SAMPLING THEOREM IN ABSTRACT HARMONIC ANALYSIS

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In the literature on information theory (see e.g. [1]) as to sampling theorem it is referred to the assertion roughly stated as follows:

If the Fourier transform $\hat{f}(\gamma)$ of a function $f(x)$ vanishes for $|\gamma| > \alpha > 0$ then $f(x)$ is completely determined by its values at $\ldots, -2h, -h, 0, h, 2h, \ldots$ where $h = \pi/\alpha$, in fact the formula

$$f(x) = \sum_{n=-\infty}^{\infty} f(nh) \frac{\sin \alpha(x - nh)}{\alpha(x - nh)} \tag{1}$$

holds.

The origin of this theorem can hardly be traced. It, or some of its analogues, was published virtually independently by several autors, e.g. [2], [3], [4].

The aim of this note is to establish and to prove the sampling theorem in terms of abstract harmonic analysis. The rôle of real line will be played by an arbitrary locally compact Abelian group and the rôle of integral multiples of h by its discrete subgroup. From the so obtained general proposition besides the sampling theorem just mentioned some more general statements concerning functions on real line follow.

The proof of generalised sampling theorem, given in this paper, is based on some relatively elementary properties of groups and Fourier transforms on groups treated e.g. in [6] or in the first two chapters of [7]. The concepts and facts used without reference are to be found there.

Let G be a locally compact Abelian group (written additively) and Γ its dual group. The value of a character $\gamma \in \Gamma$ in a point $x \in G$ will be written as (x, γ).

Suppose H be a discrete subgroup of G with discrete annihilator $\Lambda = \{\gamma : (y, \gamma) = 1$ for all $y \in H\}$. For $\gamma \in \Gamma$ we denote by $[\gamma]$ the coset of Λ which contains γ, i.e. $[\gamma] = \gamma + \Lambda$. If $y \in H$ then $(y, [\gamma])$ denotes, of course, the constant value of (y, γ) on the coset $[\gamma]$.

The Haar measure on G, resp. Γ, etc. will be denoted by m_G, resp. m_Γ, etc. We normalize $m_{\Gamma/\Lambda}$ so that $m_{\Gamma/\Lambda}(\Gamma/\Lambda) = 1$. This is possible since Γ/Λ is compact being the dual group of the discrete group H. Let be further $m_\Lambda(\{\lambda\}) = 1$ for $\lambda \in \Lambda$, $m_H(\{y\}) = 1$ for $y \in H$. We normalize the Haar measure m_Γ so that the formula

$$\int_\Gamma F(\gamma)\,\mathrm{d}m_\Gamma(\gamma) = \int_{\Gamma/\Lambda} \mathrm{d}m_{\Gamma/\Lambda}([\gamma]) \sum_{\lambda \in \Lambda} F(\gamma + \lambda) \tag{2}$$

for every non-negative Baire (and every integrable) function F on Γ holds. Such a normalization is possible (see [6], § 33 A; [7], § 27.3). Finally the Haar measure m_G on G let be adjusted so that the inversion formula for Fourier transform holds, i.e. by the relations

$$\hat{f}(\gamma) = \int_G (-x, \gamma) f(x)\,\mathrm{d}m_G(x), \quad f(x) = \int_\Gamma (x, \gamma) \hat{f}(\gamma)\,\mathrm{d}m_\Gamma(\gamma) \tag{3}$$

the Fourier transform and its inverse is given. The Plancherel theorem asserts that by the relations (3) an isometry $f \to \hat{f}$ of a dense subset in $L^2(G)$ onto a dense subset in $L^2(\Gamma)$ is defined. This isometry can be extended by continuity (in the only possible way) to be an unitary equivalence (so called Fourier-Plancherel transform) between $L^2(G)$ and $L^2(\Gamma)$. We conserve the notation $\hat{f}$ for the Fourier-Plancherel transform of an arbitrary function $f \in L^2(G)$.

Let further Ω be a (Baire) measurable subset of Γ containing exactly one element from every coset of Λ, i.e. $\Omega \cap (\gamma + \Lambda)$ consists of a single point for every $\gamma \in \Gamma$.

The set Ω may be intuitively treated as representing the group Γ/Λ. The situation in the classical sampling theorem mentioned in the introduction corresponds to the case $G = \Gamma = (-\infty, \infty)$, $H = \{\dots, -2h, -h, 0, h, 2h, \dots\}$ and $\Omega = (-\alpha, \alpha\rangle$.

Put

$$\varphi(x) = \int_\Omega (x, \gamma)\,\mathrm{d}m_\Gamma(\gamma). \tag{4}$$

Lemma. *The function φ is by* (4) *defined everywhere on G. It is continuous, positive-definite and belongs to $L^2(G)$. Its norm in $L^2(G)$ is* 1. *We have $\varphi(0) = 1$. If $y \in$ H, $y \neq 0$, then $\varphi(y) = 0$ and*

$$\int_G \varphi(x)\,\overline{\varphi(x - y)}\,\mathrm{d}m_G(x) = 0. \tag{5}$$

Proof. If we choose for F in (2) the characteristic function χ_Ω of Ω, we obtain $m_\Gamma(\Omega) = 1$ since $\Sigma_{\lambda \in \Lambda} \chi_\Omega(\gamma + \lambda) = 1$ for all $\gamma \in \Lambda$. Thus the integral in (4) exists for all $x \in G$. The equality (4) means that φ is the inverse Fourier transform of χ_Ω. Since χ_Ω is integrable and non-negative, φ is continuous and positive-definite. (The last assertion is a consequence of the Bochner-Weil theorem. The positive-definiteness will not be used in the following.) χ_Ω also belongs to $L^2(\Gamma)$, hence the square of φ is integrable and the Plancherel theorem implies that $\|\varphi\| = \|\chi_\Omega\| = 1$.

To prove (5) note, that $\varphi(x - y)$ (for fixed y) is the inverse Fourier transform of $(-y, \gamma)\chi_\Omega(\gamma)$. Using again the Plancherel theorem and (2) we obtain for all $y \in H$:

$$\int_G \varphi(x)\varphi(x - y)\, dm(x) = \int_\Gamma \chi_\Omega(\gamma)(-y, \gamma)\chi_\Omega(\gamma)\, dm_\Gamma(\gamma) = \int_\Gamma (y, \gamma)\chi_\Omega(\gamma)\, dm_\Gamma(\gamma) =$$
$$= \int_{\Gamma/\Lambda} (y, [\gamma])\, dm_{\Gamma/\Lambda}([\gamma]).$$

Since $(y, [\gamma])$ is (as a function of $[\gamma]$) a character of compact group Γ/Λ the last integral vanishes for every non-zero $y \in H$. It follows at once that $\varphi(y) = 0$ for $y \in H$, $y \neq 0$. The equality $\varphi(0) = 1$ is clear.

Theorem. *Suppose $f \in L^2(G)$ and $\hat{f}(\gamma) = 0$ for almost all $\gamma \notin \Omega$. Then f is equal almost everywhere to a continuous function. If f itself is continuous then*

$$f(x) = \sum_{y \in H} f(y)\varphi(x - y) \tag{6}$$

uniformly on G and in the sense of the convergence in $L^2(G)$. Furtheremore

$$\|f\|^2 = \sum_{y \in H} |f(y)|^2. \tag{7}$$

Proof. Since H is the dual group of Γ/Λ, every character of Γ/Λ may be written as $(y, [\gamma])$ for some $y \in H$. The set of all characters forms a complete orthonormal family of functions in $L^2(\Gamma/\Lambda)$ (see [7], § 38 C).

Denote $[E] = \{[\gamma] : \gamma \in E\}$ for $E \subset \Gamma$. Putting $F = \chi_E$ in (2) we get $m_\Gamma(E) = m_{\Gamma/\Lambda}([E])$ for all measurable sets $E \subset \Omega$. Denoting $F_1([\gamma]) = F(\gamma)$ for a function F on Ω it follows that $F \in L^1(\Omega)$ if and only if $F_1 \in L^1(\Gamma/\Lambda)$ and $\int_\Omega F(\gamma)\, dm_\Gamma(\gamma) = \int_{\Gamma/\Lambda} F_1([\gamma])\, dm_{\Gamma/\Lambda}([\gamma])$. Thus $F \in L^2(\Omega)$ if and only if $F_1 \in L^2(\Gamma/\Lambda)$. We conclude that the characters (y, γ) for $y \in H$ (more precisely the partial functions restricted to $\gamma \in \Omega$) form a complete orthonormal family in $L^2(\Omega)$.

The assumption and the Plancherel theorem implies that $\hat{f} \in L^2(\Gamma)$. Since $\hat{f}(\gamma) = 0$ for almost all $\gamma \notin \Omega$, there exist numbers a_y so that

$$\hat{f}(\gamma) = \sum_{y \in H} a_y(y, \gamma)\chi_\Omega(\gamma) \tag{8}$$

in the sense of convergence in $L^2(\Gamma)$.

Put $\varphi_y(x) = \varphi(x - y)$ for $y \in H$. By lemma the functions φ_y are orthonormal; $\hat{\varphi}(\gamma) = \hat{\varphi}_0(\gamma) = \chi_\Omega(\gamma)$ and $\hat{\varphi}_y(\gamma) = (-y, \gamma)\chi_\Omega(\gamma)$. The Fourier-Plancherel transform being unitary we get from (8)

$$f = \sum_{y \in H} a_y \varphi_{-y}$$

in $L^2(G)$ and, consequently, $\|f\|^2 = \Sigma_{y \in H} |a_y|^2$.

Since $m_\Gamma(\Omega)$ is finite, $L^1(\Omega) \supset L^2(\Omega)$. Hence $\hat{f} \in L^1(\Gamma)$ and

$$f(x) = \int_\Gamma (x, \gamma) \hat{f}(\gamma) \, \mathrm{d}m_\Gamma(\gamma)$$

almost everywhere on G. Since the integral on the right hand is a continuous function, f is equivalent to continuous function.

The convergence in $L^2(\Omega)$ implies that in $L^1(\Omega)$. It follows, that the equality (8) holds in $L^1(\Gamma)$ too. Hence if f itself is continuous, we have

$$f(x) = \int_\Gamma (x, \gamma) \hat{f}(\gamma) \, \mathrm{d}m_\Gamma(\gamma) = \int_\Gamma (x, \gamma) \, \mathrm{d}m_\Gamma(\gamma) \sum_{y \in H} a_y(y, \gamma) \chi_\Omega(\gamma) =$$
$$= \sum_{y \in H} a_y \int_\Gamma (x + y, \gamma) \chi_\Omega(\gamma) \, \mathrm{d}m_\Gamma(\gamma) = \sum_{y \in H} a_y \varphi_{-y}(x)$$

for all $x \in G$, i.e.

$$f(x) = \sum_{y \in H} a_{-y} \varphi_y(x). \tag{9}$$

The interchange of integration and summation is based on the convergence in $L^1(\Gamma)$ of the sum in (8) and on the fact that the bounded function (x, γ) does not violate this convergence. The equality (9) holds also uniformly on G, since for an arbitrary set $H_1 \subset H$ we have

$$\Big|f(x) - \sum_{y \in H_1} a_y \varphi_{-y}(x)\Big| = \Big|\int_\Gamma (x, \gamma) \, \mathrm{d}m_\Gamma(\gamma) \sum_{y \in H - H_1} a_y(y, \gamma) \chi_\Omega(\gamma)\Big| \leq$$
$$\leqq \int_\Gamma \Big| \sum_{y \in H - H_1} a_y(y, \gamma) \chi_\Omega(\gamma)\Big| \, \mathrm{d}m_\Gamma(\gamma)$$

and the integral on the right hand may be made arbitrarily small by the suitable choice of H_1.

If we put $x = y_0$ in (9) for some $y_0 \in H$, by lemma we have $f(y_0) = a_{-y_0}$. The proof is complete.

Choosing $G = \Gamma = (-\infty, \infty)$, $\Omega = (-\alpha, \alpha\rangle$ and, consequently, $H = \{\ldots, -2h, -h, 0, h, 2h, \ldots\}$ with $h\alpha = \pi$ for the function from (4) we get $\varphi(x) = (\sin \alpha x)/(\alpha x)$. Hence, if $f \in L^2(-\infty, \infty)$ and $\hat{f}(\gamma) = 0$ for $|\gamma| > \alpha$, we obtain (1).

But from the theorem just proved we may deduce more. If $\hat{f}(\gamma) = 0$ outside an arbitrary measurable set Ω of numbers pairwise incongruent modulo 2α, then $f(x)$ is completely determined by its values belonging to H. The function $(\sin \alpha x)/(\alpha x)$ in formula (1) must be of course replaced by $\varphi(x) = \int_\Omega e^{i\gamma y} d\gamma$. E.g. if $\hat{f}(\gamma) = 0$ for $|\gamma| \leqq 4\alpha$ and $|\gamma| \geqq 5\alpha$ then f is determined by its values on H. Using the classical formulation of sampling theorem, however, it were necessary to determine its values in points $\ldots, -\frac{2}{5}h, -\frac{1}{5}h, 0, \frac{1}{5}h, \frac{2}{5}h, \ldots$ It is clear from these notes that the converse of the qualitative part of sampling theorem i.e. of the assertion „if the spectrum of a function is concentrated in $\langle -\alpha, \alpha \rangle$ then the function is determined by its values in $\ldots, -2h, -h, 0, h, 2h, \ldots$" is not true. A function may possess an unbounded spectrum and depend only upon its values on H.

It follows, further, that it is impossible to prove the sampling theorem for functions on the real line in such a generality as follows from the theorem proved in this paper by the means of the theory of interpolation of entire functions. In fact, by the well-known Paley-Wiener theorem a function $f \in L^2(-\infty, \infty)$ is an entire one of exponential type if and only if its Fourier transform vanishes outside a compact set. Let us note that we have proved the uniform convergence of (1) on the whole interval $(-\infty, \infty)$ and by the means of the theory of functions of complex variable we can prove the uniform convergence only on bounded subsets of $(-\infty, \infty)$. (See [5].)

If we choose for G the multiplicative group of complex numbers z with $|z| = 1$ for H the group of all roots of the equation $z^n - 1 = 0$, we get a formula due to Cauchy obtained in [2] by the means of the Lagrange's interpolation formula.

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ТЕОРЕМА ОТСЧЕТОВ В АБСТРАКТНОМ ГАРМОНИЧЕСКОМ АНАЛИЗЕ

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Резюме

Пусть G — локально компактная коммутативная группа и Γ — ее группа (непрерывных) характеров. Пусть H — дискретная подгруппа группы G, аннулятор Λ которой (множество характеров, значение которых равно 1 на всем H) тоже дискретный. Пусть, далее, Ω — измеримое подмножество Γ, содержащее из каждого класса смежности по Λ равно один элемент. Определим функцию φ равенством (4), где m_Γ надлежащим образом нормированная мера Хаара на Γ. В статье доказывается следующая

Теорема. Пусть f — функция из $L^2(G)$ преобразование Фурье которой обращается в нуль вне множества Ω. Тогда f почти всюду равна некоторой непрерывной функции. Если f сама непрерывна, то имеет место (6) равномерно на G и в смысле сходимости в $L^2(G)$; кроме того справедлива формула (7).

Если положим $G = \Gamma = (-\infty, \infty)$, $H = \{\ldots, -2h, -h, 0, h, 2h, \ldots\}$ $(h > 0)$ и $\Omega = \langle -\alpha, \alpha)$, где $\alpha h = \pi$, то из этой теоремы вытекает справедливость теоремы, известной в литературе по теории информации под названием теоремы отсчетов Котельникова. В этом случае равенство (6) получает вид (1).