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A NOTE TO A BIFURCATION RESULT OF H. KIELHÖFER FOR THE WAVE EQUATION

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Summary. A modification of a classical number-theoretical theorem on Diophantine approximations is used for generalizing H. Kielhöfer's result on bifurcations of nontrivial periodic solutions to nonlinear wave equations.

Keywords: Diophantine approximations, wave equation, periodic solution, bifurcation.

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In [1] (and in [2] in collaboration with P. Kötzner) H. Kielhöfer studied the bifurcation of time-periodic solutions for the equation

$$u_{tt} - u_{xx} = f(\lambda, x, u), \quad (t, x) \in \mathbb{R} \times (0, \pi),$$

with homogeneous Dirichlet or Neumann boundary conditions, where the period $P > 0$ is of the form $P = 2\pi/\sqrt{(n^2 - \varrho)}$, $n \in \mathbb{N}$, $\varrho = f_u(\lambda_0, x, 0)$ with λ_0 fixed. The methods used in [1], [2] require to choose the numbers ϱ, n in order to fulfil the following conditions:

(1) the set $S = \{(k, j) \in \mathbb{N} \times \mathbb{Z};\ k^2 - j^2(n^2 - \varrho) - \varrho = 0\}$ is finite,

(2) there exists $\delta > 0$ such that $|k^2 - j^2(n^2 - \varrho) - \varrho| \geqq \delta$ for $(k, j) \in (\mathbb{N} \times \mathbb{Z}) \setminus S$.

Investigating the case of ϱ rational, H. Kielhöfer finds a dense subset $\Lambda \subset \mathbb{R}$ such that for $\varrho \in \Lambda$ the conditions (1), (2) are satisfied for an appropriate choice of $n \in \mathbb{N}$.

The aim of this note is to prove the following theorem.

Theorem. *There exists an uncountable dense subset $\tilde{\Lambda} \subset \mathbb{R}$ of irrational numbers such that for $\varrho \in \tilde{\Lambda}$ and for some $n \in \mathbb{N}$ the conditions* (1), (2) *hold.*

The proof is based on the following lemma (cf. [3], Theorem III of Chapter II). For $\alpha \in \mathbb{R}$ we denote $\|\alpha\| = \inf\{|\alpha - k|, k \in \mathbb{Z}\}$.

Lemma 1. *For each $\varepsilon > 0$ there exists an uncountable set $B_\varepsilon \subset (0, \varepsilon)$ such that $\beta \in B_\varepsilon \Rightarrow \liminf_{j \to \infty} j\|j\beta\| \geqq \frac{1}{4}$.*

Proof of the Theorem. We choose a fixed $\eta < \frac{1}{4}$. For $\varepsilon = \eta/2$, $n \in \mathbb{N}$ and $\beta \in B_\varepsilon$ we put $\alpha = n \mp \beta$, $\varrho = \pm 2n\beta - \beta^2$. We have $\alpha^2 = n^2 - \varrho$ and $\|j\alpha\| = \|j\beta\|$ for $j \in \mathbb{Z}$. Indeed, by symmetry we can assume $j \in \mathbb{N}$. Let us denote

$$(3) \qquad \zeta(k, j) = |k^2 - j^2(n^2 - \varrho) - \varrho| = |k^2 - \alpha^2 j^2 - \varrho| .$$

We have $\zeta(k, j) = 0 \Leftrightarrow (k, j) = (n, 1)$, hence (1) is satisfied. For proving (2) we construct the sets

$$M_1 = \{(k, 1);\ k \in \mathbb{N} \setminus n\} ,$$
$$M_2 = \{(k, j) \in \mathbb{N}^2;\ j|k - \alpha j| \leqq \eta\} ,$$
$$M_3 = \{(k, j) \in \mathbb{N}^2;\ j \geqq 2,\ |k - \alpha j| \geqq \alpha j\} .$$

We have $\inf_{M_1} \zeta(k, j) \geqq 1$ and $\inf_{M_2} \zeta(k, j) > 0$, since by Lemma 1 the set M_2 is finite. For $(k, j) \in M_3$ we obtain from (3)

$$\zeta(k, j) \geqq |k - \alpha j|\,|k + \alpha j| - |\varrho| \geqq \alpha^2 j^2 - |\varrho| > 2 .$$

It remains to investigate the case $(k, j) \in \mathbb{N}^2 \setminus (M_1 \cup M_2 \cup M_3 \cup \{n, 1\})$. Now, (3) yields

$$\zeta(k, j) \geqq 2\alpha j|k - \alpha j| - |k - \alpha j|^2 - |\varrho| \geqq$$
$$\geqq 2\alpha\eta - \eta^2/j^2 - |\varrho| \geqq \eta n - \eta^2$$

and (2) is verified. The theorem is proved if we put

$$\tilde{\Lambda} = \{\varrho \in \mathbb{R};\ \varrho = \pm 2n\beta - \beta^2,\ n \in \mathbb{N},\ \beta \in B_\varepsilon\} .$$

Remarks. 1. The sets B_ε contain in particular the numbers $(z + \varphi_m)^{-1}$, where $z \in \mathbb{N}$ is sufficiently large and φ_m is a root of the Markoff form F_m. We have for instance $\varphi_1 = \frac{1}{2}(\sqrt{(5)} - 1)$, $\varphi_2 = \sqrt{(2)} - 1$ etc.

2. Another sequence of numbers $\varrho \in \mathbb{R}$ satisfying the Theorem is given again by the formula $\varrho = \pm 2n\beta - \beta^2$, where $\beta = (b - \sqrt{(b^2 - 1)})/a$, $a, b, n \in \mathbb{N}$, $b \geqq 2$, $n > b/a$. Here we use the elementary inequality $j\|j\alpha\| = j\|j\beta\| \geqq [2a\sqrt{(b^2 - 1)}]^{-1} \cdot (1 - [4j^2(b^2 - 1)]^{-1})$.

3. The Lebesgue measure of the sets B_ε is zero ([3]).

References

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Souhrn

POZNÁMKA K BIFURKAČNÍMU VÝSLEDKU H. KIELHÖFERA PRO VLNOVOU ROVNICI

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Varianta klasického číselně teoretického výsledku o diofantických aproximacích je užita k zobecnění práce H. Kielhöfera o bifurkaci netriviálních periodických řešení nelineární vlnové rovnice.

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