem* 3. *Then* ϱ *is the relation of equality on* $\mathscr{L}(\mathfrak{A})$.

Proof. Any monogeneous sublattice of a lattice consists only of one element and any one-element sublattice is minimal. Therefore (iii) and also (i) holds in Theorem 11.

Theorem 15. *Let $\mathfrak{A}$ be a finite group, let ϱ^* be the relation defined in Theorem* 6. *Then ϱ^* is the relation of equality on $\mathscr{L}(\mathfrak{A})$ if and only if the order of any element of $\mathfrak{A}$ is not divisible by the square of a prime number.*

Proof. If the order of any element of $\mathfrak{A}$ is not divisible by the square of a prime number, then the monogeneous (cyclic) subgroup of $\mathfrak{A}$ generated by a non-unit element a has the order which is either a prime number or a product of pairwise different prime numbers. In the first case it is quasi-minimal, in the second case it can be expressed as a direct product of groups of prime orders; these groups are quasi-minimal subgroups of $\mathfrak{A}$ and their direct product is generated by their union. Thus (iii) in Theorem 11 holds and so does (i). On the other hand, if some element b of $\mathfrak{A}$ has the order p^2r, where p is a prime number, r is a positive integer, then b^r has the order p^2; let $\mathfrak{B}$ be the cyclic subgroup of $\mathfrak{A}$ generated by b^r. The subgroup $\mathfrak{B}$ is not quasi-minimal; it contains a quasi-minimal proper subgroup $\mathfrak{B}'$ generated by the element b^{pr}. $\mathfrak{B}'$ is a unique quasi-minimal subgroup of $\mathfrak{B}$, therefore $\mathscr{M}(\mathfrak{B}) = \{\mathfrak{B}'\}$. As $\mathfrak{B}$ contains only one quasi-minimal subgroup as a proper subgroup, it evidently cannot be generated by a union of quasi-minimal subgroups. According to Theorem 11 the relation ϱ^* is not the relation of equality on $\mathscr{L}(\mathfrak{A})$.

The Theorems 13, 14, 15 show that in these cases the lattice $\mathscr{L}(\mathfrak{A})$ is uniquely determined by the graph $G(\mathfrak{A})$ or $G^*(\mathfrak{A})$.

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