Mathematica Bohemica

Miroslav Bosák
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Mathematica Bohemica, Vol. 119 (1994), No. 1, 21–42

Persistent URL: http://dml.cz/dmlcz/126204

BIFURCATION OF STATIONARY SOLUTIONS TO QUASIVARIATIONAL INEQUALITIES

MIROSLAV BOSÁK, Prague

(Received August 10, 1992)

Summary. Bifurcation and eigenvalue theorems are proved for a certain type of quasivariational inequalities using the method of a jump in the Leray-Schauder degree.

Keywords: bifurcation problems, variational inequalities, quasivariational inequalities, eigenvalue problems, partial differential inequalities, unilateral problems

AMS classification: 35J85, 49J40

1. INTRODUCTION

Let $A\colon H \to H$ be a completely continuous linear operator on a real Hilbert space H (with the inner product $(\cdot,\cdot)$ and norm $|\cdot|$), let $G\colon \mathbf{R} \times H \to H$ be a completely continuous (nonlinear) mapping satisfying

$$\lim_{u\to 0} \frac{G(\lambda, u)}{|u|} = 0 \quad \text{uniformly on compact } \lambda\text{-intervals,} \tag{1.1}$$

and let $\{K(u);\, u \in H\}$ be a system of closed convex subsets of H.

We are interested in examining bifurcation from the origin of the solutions of the quasivariational inequality

$$u \in K(u)\colon \quad (\lambda u - Au - G(\lambda, u), v - u) \geqslant 0 \qquad \text{for all } v \in K(u); \tag{1.2}$$

that is, we are looking for values $\lambda > 0$ (bifurcation points of Ineq. (1.2)) such that $\lambda_n \to \lambda$, $0 \neq u_n \to 0$ for some solutions $[\lambda_n, u_n] \in \mathbf{R} \times H$ of (1.2).

The first major works about quasivariational inequalities appeared in the first half of the 1970's. Among others we mention Bensoussan [3], Bensoussan, Goursat, Lions [4], Friedman [7], Baiocchi, Capelo [2]. In the papers Joly, Mosco [10] and Mosco [17] existence (not bifurcation) theorems were proved for a certain type of quasivariational inequalities. Alternatively, the bifurcation problem for the inequality (1.2) with $K(u) \equiv K$, $K \subset H$ a closed convex cone with its vertex at zero has been extensively studied over the last 15 years. Miersemann [15], [16] proved bifurcation theorems for variational inequalities for the case of a potential operator. At the same time, Kučera [11], [12], [13] successfully treated the nonsymmetric case using a method based on Dancer's global bifurcation theorem. Kučera's results were later improved and extended by Quittner [19], [20], who developed a more efficient and simpler method based on a jump in the Leray-Schauder degree. The aim of the present paper is to show that most of these results remain valid if we let K vary with u, provided the mapping $u \longrightarrow K(u)$ is in a certain sense continuous. We prove the existence of a bifurcation point $\lambda \in (\lambda_1, \lambda_2)$ of Ineq. (1.2), where $\lambda_1 < \lambda_2$ are positive eigenvalues of A satisfying certain assumptions (see Section 4, Theorems 1,2,3). Also, under an additional assumption on the system $\{K(u)\}$, the existence of a bifurcation point $\lambda > \lambda_0$ is proved, where λ_0 is a positive eigenvalue of A (Theorem 4). This theorem is of particular interest when λ_0 is the largest eigenvalue of a symmetric operator A; in this case the theorem ensures the existence of an eigenvalue λ of (1.4) (see also Remark 5) that is larger than the first eigenvalue of A. (Recall that this is never the case when $K(u) \equiv K$, $u \in H$, i.e. when (1.4) is a standard variational inequality, and A is symmetric.) Some of our results, namely Theorems 3,4, deal with the situation when $\operatorname{int} K(u) = \emptyset$ which is important in the applications. Our approach is a modification of the method used by P. Quittner and can be briefly described as follows: Ineq. (1.2) with $\lambda > 0$ is rewritten as

$$(1.3) \qquad \lambda u - P_{\lambda u}(Au + G(\lambda, u)) = 0,$$

where $P_u : H \longrightarrow K(u)$ is the projection onto the convex set $K(u)$. To prove that there is at least one bifurcation point of (1.3) between two values λ_1, λ_2 (see Proposition 2) we show that there is a jump in the degree of the mapping $u \longrightarrow \lambda u - P_{\lambda u} Au$ which corresponds to the linearized inequality

$$(1.4) \qquad u \in K(u): \qquad (\lambda u - Au, v - u) \geqslant 0 \quad \text{for all } v \in K(u).$$

To determine this degree we give a series of lemmas in Section 3. Finally, an interpretation of our theorems concerning partial differential equations with unilateral conditions can be found in Section 4.

2. Preliminaries

Let us summarize the notation used throughout the paper:

$(\cdot,\cdot)$, $|\cdot|$ denote the inner product and the norm on H,
$P_u\colon H \to K(u)$ is the projection onto $K(u)$ with respect to $(\cdot,\cdot)$,
$T_0(\lambda, u) = \lambda u - P_{\lambda u} A u$,
$B_r(u)$ is the ball in H with centre u and radius $r > 0$,
$S = \partial B_1(0)$,
$\mathrm{d}(\lambda) = \deg(T_0(\lambda,\cdot), B_1(0))$ (see Remark 6),
$\sigma_+(A)$ is the set of all positive eigenvalues of A,
$E(\lambda) = \mathrm{Ker}(\lambda I - A)$,
$E^*(\lambda) = \mathrm{Ker}(\lambda I - A^*)$,
$K = \{u \in H\,;\ u \text{ satisfies } (2.9)\}$,
$K^a = \{u \in H\,;\ (\exists D \subset H, \overline{D} = H)(\forall w \in D)(\exists t > 0)(u \pm tw \in K)\}$,
$u_n \to u$, $u_n \rightharpoonup u$ denote the strong and the weak convergence in H, respectively.

Let $\{K(u)\,;\ u \in H\}$ be a system of closed convex subsets of H with the following properties:

(2.1) $\quad K(u) \neq H, K(u) \neq \emptyset \quad$ for each $\quad u \in H$,

(2.2) $\quad K(\lambda u) = \lambda K(u) \quad$ for all $\quad u \in H, \lambda > 0$,

(2.3) $\quad$ if $u_n \rightharpoonup u, v_n \rightharpoonup v, v_n \in K(u_n) \quad$ then $\quad v \in K(u)$,

(2.4) $\quad$ if $u_n \rightharpoonup u, v \in K(u)$ then there exist $v_n \in K(u_n), v_n \to v$,

(2.5) $\quad$ if $u_n \to u, v_n \to v, v_n \in \partial K(u_n) \quad$ then $\quad v \in \partial K(u)$.

Remark 1. Note that $u = 0$ is a solution of (1.2) for all $\lambda > 0$. Indeed, we obtain easily from (2.1), (2.2), (2.3) that $0 \in K(0)$ and it follows from (1.1), and from the continuity of the mapping G that $G(\lambda, 0) = 0$.

Remark 2. Let $f \in H$, $\lambda > 0$. By virtue of (2.2) the inequality

(2.6) $$u \in K(u)\colon \qquad (\lambda u - Au - f, v - u) \geqslant 0 \text{ for all } v \in K(u)$$

can be rewritten as

$$u \in K(u)\colon \qquad (\lambda u - (Au + f), v - \lambda u) \geqslant 0 \text{ for all } v \in K(\lambda u).$$

Since the projection $P_u z$ of $z \in H$ onto $K(u)$ is the only element of the set $K(u)$ that satisfies

$$(P_u z - z, v - P_u) \geqslant 0 \text{ for all } v \in K(u),$$

the inequality (2.6) is equivalent to the equation

$$\lambda u = P_{\lambda u}(Au + f).$$

In particular, Ineq. (1.2) is equivalent to Eq. (1.3).

Proposition 1. *If $u_n \rightharpoonup u$, $z_n \to z$ then $P_{u_n} z_n \to P_u z$.*

Proof. First realize that $|P_{u_n} z_n| \leqslant C$, $n = 1, 2, \ldots$ Indeed, since $K(u)$ is nonempty, we can choose $v \in K(u)$ and we obtain from (2.4) a sequence $v_n \in K(u_n)$ such that $v_n \to v$. Hence

$$|z_n - P_{u_n} z_n| \leqslant |z_n - v_n| \tag{2.7}$$

and $P_{u_n} z_n = z_n + (P_{u_n} z_n - z_n)$ is bounded. Thus we can suppose $P_{u_n} z_n \rightharpoonup w \in H$. Repeating the same argument we obtain from (2.7) for any $v \in K(u)$:

$$|z - w| \leqslant \liminf_{n\to\infty} |z_n - P_{u_n} z_n| \leqslant \limsup_{n\to\infty} |z_n - P_{u_n} z_n| \leqslant |z - v|. \tag{2.8}$$

Moreover, (2.3) implies $w \in K(u)$ and thus we conclude from (2.8) that $w = P_u z$. Now we put $v = w$ in (2.8) to get $\lim\limits_{n\to\infty} |z_n - P_{u_n} z_n| = |z - w|$. Hence $z_n - P_{u_n} z_n \to z - w = z - P_u z$. □

Remark 3. Let $u_n \rightharpoonup u$, $v_n \to v$, $v \in \operatorname{int} K(u)$. Then $v_n \in \operatorname{int} K(u_n)$ for n sufficiently large. Indeed, Proposition 1 implies $P_{u_n} v_n \to P_u v$ and if $v_n \notin \operatorname{int} K(u_n)$ we would have $P_{u_n} v_n \in \partial K(u_n)$. Then it would follow from (2.5) that $v = P_u v \in \partial K(u)$.

Remark 4. As a result of Proposition 1 we obtain the following assertion: Let $u_n \rightharpoonup u$, $v_n \to v$, $\lambda_n \to \lambda \neq 0$ and $\lambda_n u_n = P_{\lambda_n u_n}(Au_n + v_n)$. Then $u_n \to u$ and $\lambda u = P_{\lambda u}(Au + v)$.

Remark 5. We say that a number $\lambda \in \mathbf{R}$ is an eigenvalue of the inequality (1.4) if there exists a nonzero solution u of (1.4). The solution u is then called an eigenvector of (1.4). It follows from Remarks 2, 4 that under the assumption (1.1) any bifurcation point $\lambda > 0$ of (1.2) is an eigenvalue of (1.4).

Remark 6. Let D be a bounded open region in H, $\lambda > 0$, $T(\lambda, u) = \lambda u - P_{\lambda u}(Au + G(\lambda, u))$ and let $T(\lambda, u) \neq 0$ for all $u \in \partial D$. It follows from Proposition 1 that the Leray-Schauder degree – $\deg(T(\lambda, \cdot), D)$ – of the mapping $T(\lambda, \cdot)\colon H \to H$ with respect to 0 is defined. See [9] for the definition as well as for simple properties of this degree. Further, let us denote $\mathrm{d}(\lambda) = \deg(T_0(\lambda, \cdot), B_1(0))$ where

$$T_0(\lambda, u) = \lambda u - P_{\lambda u} Au.$$

Note that $\mathrm{d}(\lambda)$ is defined iff $\lambda > 0$ is not an eigenvalue of (1.4) and that in this case we have $\mathrm{d}(\lambda) = \deg(T_0(\lambda, \cdot), B_R(0))$ for all $R > 0$.

The following two propositions follow from the basic properties of the Leray-Schauder degree. Their proofs are similar to the case $K(u) \equiv K$ which can be found in [20].

Proposition 2. *Assume that* $0 < \lambda_1 < \lambda_2$, λ_1, λ_2 *are not eigenvalues of* (1.4). *If* $\mathrm{d}(\lambda_1) \neq \mathrm{d}(\lambda_2)$ *then there is a bifurcation point of Ineq.* (1.2) *in the interval* (λ_1, λ_2).

Proposition 3. *Let* $f \in H$, $\lambda > 0$ *be fixed, with* λ *not an eigenvalue of* (1.4), $T(\lambda, u) = \lambda u - P_{\lambda u}(Au + f)$. *Then there exists* $R_0 > 0$ *such that*

$$\deg(T(\lambda, \cdot), B_R(0)) = \mathrm{d}(\lambda) \quad \textit{for all } R > R_0.$$

In particular, if the inequality (2.6) *has no solution then* $\mathrm{d}(\lambda) = 0$.

The points $u \in H$ with the following property will be important in our considerations:

$$v \in K(v) \Longrightarrow v + u \in K(v); \tag{2.9}$$

we denote

$$K = \{u \in H\,;\, u \text{ satisfies } (2.9)\}.$$

It is readily verified that by virtue of the assumption (2.2), K is a closed convex cone with its vertex at zero. Notice that in the constant case $K(u) = K(0)$ for all $u \in H$ we have $K(u) = K$. Further, following Quittner [19], we define

$$K^a = \{u \in H\,;\, (\exists D \subset H, \overline{D} = H)(\forall w \in D)(\exists t > 0)(u \pm tw \in K)\}.$$

The following simple lemma, for variational inequalities first proved by Kučera [11] and later generalized by Quittner [19], plays a key role in our method. It provides a sufficient condition that the eigenvectors of Ineq. (1.4) corresponding to a given eigenvalue λ of A, are exactly the eigenvectors $u \in K(u)$ of the operator A.

Proposition 4. *Let* $\lambda_0 \in \sigma_+(A)$ *be such that* $K^a \cap E^*(\lambda_0) \neq \emptyset$. *Then any eigenvector of Ineq.* (1.4) *corresponding to* λ_0 *satisfies* $\lambda_0 u = Au$, *i.e.* $u \in E(\lambda_0)$.

P r o o f. Let $u \in K(u)$ be an eigenvector of (1.4) corresponding to λ_0 and let $u^* \in K^a \cap E^*(\lambda_0)$. Then for any $w \in D$ there exists $t > 0$ such that $u^* \pm tw \in K$. Hence, $u + u^* \pm tw \in K(u)$ and this choice of v in (1.4) yields

$$(\lambda_0 u - Au, u^* \pm tw) \geqslant 0.$$

Since $(\lambda_0 u - Au, u^*) = 0$ we have

$$(\lambda_0 u - Au, \pm tw) \geqslant 0,$$
$$(\lambda_0 u - Au, w) = 0.$$

The statement now follows from the fact that $\overline{D} = H$. □

3. Determination of $\mathrm{d}(\lambda)$

As we have mentioned above the method we use to prove bifurcation for Ineq. (1.2) is based on a jump in the degree, i.e. on Proposition 2. The following lemmas give several ways to determine the degree $\mathrm{d}(\lambda)$ (see Remark 6).

Throughout this section let ε denote a sufficiently small positive number.

Lemma 1. *Let $\lambda_0 \in \sigma_+(A)$ and $u_0^* \in \operatorname{int} K \cap E^*(\lambda_0)$. Assume*

(3.1) $$u \notin \partial K(u) \quad \text{for all } 0 \neq u \in E(\lambda_0).$$

We assert

(a) *if*

(3.2) $$(u_0^*, u_0) > 0 \quad \text{for some } u_0 \in E(\lambda_0) \cap \operatorname{int} K(u_0)$$

then $\mathrm{d}(\lambda) \neq 0$ *for all* $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$,

(b) *if*

(3.3) $$(u_0^*, u_0) < 0 \quad \text{for some } u_0 \in E(\lambda_0) \cap \operatorname{int} K(u_0)$$

then $\mathrm{d}(\lambda) \neq 0$ *for all* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$.

P r o o f. In order to prove part (a) of the lemma let us verify the following points (I), (II):

(I) There are no eigenvalues of Ineq. (1.4) in the set $(\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon)$.

Let us assume that there exist sequences $\lambda_n \to \lambda_0$, $\lambda_n \neq \lambda_0$, $0 \neq u_n \in K(u_n)$ such that

(3.4) $$(\lambda_n u_n - Au_n, v - u_n) \geqslant 0 \quad \text{for all } v \in K(u_n).$$

Remark 2 yields

(3.5) $$\lambda_n u_n = P_{\lambda_n u_n} Au_n,$$

and we can suppose $|u_n| = 1$, $u_n \rightharpoonup u \in H$. Remark 4 implies $u_n \to u$, $\lambda_0 = P_{\lambda_0 u}Au$, and by Proposition 4, $u \in E(\lambda_0)$. On the other hand, $u_n \in \partial K(u_n)$ for all large n since otherwise u_n would satisfy $\lambda_n u_n = Au_n$, and λ_n would be eigenvalues of A. Hence $u \in \partial K(u)$ by the property (2.5). Since u is nonzero this contradicts (3.1) and (I) is proved.

(II) The inequality

$$u \in K(u): \qquad (\lambda u - Au - (\lambda - \lambda_0)u_0, v - u) \geqslant 0 \quad \text{for all } v \in K(u) \tag{3.6}$$

has for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ the only solution u_0.

Let $\lambda_n \searrow \lambda_0$, $u_n \in K(u_n)$, $u_n \neq u_0$,

$$(\lambda_n u_n - Au_n - (\lambda_n - \lambda_0)u_0, v - u_n) \geqslant 0 \quad \text{for all } v \in K(u_n). \tag{3.7}$$

Since $u_0^* \in K$, we have $u_n + u_0^* \in K(u_n)$ for all n. Setting $v = u_n + u_0^*$ in (3.7) we obtain

$$(\lambda_n u_n - Au_n - (\lambda_n - \lambda_0)u_0, u_0^*) \geqslant 0.$$

We have $(Au_n, u_0^*) = (u_n, Au_0^*) = \lambda_0(u_n, u_0^*)$ and, consequently,

$$\begin{aligned} ((\lambda_n - \lambda_0)u_n - (\lambda_n - \lambda_0)u_0, u_0^*) &\geqslant 0, \\ (\lambda_n - \lambda_0)(u_n - u_0, u_0^*) &\geqslant 0, \\ (u_n - u_0, u_0^*) &\geqslant 0, \\ (u_n, u_0^*) \geqslant (u_0, u_0^*) &> 0. \end{aligned}$$

Hence

$$|u_n| \geqslant \varepsilon > 0, \quad n = 1, 2, \ldots \tag{3.8}$$

By Remark 2

$$\lambda_n u_n = P_{\lambda_n u_n}(Au_n + (\lambda_n - \lambda_0)u_0). \tag{3.9}$$

Putting $w_n = \frac{u_n}{|u_n|}$ and using (2.2) we rewrite (3.9) as

$$\lambda_n w_n = P_{\lambda_n w_n}\Big(Aw_n + (\lambda_n - \lambda_0)\frac{u_0}{|u_n|}\Big). \tag{3.10}$$

Assuming $w_n \rightharpoonup w \in H$ and using Remark 4 we obtain from (3.8), (3.10) $\lambda_0 w = P_{\lambda_0 w}Aw$ together with $w_n \to w$. Proposition 4 gives $w \in E(\lambda_0)$. Moreover, Ineq. (3.7) ensures $u_n \in \partial K(u_n)$ for n sufficiently large. Indeed, let $u_n \in \operatorname{int} K(u_n)$. Then (3.7) would imply $\lambda_n u_n - Au_n = (\lambda_n - \lambda_0)u_0$ and, since λ_n is not an eigenvalue of A for n large, we would have $u_n = u_0$. This is a contradiction and therefore $u_n \in \partial K(u_n)$, i.e. $w_n \in \partial K(w_n)$. Thus (2.5) implies $w \in \partial K(w)$ which contradicts (3.1). The proof of (II) is complete.

To complete the proof of Lemma 1 we define

$$T(\lambda, u) = \lambda u - P_{\lambda u}(Au + (\lambda - \lambda_0)u_0). \tag{3.11}$$

It follows from (II) that for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$

$$\deg(T(\lambda, \cdot), B_R(0) \setminus \overline{B_r(u_0)}) = 0 \tag{3.12}$$

where $r > 0$ is sufficiently small, $R > 0$ sufficiently large. By the additivity property of the degree we have

$$\deg(T(\lambda, \cdot), B_R(0)) = \deg(T(\lambda, \cdot), B_r(u_0)) + \deg(T(\lambda, \cdot), B_R(0) \setminus \overline{B_r(u_0)}).$$

Since $\lambda_0 u_0 \in \operatorname{int} K(\lambda_0 u_0)$ we obtain from Remark 3 that there is $r > 0$ such that $Au + (\lambda - \lambda_0)u_0 \in K(\lambda u)$ for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$, $u \in B_r(u_0)$. Hence

$$T(\lambda, u) = \lambda u - Au - (\lambda - \lambda_0)u_0 = \lambda(u - u_0) - A(u - u_0)$$

for such λ and u. On the other hand, the element u_0 is the only solution of the equation $\lambda u - Au = (\lambda - \lambda_0)u_0$. Using Schauder's formula (see e.g. [18]) we get

$$\deg(T(\lambda, \cdot), B_r(u_0)) = \deg(\lambda I - A, B_r(0)) = (-1)^{\beta(\lambda_0)}, \tag{3.13}$$

where

$$\beta(\lambda_0) = \sum_{\lambda > \lambda_0} \dim\left(\bigcup_{p=1}^{\infty} \operatorname{Ker}(\lambda I - A)^p\right). \tag{3.14}$$

Consequently, $\deg(T(\lambda, \cdot), B_R(0)) \neq 0$ for all R sufficiently large. The assertion (a) now follows from Proposition 3 as $\deg(T(\lambda, \cdot), B_R(0)) = \mathrm{d}(\lambda)$ for large values of R.

To prove the part (b) we need to show that u_0 is the only solution of (3.6) for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$. As above we proceed by contradiction. Let $\lambda_n \nearrow \lambda_0$, $u_n \neq u_0$, $u_n \in K(u_n)$ satisfy (3.7). Since $u_0^* \in K$ we can take $v = u_n + u_0^* \in K(u_n)$ in (3.7) to obtain

$$\begin{aligned} (\lambda_n u_n - Au_n - (\lambda_n - \lambda_0)u_0, u_0^*) &\geqslant 0, \\ (\lambda_n - \lambda_0)[(u_n, u_0^*) - (u_0, u_0^*)] &\geqslant 0, \\ (u_n, u_0^*) \leqslant (u_0, u_0^*) &< 0, \end{aligned}$$

and therefore $|u_n| > \varepsilon > 0$, $n = 1, 2, \ldots$. The rest of the proof follows the same lines as that of (a). $\square$

Remark 7. Let $K(u) \equiv K$, $u \in H$ in Lemma 1. Then it can be easily verified that the assumption $\operatorname{int} K \cap E^*(\lambda_0) \neq \emptyset$ together with (3.1) imply $\dim E(\lambda_0) = 1$. So, Lemma 1 can be used only with simple eigenvalues λ_0 if (1.2) is a variational inequality. This is not true in general and one can easily construct examples of quasivariational inequalities in $\mathbf{R}^3$ with multiple eigenvalues λ_0 of the operator A satisfying the assumptions of Lemma 1. Nevertheless, we shall prove the following lemma which admits multiple eigenvalues even in the constant case $K(u) \equiv K$. (Cf. [19], Theorem 4.)

Remark 8. Let $\lambda_0 \in \sigma_+(A)$, $K^a \cap E^*(\lambda_0) \neq \emptyset$. Then the closed convex cone with its vertex at the origin $K \cap E^*(\lambda_0)$ is not a linear space. Indeed, if $-u_0 \in K$ for an element $u_0 \in K^a$ we would have $K = H$ which would contradict (2.1). By [20], Lemma 2, there exists an element $u_1^* \in K \cap E^*(\lambda_0)$ such that

$$u_1^* \neq 0, \ (u_1^*, u^*) \geqslant 0 \quad \text{for all } u^* \in K \cap E^*(\lambda_0). \tag{3.15}$$

Lemma 2. *Let $\lambda_0 \in \sigma_+(A)$ be such that $\operatorname{int} K \cap E^*(\lambda_0) \neq \emptyset$. Assume*

$$\forall\, u \in E(\lambda_0) \cap \partial K(u) \cap S \ \exists\, u^* \in E^*(\lambda_0) \cap K : (u, u^*) < 0 \tag{3.16}$$

and

$$u \in \operatorname{int} K(u) \quad \text{for all } u \in E(\lambda_0) \cap (E^*(\lambda_0)^{\perp} \oplus \{c u_1^*;\, c \geqslant 0\}) \cap S, \tag{3.17}$$

where $u_1^* \in K \cap E^*(\lambda_0)$ is a vector satisfying (3.15). Then $\mathrm{d}(\lambda) \neq 0$ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

Proof. We shall first prove that

(I) Ineq. (1.4) has no eigenvalues in $(\lambda_0, \lambda_0 + \varepsilon)$.

Assume that there exist sequences $\lambda_n \searrow \lambda_0$, $u_n \neq 0$, $u_n \in K(u_n)$ that satisfy (3.4). As in the proof of Lemma 1 we get $u_n \to u$, $|u_n| = 1$, $u \in E(\lambda_0) \cap \partial K(u)$. By the assumption (3.16) there exists $u^* \in E^*(\lambda_0) \cap K$ such that $(u, u^*) < 0$. Putting $v = u_n + u^* \in K(u_n)$ in (3.4) we obtain

$$\begin{aligned}(\lambda_n u_n - A u_n, u^*) &\geqslant 0,\\ (\lambda_n - \lambda_0)(u_n, u^*) &\geqslant 0,\\ (u_n, u^*) &\geqslant 0.\end{aligned}$$

This contradicts $(u, u^*) < 0$, and (I) is proved.

(II) the inequality

$$(3.18) \qquad u \in K(u): \qquad (\lambda u - Au - u_1^*, v - u) \geqslant 0 \quad \text{for all } v \in K(u)$$

has for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ exactly one solution which satisfies $u \in \operatorname{int} K(u)$.

In order to prove (II) let us first consider the solutions $u \in \partial K(u)$ of (3.18). Assume that there exist $u_n \in \partial K(u_n)$, $\lambda_n \searrow \lambda_0$ satisfying

$$(3.19) \qquad (\lambda_n u_n - Au_n - u_1^*, v - u_n) \geqslant 0 \quad \text{for all } v \in K(u_n).$$

Since $u_1^* \in K$ we can put $v = u_n + u_1^*$ in (3.19) to obtain

$$(u_n, u_1^*) \geqslant (\lambda_n - \lambda_0)^{-1} |u_1^*|^2,$$

which implies $|u_n| \to \infty$. Further, putting $w_n = \frac{u_n}{|u_n|}$ we can rewrite (3.19) as

$$\lambda_n w_n = P_{\lambda_n w_n}\Big(Aw_n + \frac{u_1^*}{|u_n|}\Big).$$

We can assume $w_n \rightharpoonup u$ and Remark 4 yields $w_n \to u \in S$, $\lambda_0 u = P_{\lambda_0 u} Au$. Moreover, taking into account Proposition 4 and the property (2.5), we get $u \in E(\lambda_0) \cap \partial K(u)$. By the assumption (3.16), there exists an element $u^* \in K \cap E^*(\lambda_0)$ with $(u, u^*) < 0$. On the other hand, the choice $v = u_n + u^*$ in (3.19) together with (3.15) yield

$$(\lambda_n - \lambda_0)(u_n, u^*) \geqslant (u_1^*, u^*) \geqslant 0,$$

which implies $(u, u^*) \geqslant 0$. Hence a contradiction and there is no solution $u \in \partial K(u)$ of (3.18). Further, any solution $u \in \operatorname{int} K(u)$ of (3.18) satisfies $\lambda u - Au = u_1^*$. Thus, it is sufficient to prove that the (unique) solution of this equation with $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ satisfies $u \in \operatorname{int} K(u)$. It was proved in [19], p. 291 that if $\lambda_n \searrow \lambda_0$, $\lambda_0 \in \sigma_+(A)$, $u_1^* \in E^*(\lambda_0)$ and $\lambda_n u_n - Au_n = u_1^*$, $n = 1, 2, \dots$ then

$$\frac{u_n}{|u_n|} \to u \in E(\lambda_0) \cap (E^*(\lambda_0)^\perp \oplus \{cu_1^*;\, c \geqslant 0\})$$

for a suitable subsequence of $\{u_n\}$. By (3.17), $u \in \operatorname{int} K(u)$ and therefore $u_n \in \operatorname{int} K(u_n)$ for n large, which completes the proof of (II).

In the rest of the proof we proceed as in Lemma 1; putting

$$(3.20) \qquad T(\lambda, u) = \lambda u - P_{\lambda u}(Au + u_1^*)$$

we prove $\deg(T(\lambda, \cdot), B_R(0)) = (-1)^{\beta(\lambda_0)}$ for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ and R sufficiently large. The assertion then follows from Proposition 3. □

Lemma 3. *Let* $\lambda_0 \in \sigma_+(A)$ *be such that* $K^a \cap E^*(\lambda_0) \neq \emptyset$. *Then we assert*
(a) *if*

$$(3.21) \qquad \forall u \in E(\lambda_0) \cap K(u) \cap S \ \exists u^* \in E^*(\lambda_0) \cap K : (u, u^*) > 0$$

then $\mathrm{d}(\lambda) = 0$ *for all* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$,
(b) *if*

$$(3.22) \qquad \forall u \in E(\lambda_0) \cap K(u) \cap S \ \exists u^* \in E^*(\lambda_0) \cap K : (u, u^*) < 0$$

then $\mathrm{d}(\lambda) = 0$ *for all* $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

Proof. We will confine ourselves to proving part (a), the proof of part (b) being similar. As in Lemma 2 one can prove

(I) Ineq. (1.4) has no eigenvalue λ in $(\lambda_0 - \varepsilon, \lambda_0)$.

Further, we shall consider Ineq. (3.18), where $0 \neq u_1^* \in E^*(\lambda_0) \cap K$ is a vector satisfying (3.15) (see Remark 8), and we shall prove

(II) Ineq. (3.18) has no solution for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$.

Assume there exist $u_n \in K(u_n), \lambda_n \nearrow \lambda_0$ satisfying (3.19). As in the proof of Lemma 2 we get $\frac{u_n}{|u_n|} \to u \in E(\lambda_0) \cap K(u) \cap S$. By the assumption (3.21), there exists $u^* \in E^*(\lambda_0) \cap K$ such that $(u, u^*) > 0$. Putting $v = u_n + u^*$ in (3.19) we obtain

$$(\lambda_n - \lambda_0)(u_n, u^*) \geqslant (u_1^*, u^*) \geqslant 0,$$

which implies $(u, u^*) \leqslant 0$. This is a contradiction and (II) is proved.

Finally, defining $T(\lambda, u)$ by (3.20), we obtain from (I), (II) and from Proposition 3

$$\mathrm{d}(\lambda) = \deg(T(\lambda, \cdot), B_R(0)) = 0$$

for $\lambda \in (\lambda_0 - \varepsilon, \lambda_0)$ and R sufficiently large. □

Lemma 4. *Let* $\lambda_0 \in \sigma_+(A)$, $E(\lambda_0) \cap \{u \in H\,;\, u \in K(u)\} = \{0\}$, $E^*(\lambda_0) \cap K^a \neq \emptyset$. *Then* $\mathrm{d}(\lambda) = 0$ *for all* $\lambda \in (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.

Proof. By the preceding lemma, $\mathrm{d}(\lambda) = 0$ for λ close to λ_0, $\lambda \neq \lambda_0$. In particular, there is no eigenvalue of (1.4) in $(\lambda_0 - \varepsilon, \lambda_0) \cup (\lambda_0, \lambda_0 + \varepsilon)$. Further, it follows from Proposition 4 and from $E(\lambda_0) \cap \{u \in H\,;\, u \in K(u)\} = \{0\}$ that λ_0 is not an eigenvalue of (1.4) and $\mathrm{d}(\lambda_0)$ is defined. If $\mathrm{d}(\lambda_0) \neq 0$ we would obtain a contradiction with Proposition 2, putting $\lambda_1 = \lambda_0 - \frac{1}{2}\varepsilon$, $\lambda_2 = \lambda_0$. See also Remark 5. □

Being a generalization of [19], Theorem 5, the following lemma admits operators with multiple eigenvalues as well as systems of sets $K(u)$ with empty interiors. The important assumption is that A is symmetric.

Lemma 5. *Let A be a symmetric operator, $\lambda_0 \in \sigma_+(A)$ and $u_0^* \in K^a \cap E(\lambda_0)$. In addition, let there exist $u_0 \in E(\lambda_0) \cap K(u_0) \cap S$ such that*

$$(u_0, u_0^*) > 0, \tag{3.23}$$

and

$$\left.\begin{array}{l} w \in \bigcup\limits_{\lambda \in \mathbb{R}} E(\lambda) \\ u_n \in H, u_n \to u \\ u \in K(u) \cap E(\lambda_0) \cap S \end{array}\right. \Longrightarrow \; u_0 \pm tw \in K(u_n) \quad \begin{array}{l} \text{for some } t > 0, \\ \text{all } n \text{ large.} \end{array} \tag{3.24}$$

Then $\mathrm{d}(\lambda) \neq 0$ *for all* $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

Remark 9. It follows from Remark 3 that (3.24) is satisfied if

$$u_0 \in \operatorname{int} K(u) \quad \text{for all } u \in E(\lambda_0) \cap K(u) \cap S. \tag{3.25}$$

If H is finite dimensional then (3.24), (3.25) are equivalent.

Remark 10. Note that if $K(u)$ is independent of u, i.e. if $K(u) = K$ for all $u \in H$, then all assumptions of Lemma 5 reduce to the following one: there exists an element $u_0 \in E(\lambda_0)$ such that for arbitrary $w \in \bigcup\limits_{\lambda \in \mathbb{R}} E(\lambda)$ there is $t > 0$ with $u_0 \pm tw \in K$. Cf. Theorem 5, [18].

Proof. We shall consider again the inequality (3.6) and prove

(I) Ineq. (3.6) has the only solution u_0 for $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

Assume there exist sequences $\lambda_n \searrow \lambda_0, u_n \neq u_0$ that satisfy (3.7). Using (3.23) we proceed as in the proof of Lemma 1 to obtain (3.8), (3.10) together with

$$w_n = \frac{u_n}{|u_n|} \to u \in E(\lambda_0) \cap K(u) \cap S.$$

We shall prove $u = u_0$. Rewriting (3.7) we get

(3.26)

$$w_n \in K(w_n): \qquad \left(\lambda_n w_n - Aw_n - \frac{(\lambda_n - \lambda_0)}{|u_n|} u_0, v - w_n\right) \geqslant 0 \quad \text{for all } v \in K(w_n).$$

Let $\pi: H \to H_0$ be the projection onto the space

$$H_0 = \bigoplus_{\lambda \geqslant \lambda_0} E(\lambda).$$

Since H_0 is finitedimensional, the following fact follows from (3.24) which we shall use several times in the sequel:

Let u_n, u be as in (3.24) and $v_n \in H_0$, $n = 1, 2, \ldots$. Then $u_0 + v_n \in K(u_n)$ for n large.

Hence $u_0 + \pi(w_n - u) \in K(w_n)$ for n large. Putting $v = u_0 + \pi(w_n - u)$ in (3.26) we get

$$\begin{aligned} 0 &\leqslant \Big(\lambda_n w_n - Aw_n - \frac{\lambda_n - \lambda_0}{|u_n|} u_0, u_0 - u + (\pi - I)w_n\Big) \\ &= \Big(\lambda_n w_n - Aw_n - \frac{\lambda_n - \lambda_0}{|u_n|} u_0, u_0 - u\Big) \\ &\quad + (\lambda_n w_n - Aw_n, (\pi - I)w_n) - \frac{\lambda_n - \lambda_0}{|u_n|}(u_0, (\pi - I)w_n). \end{aligned}$$

Since $u_0 \in H_0$, the last term equals zero. Also, $(Aw_n, u_0 - u) = \lambda_0(w_n, u_0 - u)$ and therefore

$$0 \leqslant (\lambda_n - \lambda_0)\Big(w_n - \frac{u_0}{|u_n|}, u_0 - u\Big) + (\lambda_n w_n - Aw_n, (\pi - I)w_n). \tag{3.27}$$

Let us prove that the second term in (3.27) is $\leqslant 0$. Indeed, let $\lambda \geqslant \lambda_0$, $w \in H$ be arbitrary. Let $\{u_{(s)}\}$ be an orthonormal basis of the eigenvectors of A, $w = \sum_s c_s u_{(s)}$. Then

$$\pi w = \sum_{\lambda_s \geqslant \lambda_0} c_s u_{(s)}, \quad (\pi - I)w = -\sum_{\lambda_s < \lambda_0} c_s u_{(s)},$$
$$\lambda w - Aw = \sum_s (\lambda - \lambda_s) c_s u_{(s)}.$$

Thus we get

$$\begin{aligned} (\lambda w - Aw, (\pi - I)w) &= \Big(\sum_s (\lambda - \lambda_s) c_s u_{(s)}, -\sum_{\lambda_s < \lambda_0} c_s u_{(s)}\Big) \\ &= -\sum_{\lambda_s < \lambda_0} (\lambda - \lambda_s)(c_s)^2 \leqslant 0 \quad \text{for all } \lambda \geqslant \lambda_0,\ w \in H. \end{aligned} \tag{3.28}$$

Moreover, equality in (3.28) occurs if and only if $w \in H_0$. Thus (3.27) yields

$$0 \leqslant (w_n, u_0 - u) - \Big(\frac{u_0}{|u_n|}, u_0 - u\Big).$$

Since $|u_0| = |u| = 1$, we get $(u_0, u_0 - u) \geqslant 0$ and so $0 \leqslant (w_n, u_0 - u)$. This implies $0 \leqslant (u, u_0 - u)$ and $u = u_0$. By virtue of (3.28), the second term in (3.27) is zero and therefore $w_n \in H_0$. Hence $A(w_n - u_0) \in H_0$, $n = 1, 2, \ldots$. As above we get using (3.8) and (3.24)

$$u_0 + \lambda_0^{-1} A(w_n - u_0) + \lambda_0^{-1} \frac{\lambda_n - \lambda_0}{|u_n|} u_0 \in K(\lambda_0^{-1} \lambda_n w_n)$$

for n large. Hence

$$Aw_n + \frac{\lambda_n - \lambda_0}{|u_n|} u_0 = \lambda_0 u_0 + A(w_n - u_0) + \frac{\lambda_n - \lambda_0}{|u_n|} u_0 \in K(\lambda_n w_n)$$

and (3.10) implies

$$\lambda_n w_n - Aw_n = \frac{\lambda_n - \lambda_0}{|u_n|} u_0.$$

Since $\lambda_n I - A \colon H \to H$ is an isomorphism for n large, the last equation has the only solution $w_n = \frac{u_0}{|u_n|}$. Finally, $|w_n| = |u_0| = 1$ implies $u_n = w_n = u_0$. Hence a contradiction and the assertion (I) is proved.

(II) Ineq. (1.4) has no eigenvalue in the interval $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$.

Assume there exist sequences $u_n \in H$, $|u_n| = 1$, $\lambda_n \searrow \lambda_0$ such that (3.4) holds. As in the proof of Lemma 1 we obtain (3.5) together with $u_n \to u$, $u \in E(\lambda_0) \cap K(u) \cap S$. Since π is a projection onto the finite dimensional space H_0, the assumption (3.24) yields $u_0 + \pi(u_n - u) \in K(u_n)$ for n large and so we get from (3.4)

$$\begin{aligned} 0 &\leqslant (\lambda_n u_n - Au_n, u_0 + \pi(u_n - u) - u_n) \\ &= (\lambda_n u_n - Au_n, u_0 - u + (\pi - I)u_n) \\ &= (\lambda_n u_n - Au_n, u_0 - u) + (\lambda_n u_n - Au_n, (\pi - I)u_n) \\ &= (\lambda_n - \lambda_0)(u_n, u_0 - u) + (\lambda_n u_n - Au_n, (\pi - I)u_n). \end{aligned} \tag{3.29}$$

By (3.28), the last term is $\leqslant 0$, which implies $(u_n, u_0 - u) \geqslant 0$. Hence $(u, u_0 - u) \geqslant 0$, and $|u| = 1$, $|u_0| = 1$ yield $u_0 = u$. Using again (3.28) for $w = u_n$ together with (3.29) we get $c_s = 0$ for all $\lambda_s < \lambda_0$ and therefore $u_n = \pi u_n$. Hence $A(u_n - u_0) \in H_0$ and by a similar argument as above we get from (3.24) $Au_n = \lambda_0 u_0 + A(u_n - u_0) \in K(\lambda_n u_n)$ for all n large. Thus (3.5) yields $\lambda_n u_n = Au_n$. This contradicts $\lambda_n \searrow \lambda_0$ and (II) is proved.

(III) For any $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ there exists $r(\lambda) > 0$ such that $\deg(T(\lambda, \cdot), B_r(u_0)) \neq 0$ for all $r \in (0, r(\lambda))$, where $T(\lambda, u)$ is given by (3.11).

Let $\varepsilon > 0$ be such that there is no eigenvalue of A in $(\lambda_0, \lambda_0 + \varepsilon)$. We take a fixed value $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ and put $f = (\lambda - \lambda_0)u_0$,

$$H(t,u) = \lambda u - tP_{\lambda u}(Au + f) - (1-t)(Au + f)$$

for $u \in H, 0 \leqslant t \leqslant 1$. Then

$$H(1,u) = T(\lambda, u),$$
$$H(0,u) = \lambda u - Au - (\lambda - \lambda_0)u_0.$$

Obviously, the equation $H(0,u) = 0$ has the only solution $u = u_0$. Using Schauder's formula (see [18]) we get

$$\deg(H(0,\cdot), B_r(u_0)) = \deg(\lambda I - A, B_r(0)) = (-1)^{\beta(\lambda_0)} \quad \text{for all } r > 0,$$

where $\beta(\lambda_0)$ is from (3.14). To prove $\deg(H(1,\cdot), B_r(u_0)) = \deg(H(0,\cdot), B_r(u_0))$ for $r > 0$ small it is sufficient to show

$$H(t,u) \neq 0, \quad \text{for all } 0 \leqslant t \leqslant 1,\ u \in B_r(u_0). \tag{3.30}$$

Assume $H(t_n, u_n) = 0$ for $u_n \to u_0$, $t_n \in [0,1]$, $u_n \neq u_0$, $n = 1, 2, \ldots$. We have

$$\lambda u_n - t_n P_{\lambda u_n}(Au_n + f) - (1 - t_n)(Au_n + f) = 0,$$
$$\lambda(u_n - u_0) = t_n(P_{\lambda u_n}(Au_n + f) - (Au_n + f)) + A(u_n - u_0).$$

Let

$$w_n = \frac{u_n - u_0}{|u_n - u_0|}, \quad z_n = \frac{P_{\lambda u_n}(Au_n + f) - (Au_n + f)}{|u_n - u_0|}.$$

Then

$$\lambda w_n = t_n z_n + Aw_n. \tag{3.31}$$

Let $u_n - u_0 = \sum_s c_s^n u_{(s)}$, $n = 1, 2, \ldots$, $s_0 > 0$ a fixed integer. Then

$$Au_n + f = \lambda u_0 + A(u_n - u_0) = \lambda u_0 + \sum_s \lambda_s c_s^n u_{(s)}$$
$$= \lambda u_0 + \sum_{s=1}^{s_0} \lambda_s c_s^n u_{(s)} + \sum_{s > s_0} \lambda_s c_s^n u_{(s)}.$$

Since $u_n \dot{\rightarrow} u_0 \in K(u_0) \cap E(\lambda_0) \cap S$ we get from (3.24) that

$$u_0 + \lambda^{-1} \sum_{s=1}^{s_0} \lambda_s c_s^n u_{(s)} \in K(u_n)$$

for all n large. Thus $\lambda u_0 + \sum\limits_{s=1}^{s_0} \lambda_s c_s^n u_{(s)} \in K(\lambda u_n)$ and by the definition of the projection $P_{\lambda u}\colon H \rightarrow K(\lambda u)$

$$|Au_n + f - P_{\lambda u_n}(Au_n + f)| \leqslant \Big|\sum_{s>s_0} \lambda_s c_s^n u_{(s)}\Big| \leqslant |\lambda_{s_0}| \cdot |u_n - u_0|.$$

We have proved $|z_n| \leqslant |\lambda_{s_0}|$ for n sufficiently large. Since $\lambda_s \rightarrow 0$ as $s \rightarrow +\infty$, we get $z_n \rightarrow 0$. We can suppose $w_n \rightharpoonup w$ and (3.31) implies $w_n \rightarrow w \neq 0$, $\lambda w = Aw$. This is a contradiction since $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ is not an eigenvalue of A. Thus (3.30) holds and (III) is proved.

The rest of the proof is similar to the proof of Lemma 1. □

Lemma 6. *Let the system* $\{K(u)\}$, *in addition to the properties* (2.1)–(2.5), *satisfy*

(3.32) $$u = tP_u 0 \text{ for some } 0 \leqslant t \leqslant 1 \Longrightarrow u = 0.$$

Then $\mathrm{d}(\lambda) = 1$ *for all large* λ.

Remark 11. Note that (3.32) holds if $0 \in K(u)$ for all $u \in H$.

Proof. We put $H(t, u) = \lambda u - tP_{\lambda u} Au$ and prove

$$H(t, u) \neq 0 \quad \text{for all } 0 \leqslant t \leqslant 1,\ u \in S,\ \lambda \text{ large}.$$

Assume on the contrary that there exist sequences u_n, t_n, λ_n such that $|u_n| = 1$, $0 \leqslant t_n \leqslant 1$, $\lambda_n \rightarrow +\infty$ and

$$\lambda_n u_n = t_n P_{\lambda_n u_n} Au_n, \quad n = 1, 2, \ldots.$$

Dividing by λ_n we obtain

$$u_n = t_n P_{u_n}\Big(\frac{1}{\lambda_n} Au_n\Big).$$

Assuming $u_n \rightharpoonup u$, $t_n \rightarrow t \in [0, 1]$ we get from Proposition 1 and from the complete continuity of A that $u_n \rightarrow u$, $u = tP_u 0$. Hence $|u| = 1$ and simultaneously $u = 0$ by (3.32). □

4. Existence of bifurcation points

Using our lemmas from the preceding section together with Proposition 2 we obtain bifurcation points of the quasivariational inequality (1.2). For instance, if $0 < \lambda_1 < \lambda_2$ are such that $\mathrm{d}(\lambda) = 1$ for $\lambda \in (\lambda_1, \lambda_1 + \varepsilon)$ and $\mathrm{d}(\lambda) = 0$ for $\lambda \in (\lambda_2 - \varepsilon, \lambda_2)$ then there exists a bifurcation point λ of Ineq. (1.2) in the interval (λ_1, λ_2). Thus we obtain a bifurcation point of (1.2) between two positive eigenvalues of the operator A satifying certain assumptions. Similarly, if we prove $\mathrm{d}(\lambda) = 0$ for all $\lambda \in (\lambda_0, \lambda_0 + \varepsilon)$ and $\mathrm{d}(\lambda) = 1$ for λ sufficiently large we come up with a bifurcation of (1.2) that is larger than a given eigenvalue λ_0 of A. The following theorem is an immediate consequence of Lemma 1 (a), Lemma 3 (a) and of Proposition 2.

Theorem 1. *Let $0 < \lambda_1 < \lambda_2$ be two eigenvalues of the operator A such that*

$$\exists\, u_0 \in E(\lambda_1) \cap \operatorname{int} K(u_0)\ \exists\, u_0^* \in E^*(\lambda_1) \cap \operatorname{int} K : (u_0, u_0^*) > 0, \tag{4.1}$$

$$u \notin \partial K(u) \quad \text{for all } 0 \neq u \in E(\lambda_1), \tag{4.2}$$

$$E^*(\lambda_2) \cap \operatorname{int} K \neq \emptyset, \tag{4.3}$$

$$\forall\, u \in E(\lambda_2) \cap K(u) \cap S\ \ \exists\, u^* \in E^*(\lambda_2) \cap K : (u, u^*) > 0. \tag{4.4}$$

Then there exists a bifurcation point $\lambda \in (\lambda_1, \lambda_2)$ of Ineq. (1.2).

Using Lemma 1 (b) together with Lemma 3 (b) we can see that Theorem 1 remains valid if we swap the roles of λ_1 and λ_2 and reverse the inequalities in both (4.1) and (4.4). Further, Lemma 2 together with Lemma 3 (a) give

Theorem 2. *Let $0 < \lambda_1 < \lambda_2$ be two eigenvalues of the operator A and let $u_1^* \in K \cap E^*(\lambda_1)$ be a nonzero element satisfying*

$$(u_1^*, u) \geqslant 0 \text{ for all } u \in K \cap E^*(\lambda_1).$$

Let the following hold:

$$\operatorname{int} K \cap E^*(\lambda_1) \neq \emptyset,$$
$$\forall\, u \in \partial K(u) \cap E(\lambda_1) \cap S\ \exists\, u^* \in K \cap E^*(\lambda_1) : (u, u^*) < 0,$$
$$u \in \operatorname{int} K(u) \text{ for all } u \in E(\lambda_1) \cap (E^*(\lambda_1)^{\perp} \oplus \{cu_1^*;\ c \geqslant 0\}) \cap S,$$
$$\operatorname{int} K \cap E^*(\lambda_2) \neq \emptyset,$$
$$\forall\, u \in K(u) \cap E(\lambda_2) \cap S\ \exists\, u^* \in K \cap E^*(\lambda_2) : (u, u^*) > 0.$$

Then there exists a bifurcation point $\lambda \in (\lambda_1, \lambda_2)$ of Ineq. (1.2).

The following theorem uses Lemma 5 and Lemma 3 (a). It admits systems of convex sets $K(u)$ with empty interiors but holds only for symmetric operators.

Theorem 3. *Let $A\colon H \to H$ be a symmetric operator, $0 < \lambda_1 < \lambda_2$ two eigenvalues of A. Let there exist two elements $u_1 \in K(u_1) \cap E(\lambda_1) \cap S$, $u_1^* \in K^a \cap E(\lambda_1)$ such that*

$$(u_1, u_1^*) > 0,$$

$$\left.\begin{array}{l} w \in \bigcup\limits_{\lambda \in \mathbf{R}} E(\lambda) \\ u_n \in H, u_n \to u \\ u \in K(u) \cap E(\lambda_1) \cap S \end{array}\right. \implies u_1 \pm tw \in K(u_n) \quad \begin{array}{l} \text{for some } t > 0, \\ \text{all } n \text{ large.} \end{array}$$

Moreover, let

$$K^a \cap E(\lambda_2) \neq \emptyset,$$
$$\forall u \in K(u) \cap E(\lambda_2) \cap S \ \exists u^* \in K \cap E(\lambda_2)\colon (u, u^*) > 0.$$

Then there is at least one bifurcation point λ of Ineq. (1.2) in the interval (λ_1, λ_2).

As we have pointed out in Introduction it is well known that if A is symmetric and (1.4) is a variational inequality then no eigenvalue of (1.4) (and therefore no bifurcation point of (1.2)) can be larger than the largest eigenvalue of A. The next theorem, based on Lemma 3 (b) and Lemma 6, shows that this is no longer the case if we let $K(u)$ vary with u. Indeed, the quasivariational inequality (1.4) can have an eigenvalue $\lambda > \lambda_0$, λ_0 being the first eigenvalue of A. See also Example 2.

Theorem 4. *Let $\lambda_0 > 0$ be an eigenvalue of A, $K^a \cap E^*(\lambda_0) \neq \emptyset$ and let the following condition hold:*

(4.5) $$\forall u \in E(\lambda_0) \cap K(u) \cap S \ \exists u^* \in E^*(\lambda_0) \cap K : (u, u^*) < 0.$$

Moreover, let

(4.6) $$u = tP_u 0 \text{ for some } 0 \leqslant t \leqslant 1 \implies u = 0.$$

Then there exists a bifurcation point $\lambda > \lambda_0$ of Ineq. (1.2).

Example 1. We shall give the following interpretation of Theorem 3:

Let Ω be an open subset of $\mathbf{R}^n$ with sufficiently regular boundary and let $H =$

$H^1(\Omega)$ be equipped with the scalar product $(u,v) = \sum_{i=1}^{n} \int_\Omega D^i u D^j v \,\mathrm{dx} + \int_\Omega uv \,\mathrm{dx}$. We define

$$(Au, v) = \int_\Omega uv \,\mathrm{dx} \quad \text{for all } u, v \in H,$$

$$(G(\lambda, u), v) = \int_\Omega g(\lambda, u) v \,\mathrm{dx} \quad \text{for all } u, v \in H,\ \lambda \in \mathbf{R},$$

where $g\colon \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ is a smooth function satisfying

$$\lim_{u \to 0} \frac{g(\lambda, u)}{|u|} = 0 \text{ uniformly on compact } \lambda\text{-intervals},$$

such that the Nemyckij operator $g\colon \mathbf{R} \times H \to L^2(\Omega)$ is completely continuous. (See for instance [8] for such functions.) Then $A\colon H \to H$ and $G\colon \mathbf{R} \times H \to H$ are completely continuous, A is symmetric and (1.1) holds. Further, let Γ be an open subset of $\partial\Omega$ and $\varphi \in L^2(\partial\Omega)$ a given function. We define

$$K(u) = \left\{ v \in H^1(\Omega);\ v(x) \geqslant \int_{\partial\Omega} \varphi u \,\mathrm{dx} \text{ a.e. on } \Gamma \right\}.$$

Then $\{K(u)\}$ is a system of convex sets satisfying the conditions (2.1)–(2.5). The properties (2.3)–(2.4) follow from the well known trace theorem for $H^1(\Omega)$, see Lions, Magenes [14]. For instance, to verify (2.4) it is sufficient to put $v_n(x) = v(x) + (\int_{\partial\Omega} \varphi u_n \,\mathrm{dx} - \int_{\partial\Omega} \varphi u \,\mathrm{dx})$. In this setting, the solutions of Ineq. (1.2) for $\lambda > 0$ are the weak solutions of the problem

(4.7) $$\lambda \Delta u + (1 - \lambda) u + g(\lambda, u) = 0 \quad \text{on } \Omega$$

(4.8) $$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \setminus \Gamma$$

(4.9) $$\frac{\partial u}{\partial \nu} \geqslant 0, \quad u \geqslant \int_{\partial\Omega} \varphi u \,\mathrm{dx}, \quad \frac{\partial u}{\partial \nu}\left(u - \int_{\partial\Omega} \varphi u \,\mathrm{dx}\right) = 0 \quad \text{on } \Gamma.$$

Further, the elements of $E(\lambda)$, i.e. the eigenfunctions of A, are the solutions of the Neumann problem

(4.10) $$\Delta u + \frac{1 - \lambda}{\lambda} u = 0 \quad \text{on } \Omega$$

(4.11) $$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

We have $E(\lambda) \subset C^1(\overline{\Omega})$ and $K = \{u \in H^1(\Omega);\ u \geqslant 0 \text{ a.e. on } \Gamma\}$. We can see that $u \in K^a$ for any function $u \in H \cap C(\overline{\Omega})$ such that $u \geqslant \varepsilon$ on Γ, $\varepsilon > 0$. (It is sufficient to put $D = C^1(\overline{\Omega})$ in the definition of K^a.) Theorem 3 gives the following proposition:

Let $0 < \lambda_1 < \lambda_2$ be two eigenvalues of (4.10), (4.11). Then there exists a bifurcation point $\lambda \in (\lambda_1, \lambda_2)$ of the problem (4.7)–(4.9) provided

(i) there exist $u_1, u_1^* \in E(\lambda_1) \cap S$ such that

$$
\begin{aligned}
&(u_1, u_1^*) > 0\\
&u_1^* \geqslant \varepsilon > 0 \quad \text{on } \Gamma\\
&u_1 \geqslant \int_{\partial\Omega} \varphi u_1 \, \mathrm{dx} \quad \text{on } \Gamma\\
&u_1 > \int_{\partial\Omega} \varphi u \, \mathrm{dx} \text{ on } \Gamma \quad \text{for all } u \in E(\lambda_1) \cap S,\ u \geqslant \int_{\partial\Omega} \varphi u \, \mathrm{dx} \text{ on } \Gamma,
\end{aligned}
$$

(ii) there exists $u_2 \in E(\lambda_2)$ such that $u_2 \geqslant \varepsilon > 0$ on Γ,

(iii) for any $u \in E(\lambda_2) \cap S$ such that $u \geqslant \int_{\partial\Omega} \varphi u \, \mathrm{dx}$ on Γ there exists $u^* \in E(\lambda_2)$ such that $u^* \geqslant 0$ on Γ and $(u, u^*) > 0$.

Example 2. Using Theorem 4 we shall show that inequality (1.2) can have bifurcation points $\lambda > \lambda_0$, λ_0 being the first eigenvalue of a symmetric operator A. Let $\Omega = (0,1)$, $H = \{u \in H^1(\Omega);\ u(0) = 0\}$, $(u,v) = \int_0^1 u'v' \mathrm{d}x$ and let A, G be as in Example 1. Consider the system of convex sets in H

$$K(u) = \{v \in H^1(0,1);\ v(0) = 0,\ v(1) \geqslant \tau(u)\},$$

where $\tau\colon H \to \mathbf{R}$ is a continuous functional on H such that $\tau(\lambda u) = \lambda \tau(u)$ for $\lambda > 0$, $u \in H$. The solutions of Ineq. (1.2) are the solutions of the problem

$$\lambda u'' + u + g(\lambda, u) = 0 \ \text{ on } (0,1) \tag{4.12}$$

$$u(0) = 0 \tag{4.13}$$

$$u'(1) \geqslant 0,\ u(1) \geqslant \tau(u),\ u'(1)(u(1) - \tau(u)) = 0. \tag{4.14}$$

The elements of $E(\lambda)$ are the solutions of

$$\lambda u'' + u = 0 \ \text{ on } (0,1) \tag{4.15}$$

$$u(0) = 0,\ u'(1) = 0. \tag{4.16}$$

We have

$$K = \{u \in H^1(0,1);\ u(0) = 0, u(1) \geqslant 0\}, \quad K^a = \operatorname{int} K = \{u \in K;\ u(1) > 0\}.$$

Let λ_0 be the largest eigenvalue of the problem (4.15), (4.16). We have $\lambda_0 = 4/\pi^2$, $E(\lambda_0) = \{c \sin \frac{\pi}{2} x;\ c \in \mathbf{R}\}$ and the condition $K^a \cap E(\lambda_0) \neq \emptyset$ is fulfilled. Clearly, (4.5) is satisfied if

(i) $u(1) < \tau(u)$ for any $0 \neq u \in E(\lambda_0)$.

Indeed, in this case we have $E(\lambda_0) \cap \{u \in H; u \in K(u)\} = \{0\}$. Further, the projection $w = P_u 0$ is $w \equiv 0$ if $\tau(u) \leqslant 0$ and $w(x) = \tau(u)x$, $x \in [0, 1]$, if $\tau(u) > 0$. Thus, (4.6) is satisfied if the following condition holds:

(ii) if $u(x) = ax$, $x \in [0, 1]$, $a > 0$ then $\tau(u) < a$.

By Theorem 4, the assumptions (i), (ii) imply the existence of a bifurcation point $\lambda \in (4/\pi^2, +\infty)$ of the problem (4.12)–(4.14). To satisfy (i), (ii) one can take for instance $\tau(u) = \alpha \int_0^1 |u(x)| \, dx$ with $\frac{1}{2}\pi < \alpha < 2$.

For more examples of quasivariational inequalities see [5] where partial differential equations corresponding to systems of reaction-diffusion with unilateral conditions on the boundary were considered. Note that such inequalities involve nonsymmetric operators A.

Acknowledgement. I would like to thank M. Kučera for his support and inspiration during my work on this paper.

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Author's address: *Miroslav Bosák*, elektrotechnická fakulta ČVUT, katedra matematiky, Technická 2, 166 27 Praha 6, Czech Republic.