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Mathematica Bohemica, Vol. 119 (1994), No. 3, 255–274

Persistent URL: http://dml.cz/dmlcz/126167

SOLVABILITY OF DEGENERATE ELLIPTIC BOUNDARY VALUE PROBLEMS: ANOTHER APPROACH

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(Received May 12, 1993)

Summary. Using a Hardy-type inequality, the authors weaken certain assumptions from the paper [1] and derive existence results for equations with a stronger degeneration.

Keywords: Weighted Sobolev spaces, degenerate elliptic equations, Hardy inequality, weak solutions.

AMS classification: 35J70

0. INTRODUCTION

In the paper Drábek, Kufner, Nicolosi [1], boundary value problems for the nonlinear partial differential equation in divergence form of order $2m$,

$$\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^{\alpha}A_{\alpha}(x,u(x),\dots,D^{m}u(x))=\sum_{|\alpha|\leqslant m}(-1)^{|\alpha|}D^{\alpha}f_{\alpha}(x), \tag{0.1}$$

on a domain $\Omega\subset\mathbb{R}^N$, $N\geqslant 2$, have been investigated.

It was supposed that the corresponding differential operator was *degenerate* (or *singular*) *elliptic* and that this behaviour can be described by some weight functions $\nu_\alpha(x)$, $|\alpha|=m$, appearing in the highest order terms of this operator. This means, roughly speaking, that the functions

$$A_\alpha(x,\xi)=A_\alpha(x,\xi_0,\xi_1,\dots,\xi_m),\quad |\alpha|=m,$$

where $\xi_j=\{\xi_\beta\in\mathbb{R};\ |\beta|=j\}$ behave with respect to the components ξ_β of the vector ξ_m, which represents in (0.1) the vector $D^m u$ of all m-th order derivatives of u, like

$$\nu_\beta(x)|\xi_\beta|^{p-1},\quad x\in\Omega,\ \xi_\beta\in\mathbb{R}, \tag{0.2}$$

with $p > 1$. Thus the weight function $\nu_\beta(x)$ describes the degeneration (or singularity) and the exponent p the degree of nonlinearity.

In [1], the existence of weak solutions of boundary value problems was proved using the degree theory of monotone mappings. The solutions have been sought in a special *weighted Sobolev space*

$$W^{m,p}(\nu, \Omega) \tag{0.3}$$

normed by

$$\|u\|_{m,p,\nu} = \Bigl(\sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^p \nu_\alpha(x)\,\mathrm{d}x + \sum_{|\beta|\leqslant m-1} \int_\Omega |D^\beta u(x)|^p\,\mathrm{d}x \Bigr)^{1/p} \tag{0.4}$$

where

$$\nu = \{\nu_\alpha; |\alpha| = m\}$$

is the collection of weight functions $\nu_\alpha = \nu_\alpha(x)$ (i.e., functions measurable and positive a.e. in Ω) which describe the degeneration (or singularity)—see (0.2). Let us point out that in (0.4), the weights appear only in the highest (m-th) order derivatives.

In order to guarantee that the space $W^{m,p}(\nu, \Omega)$ and also its subspace

$$W_0^{m,p}(\nu, \Omega) \tag{0.5}$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (0.4), are well-defined, we suppose—analogously to [1]—that

$$v_\alpha \in L^1_{\mathrm{loc}}(\Omega), \quad v_\alpha^{1/(1-p)} \in L^1_{\mathrm{loc}}(\Omega), \quad |\alpha| = m. \tag{0.6}$$

A very important tool in [1] was the *compactness* of the imbedding

$$W^{m,p}(\nu, \Omega) \hookrightarrow W^{m-1,p}(\Omega) \tag{0.7}$$

where on the right hand side the *classical* (= nonweighted) Sobolev space appears. This imbedding was established in two steps. First it was shown that the *continuous* imbedding

$$W^{m,p}(\nu, \Omega) \hookrightarrow W^{m,p_1}(\Omega) \tag{0.8}$$

takes place with

$$p_1 = p\frac{s}{s+1}. \tag{0.9}$$

This step needs the additional assumption

$$\frac{1}{\nu_\alpha} \in L^s(\Omega), \quad |\alpha| = m, \tag{0.10}$$

with the condition

$$s \geqslant \frac{1}{p-1}, \tag{0.11}$$

which ensures that p_1 in (0.9) satisfies $p_1 \geqslant 1$. Then, assuming that (0.10) holds with

$$s > \frac{N}{p}, \tag{0.12}$$

the *compact* imbedding

$$W^{m,p_1}(\Omega) \hookrightarrow W^{m-1,p}(\Omega) \tag{0.13}$$

was used. Compactness of (0.7) follows from (0.8), (0.13).

Summarizing (0.11) and (0.12), we have to suppose that the weight functions ν_α satisfy (0.10) with

$$\begin{aligned} s &\geqslant \frac{1}{p-1} \quad \text{for } 1 < p < \frac{N}{N-1} \\ s &> \frac{N}{p} \quad \text{for } p \geqslant \frac{N}{N-1}. \end{aligned} \tag{0.14}$$

This condition is rather *restrictive* and excludes strong degenerations. It is the aim of this paper to show how this condition can be weakened. Using certain Hardy-type inequalities, we are able to derive compactness of the imbedding (0.7) *directly*, avoiding the (now unnecessary) introduction of the intermediate auxiliary space $W^{m,p_1}(\Omega)$. So we can obtain existence results for degenerate elliptic equations with a *stronger degeneration* than in [1]. Of course, the price we have to pay for this improvement consists in a certain decline of the growth of lower order terms (see Subsection 3.7).

The paper, which could be considered an extension of the results derived by Guglielmino, Nicolosi [2] and Drábek, Nicolosi [3], is organized as follows: In Section 1, after some preliminaries, we will indicate how the imbedding (0.7) can be realized (to be compact) in a direct way. In Section 2, we will formulate the main results; since we are looking for a weak solution in the same space as in [1]—namely in $W^{m,p}(\nu, \Omega)$ from (0.3)—and since also the method is literally the same as in [1] (with the appropriate changes in the assumptions), we can omit the proofs here and refer to [1]. In Section 3, we will give some examples which illustrate the advantage (and also the disadvantages) of our method.

1.1. The domain. We will suppose that the domain $\Omega \subset \mathbb{R}^N$ is *bounded* and satisfies the so-called *cone condition.* This enables us, among other, to use imbedding theorems for classical Sobolev spaces in Subsection 1.10. For details see, e.g., Adams [4] or Kufner, John, Fučík [5].

1.2. The function spaces. (i) Let $\omega = \omega(x)$ be a weight function on Ω. For $p > 1$ we denote by

$$L^p(\omega, \Omega) \tag{1.1}$$

the set of all measurable functions $u = u(x)$ on Ω for which the norm

$$\|u\|_{p,\omega} = \left(\int_\Omega |u(x)|^p \omega(x)\,\mathrm{d}x \right)^{1/p} \tag{1.2}$$

is finite. $L^p(\omega, \Omega)$ will be called the *weighted Lebesgue space.*

(ii) Let $\nu = \{\nu_0, \nu_1, \ldots, \nu_N\}$ be a family of weight functions on Ω. For $p > 1$, we denote by

$$W^{1,p}(\nu, \Omega) \tag{1.3}$$

the set of all functions $u \in L^p(\nu_0, \Omega)$ whose distributional derivatives $\frac{\partial u}{\partial x_i}$ belong to $L^p(\nu_i, \Omega)$, $i = 1, 2, \ldots, N$.

(iii) We will suppose that all weight functions appearing in this paper satisfy the conditions (0.6). Then $W^{1,p}(\nu, \Omega)$ is a Banach space if equipped with the norm

$$\|u\|_{1,p,\nu} = \left(\|u\|^p_{p,\nu_0} + \sum_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|^p_{p,\nu_i} \right)^{1/p} \tag{1.4}$$

and we can introduce the space

$$W_0^1(\nu, \Omega) \tag{1.5}$$

as the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (1.4).

(iv) For M a subset of the boundary $\partial\Omega$ of Ω, denote by

$$C_M^\infty(\Omega) \tag{1.6}$$

the set of all $u \in C^\infty(\bar{\Omega})$ such that $\operatorname{supp} u \cap M = \emptyset$, i.e., u vanishes in an open neighborhood of $\overline{M}$. We denote by

$$W_M \ (= W), \tag{1.7}$$

the closure of $C_M^\infty(\Omega)$ with respect to the norm (1.4). Obviously

$$W_0^{1,p}(\nu,\Omega) \subset W_M \subset W^{1,p}(\nu,\Omega) \tag{1.8}$$

and $W_0^{1,p}(\nu,\Omega)$ is in fact $W_{\partial\Omega}$. In the sequel we will often omit the subscript M in W_M.

We will suppose that *for $u \in W$, the expression*

$$|||u|||_{1,p,\nu} = \left(\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial u}{\partial x_i}(x) \right|^p \nu_i(x) \, \mathrm{d}x \right)^{1/p} \tag{1.9}$$

is a norm on W equivalent to the norm $\|u\|_{1,p,\nu}$ from (1.4).

1.3. Remark. Due to the density of $C_M^\infty(\Omega)$ in W, we will often consider only the smooth functions from $C_M^\infty(\Omega)$. Since these functions satisfy $u\big|_M = 0$, we can, *roughly speaking*, interprete the functions $u \in W$ as "those functions from $W^{1,p}(\nu,\Omega)$ which satisfy the boundary condition $u\big|_M = 0$". This shows a close connection of our considerations with the boundary value problems which will be investigated in Section 2.

1.4. Hardy type inequalities. Let $p, q > 1$. In the sequel, we will make substantial use of a special type of the inequality

$$\left(\int_\Omega |v(x)|^q \omega(x) \, \mathrm{d}x \right)^{1/q} \leqslant C \left[\sum_{i=1}^{N} \int_\Omega \left| \frac{\partial v}{\partial x_i}(x) \right|^p \nu_i(x) \, \mathrm{d}x \right]^{1/p} \tag{1.10}$$

where $\omega, \nu_1, \ldots, \nu_N$ are weight functions on Ω. We will suppose that (1.10) holds for all $v \in W$ [or, due to the density, for all $v \in C_M^\infty(\Omega)$]. This inequality, called *Hardy-type inequality* (or weighted Friedrichs/Poincaré inequality) expresses an imbedding between a weighted Sobolev and a weighted Lebesgue space of the type

$$W^{1,p}(\nu,\Omega) \hookrightarrow L^q(\omega,\Omega)$$

or, more precisely, the imbedding

$$W \hookrightarrow L^q(\omega,\Omega). \tag{1.11}$$

It is not possible to give here a detailed account of the relations between the parameters p, q, between the (optimal) weight function ω and the weight functions $\nu_1, \nu_2, \dots, \nu_N$ etc., under which inequalities of the form (1.10) hold. These relations are in the general case rather complicated and depend even in the particular cases substantially on the properties of the parameters involved, like the shape of the domain Ω, the character of the weight functions (e.g., their dependence only on some variables), the assumptions about the space W etc. A lot of information can be found in the book Opic, Kufner [6] for general as well as for special weights. We will give some examples in Subsection 1.7 and also in Section 3, but now let us simply formulate our fundamental assumption. It should be emphasized that we will be interested in the inequality (1.10) for the special case

$$p = q, \quad \omega(x) \equiv 1.$$

1.5. The main assumption. Let W be the subspace of the weighted Sobolev space $W^{1,p}(\nu, \Omega)$ defined in 1.2 (iii). We will suppose that the weight functions $\nu_1, \nu_2, \dots, \nu_N$ have the following property: the inequality

$$\int_\Omega |v(x)|^p \, dx \leqslant c_0 \sum_{i=1}^N \int_\Omega \left| \frac{\partial u}{\partial x_i}(x) \right|^p \nu_i(x) \, dx \tag{1.12}$$

holds for every function $v \in W$ with a constant $c_0 > 0$ independent of v, and the corresponding imbedding

$$W \hookrightarrow L^p(\Omega) \tag{1.13}$$

is compact.

1.6. Remark. Let us point out that assumption 1.5 is primarily an *assumption about the weight functions* $\nu_1, \nu_2, \dots, \nu_N$. As we will see later (cf. Remark 1.9), it will be mainly this assumption which will replace the condition $\frac{1}{\nu_i} \in L^s(\Omega)$ with s satisfying (0.14).

1.7. Examples. (i) The inequality (1.12) holds for every $v \in C_0^\infty(\Omega)$ [which corresponds to the choice $W = W_0^{1,p}(\nu, \Omega)$] if the weight functions have the form

$$\nu_i(x) = |g_i(x)|^p, \quad i = 1, 2, \dots, N,$$

where the vector $g = (g_1, g_2, \dots, g_N)$ satisfies $\operatorname{div} g \equiv 1$, or if

$$\nu_1(x) = \nu_2(x) = \dots = \nu_N(x) = \left(\sum_{j=1}^N \left| \frac{\partial G}{\partial x_j}(x) \right|^{p/(p-1)} \right)^{p-1}$$

where $G = G(x)$ is a solution of the equation $\Delta G = 1$ (see [6], 14.7 and 14.11).

(ii) The more general inequality (1.10) holds for every $v \in C_0^\infty(\Omega)$ with

$$\nu_1(x) = \ldots = \nu_N(x) = [\mathrm{dist}(x, \partial\Omega)]^\lambda, \quad \omega(x) = [\mathrm{dist}(x, \partial\Omega)]^\kappa$$

where

$$\lambda < p - 1$$

if and only either

$$(1.14) \qquad 1 < p \leqslant q < \infty, \quad \frac{N}{q} - \frac{N}{p} + 1 \geqslant 0, \quad \kappa \geqslant \frac{q}{p}(\beta + N) - q - N$$

or

$$1 \leqslant q < p < \infty, \quad \kappa > \frac{q}{p}(\beta + 1) - q - 1$$

(see [6], Theorem 21.5). Moreover, the corresponding imbedding (1.11) is compact if the last inequality in (1.14) is strict.

(iii) Further examples for weight functions of the type $[\mathrm{dist}(x, M)]^\lambda$ and $\varrho = \varrho(\mathrm{dist}(x, M))$ with $\varrho = \varrho(t) > 0$ and $M \subset \partial\Omega$ the set mentioned in 1.2 (iv) can be found in Kufner [7].

1.8. The main imbedding. (i) Now, we will work with the space $W^{m,p}(\nu, \Omega)$ defined in the Introduction—see (0.3), (0.4), with the family of weight functions

$$\nu = \{\nu_\alpha = \nu_\alpha(x);\, |\alpha| = m\};$$

here we adopted the multiindices notation. More precisely, we will work with its subspace V satisfying

$$(1.15) \qquad W_0^{m,p}(\nu, \Omega) \subset V \subset W^{m,p}(\nu, \Omega),$$

and we will suppose *that*

$$(1.16) \qquad D^\beta u \in W \quad \text{for } u \in V,\ |\beta| = m - 1$$

where W is the space introduced in 1.2 (iv).

(ii) Let us fix one multiindex β, $|\beta| = m - 1$, and denote by $\beta(i)$ the multiindex $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ for which

$$\alpha_j = \beta_j \quad \text{for } i \neq j,\ \alpha_i = \beta_i + 1,$$

i.e.

$$\beta(i) = (\beta_1, \beta_2, \dots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \dots, \beta_N). \tag{1.17}$$

Obviously, $|\beta(i)| = m$.

(iii) Suppose that *for every fixed* β, $|\beta| = m - 1$, *the weight functions* $\nu_i(x) = \nu_{\beta(i)}(x)$, $i = 1, 2, \dots, N$, *satisfy assumption* 1.5, i.e., the inequality

$$\int_\Omega |v(x)|^p \, dx \leqslant c_0 \sum_{i=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}(x)\right|^p \nu_{\beta(i)}(x) \, dx \tag{1.18}$$

holds for every $v \in W$.

Due to (1.16), we can take $v = D^\beta u$ in (1.18) obtaining the inequality

$$\int_\Omega |D^\beta u(x)|^p \, dx \leqslant c_0 \sum_{i=1}^N \int_\Omega |D^{\beta(i)} u(x)|^p \nu_{\beta(i)}(x) \, dx \tag{1.19}$$

since $\frac{\partial}{\partial x_i} D^\beta u = D^{\beta(i)} u$ due to the definition of $\beta(i)$—see (1.17). But (1.19) holds for every β, $|\beta| = m - 1$, and we have $|\beta(i)| = m$; consequently we obtain that

$$\sum_{|\beta|=m-1} \int_\Omega |D^\beta u(x)|^p \, dx \leqslant c_1 \sum_{|\alpha|=m} \int_\Omega |D^\alpha u(x)|^p \nu_\alpha(x) \, dx, \tag{1.20}$$

and it follows immediately that the inequality

$$\|u\|_{m-1,p} \leqslant c_2 \|u\|_{m,p,\nu}, \tag{1.21}$$

where on the left hand side the norm in the (nonweighted) Sobolev space $W^{m-1,p}(\Omega)$ appears, holds for every $u \in V$.

So, *we have derived the imbedding*

$$V \subset W^{m-1,p}(\Omega) \tag{1.22}$$

and moreover, *this imbedding is compact* due to the compactness of imbedding (1.13).

1.9. Remark. It is in fact the imbedding (1.22) which we will use in the sequel instead of the imbedding (0.7). We have derived it without introducing the "intermediate space" $W^{m,p_1}(\Omega)$; let us emphasize that the crucial step in our assumptions, which replaces this auxiliary space [and consequently, the restrictive assumption $\frac{1}{\nu_\alpha} \in L^s(\Omega)$ with s satisfying (0.14)] was the claim from 1.8 (iii) that *every N-tuple of weight functions ν_α from the family ν, which has the form*

$$\nu_{\beta(1)}, \nu_{\beta(2)}, \dots, \nu_{\beta(N)} \quad \textit{for some } \beta, \ |\beta| = m - 1,$$

satisfies assumption 1.5.

1.10. Some classical imbeddings. Since we will need some imbeddings of classical Sobolev spaces, let us recall some results concerning them.

(i) First we will deal with the space $W^{m-1,p}(\Omega)$. Let us denote by κ_2 the number

$$\kappa_2 = m - 1 - \frac{N}{p}. \tag{1.23}$$

Then the following imbeddings hold:

(i-1) For $K \in \mathbb{N}_0$ such that $m - 1 \geqslant k > \kappa_2$, we have

$$W^{m-1,p}(\Omega) \hookrightarrow W^{k,r_k}(\Omega) \tag{1.24}$$

where

$$1 \leqslant r_k \leqslant q_k = \frac{Np}{N - (m - k - 1)p}. \tag{1.25}$$

(i-2) For $k = \kappa_2 \in \mathbb{N}_0$, we have

$$W^{m-1,p}(\Omega) \hookrightarrow W^{k,r}(\Omega) \tag{1.26}$$

with r arbitrary, $1 \leqslant r < \infty$.

(i-3) For $k \in \mathbb{N}_0$ such that $0 \leqslant k < \kappa_2$, we have

$$W^{m-1,p}(\Omega) \hookrightarrow C^k(\bar{\Omega}). \tag{1.27}$$

(ii) The imbeddings (1.26) and (1.27) are compact; the imbedding (1.24) is compact if we have strict inequality $r_k < q_k$ in (1.25).

(iii) Due to assumption 1.5, we have also the *compact* imbedding (1.22) for the subspace V of $W^{m,p}(\nu, \Omega)$. Combining (1.22) with (1.24), (1.26) and (1.27), respectively, we immediately obtain the following assertion, which will be used in the next section:

The imbeddings

$$\begin{aligned} V &\hookrightarrow W^{k,r_k}(\Omega), \\ V &\hookrightarrow W^{k,r}(\Omega), \\ V &\hookrightarrow C^k(\bar{\Omega}) \end{aligned} \tag{1.28}$$

hold under the same conditions as the imbeddings (1.24), (1.26) *and* (1.27) *and, moreover, they are compact.*

2. THE EXISTENCE OF WEAK SOLUTIONS

2.1. Now we will show under what conditions there exists a weak solution $u \in W^{m,p}(\nu, \Omega)$ of a boundary value problem for the differential equation (0.1). Let us start with some assumptions about the differential operator and the right hand side.

Recall that

$$\kappa_2 = m - 1 - \frac{N}{p}. \tag{2.1}$$

2.2. Growth conditions. Let k be the number of distinct multiindinces $\beta = (\beta_1, \beta_2, \dots, \beta_N)$, β_i nonnegative integers, for which $|\beta| = \beta_1 + \beta_2 + \dots + \beta_N \leqslant m$. Denote

$$\xi = \{\xi_\beta \in \mathbb{R},\ |\beta| \leqslant m\} \in \mathbb{R}^k$$

and

$$\xi_0 = \{\xi_\beta \in \mathbb{R},\ |\beta| < \kappa_2\}. \tag{2.2}$$

Let us suppose that the functions $A_\alpha = A_\alpha(x, \xi)$, $|\alpha| \leqslant m$, which appear in equation (0.1), are *Carathéodory functions* on $\Omega \times \mathbb{R}^k$. Let $g_\alpha = g_\alpha(t)$, $|\alpha| \leqslant m$, be positive, continuous, nondecreasing functions on $(0, \infty)$; for $\kappa_2 \leqslant 0$ we take $g_\alpha(t) \equiv 1$.

Let q_β be the number q_k from (1.25) for $k = |\beta|$, i.e.

$$q_\beta = \frac{Np}{N - (m - 1 - |\beta|)p}, \tag{2.3}$$

and denote by p_β, $\kappa_2 \leqslant |\beta| \leqslant m - 1$, any number which satisfies

$$\begin{aligned} &1 < p_\beta \leqslant q_\beta \quad \text{for } \kappa_2 < |\beta| \leqslant m - 1, \\ &1 < p_\beta < \infty \quad \text{arbitrary for } |\beta| = \kappa_2. \end{aligned} \tag{2.4}$$

Finally, for $r > 1$ denote

$$r' = \frac{r}{r - 1}.$$

We will assume that the functions $A_\alpha(x, \xi)$ satisfy for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^k$ the following *growth conditions*:

(i) For $|\alpha| = m$,

$$|A_\alpha(x, \xi)| \leqslant g_\alpha(|\xi_0|)\nu_\alpha^{1/p}(x)\Big[a_\alpha(x) + \sum_{\kappa_2 \leqslant |\beta| \leqslant m-1} |\xi_\beta|^{p_\beta/p'} + \sum_{|\beta| = m} \nu_\beta^{1/p'}(x)|\xi_\beta|^{p-1}\Big] \tag{2.5}$$

with $a_\alpha \in L^{p'}(\Omega)$.

(ii) For α with $\kappa_2 < |\alpha| \leqslant m - 1$,

$$(2.6)\quad |A_\alpha(x,\xi)| \leqslant g_\alpha(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_2 \leqslant |\beta| \leqslant m-1} |\xi_\beta|^{p_\beta/p'_\alpha} + \sum_{|\beta|=m} \nu_\beta^{1/p'_\alpha}(x)|\xi_\beta|^{p/p'_\alpha}\right]$$

with $a_\alpha \in L^{p'_\alpha}(\Omega)$.

(iii) For $|\alpha| = \kappa_2$,

$$(2.7)\quad |A_\alpha(x,\xi)| \leqslant g_\alpha(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_2 \leqslant |\beta| \leqslant m-1} |\xi_\beta|^{p_\beta/\sigma} + \sum_{|\beta|=m} \nu_\beta^{1/\sigma}(x)|\xi_\beta|^{p/\sigma}\right]$$

with some $\sigma > 1$ and $a_\alpha \in L^\sigma(\Omega)$.

(iv) For $|\alpha| < \kappa_2$,

$$(2.8)\quad |A_\alpha(x,\xi)| \leqslant g_\alpha(|\xi_0|)\left[a_\alpha(x) + \sum_{\kappa_2 \leqslant |\beta| \leqslant m-1} |\xi_\beta|^{p_\beta} + \sum_{|\beta|=m} \nu_\beta(x)|\xi_\beta|^{p}\right]$$

with $a_\alpha \in L^1(\Omega)$.

2.3. The right hand side. We will suppose that the functions f_α, $|\alpha| \leqslant m$, which appear on the right hand side of equation (0.1), satisfy the following conditions (with parameters p, p_α, σ which appear in the foregoing growth conditions):

$$(2.9)\quad \begin{aligned} &f_\alpha \in L^{p'}\left(\nu_\alpha^{-\frac{1}{p-1}}, \Omega\right) && \text{if } |\alpha| = m,\\ &f_\alpha \in L^{p'_\alpha}(\Omega) && \text{if } \kappa_2 < |\alpha| \leqslant m-1,\\ &f_\alpha \in L^\sigma(\Omega) && \text{if } |\alpha| = \kappa_2,\\ &f_\alpha \in L^1(\Omega) && \text{if } |\alpha| < \kappa_2. \end{aligned}$$

2.4. The boundary value problem and the operator T. Let us consider a boundary value problem for the equation (0.1). By means of the boundary conditions, we introduce a closed subspace V of $W^{m,p}(\nu,\Omega)$ which satisfies the condition

$$W_0^{m,p}(\nu,\Omega) \subset V \subset W^{m,p}(\nu,\Omega).$$

Moreover, we will suppose that V *satisfies the conditions mentioned in Subsection* 1.8; consequently, the imbedding (1.22) and the imbeddings (1.28) hold and are even *compact*. (For example, this will be satisfied if we, roughly speaking, consider boundary conditions with *Dirichlet data* on some part M of $\partial\Omega$—see also Section 3.)

For $u, \varphi \in V$, we can introduce the operator T acting on V by the formula

$$(2.10) \qquad \langle Tu, \varphi\rangle = \sum_{|\alpha|\leqslant m} \int_\Omega [A_\alpha(x, u(x), \dots, D^m u(x)) - f_\alpha(x)] D^\alpha \varphi(x)\, dx$$

where $\langle \cdot, \cdot\rangle$ denotes the duality between V^* and V; using the properties of Nemytskii operators, we derive the following analogue of Theorem 3.5 in [1]:

2.5. Theorem. *Let us assume that the growth conditions* (2.5)–(2.8) *and the conditions* (2.9) *are satisfied. Then the operator* $T\colon V \to V^*$ *defined in* (2.10) *is bounded and continuous.*

2.6. Definition. We will say that a function $u \in V$ is a *weak solution* of our boundary value problem, described by the equation (0.1) and by the space V, if the identity

$$\sum_{|\alpha|\leqslant m} \int_\Omega A_\alpha(x, u(x), \dots, D^m u(x)) D^\alpha \varphi(x)\, dx = \sum_{|\alpha|\leqslant m} \int_\Omega f_\alpha(x) D^\alpha \varphi(x)\, dx$$

i.e., the identity

$$\langle Tu, \varphi\rangle = 0$$

holds for every $\varphi \in V$.

Now we will modify the conditions from [1] which will guarantee the *existence* of a weak solution.

2.7. The ellipticity and monotonicity. Let us write the vector $\xi \in \mathbf{R}^k$ in the form

$$\xi = (\eta, \zeta)$$

where $\eta = \{\xi_\beta; |\beta| \leqslant m-1\}$, $\zeta = \{\xi_\beta; |\beta| = m\}$. We will write $\xi_\beta = \eta_\beta$ for $\xi_\beta \in \eta$, $\xi_\beta = \zeta_\beta$ for $\xi_\beta \in \zeta$.

(i) Let $g_1 = g_1(t)$ be positive, continuous and nonincreasing on $(0, \infty)$, $g_2 = g_2(t)$ positive, continuous and nondecreasing on $(0, \infty)$. Let ξ_0 be defined by (2.2). We will suppose that for a.e. $x \in \Omega$ and for every $\xi \in \mathbf{R}^k$ the following *ellipticity condition* holds:

$$(2.11) \quad \sum_{|\alpha|=m} A_\alpha(x, \eta, \zeta)\zeta_\alpha \geqslant g_1(|\xi_0|) \sum_{|\beta|=m} \nu_\beta(x)|\zeta_\beta|^p - g_2(|\xi_0|) \sum_{\kappa_2\leqslant|\beta|\leqslant m-1} |\eta_\beta|^{p_\beta}$$

with p_β appearing in the growth conditions.

(ii) Further, we will suppose that the differential operator from (0.1) is *monotone in the principal part*, i.e., the following conditions hold for a.e. $x \in \Omega$ and for every pair (η,ζ), $(\eta,\hat{\zeta})$ from $\mathbb{R}^k$ with $\zeta \neq \hat{\zeta}$:

$$\sum_{|\alpha|=m} [A_\alpha(x,\eta,\zeta) - A_\alpha(x,\eta,\hat{\zeta})](\zeta_\alpha - \hat{\zeta}_\alpha) > 0. \tag{2.12}$$

The following *existence theorems* can be derived completely analogously as Theorems 5.3 and 5.6 in [1].

2.8. Theorem. *Let us assume that* (2.5)–(2.8), (2.9), (2.11) *and* (2.12) *hold. Moreover, assume that there exists a number $R > 0$ such that*

$$\langle Tu, u\rangle \geqslant 0$$

for all $u \in V$ such that $\|u\|_{m,p,\nu} = R$. Then the boundary value problem for the differential equation (0.1) *has at least one weak solution $u_0 \in V$ such that $\|u_0\|_{m,p,\nu} \leqslant R$.*

2.9. Theorem. *Let us assume that the coefficients $A_\alpha = A_\alpha(x,\xi)$ of the differential operator in* (0.1) *satisfy conditions* (2.5)–(2.8), (2.11) *and* (2.12). *Moreover, suppose that the following coercivity condition is fulfilled for a.e. $x \in \Omega$ and for every $\xi \in \mathbb{R}^k$:*

$$\sum_{|\alpha|\leqslant m} A_\alpha(x,\xi)\xi_\alpha \geqslant c_1 \sum_{|\alpha|=m} \nu_\alpha(x)|\xi_\alpha|^p + c_2|\xi_\theta|^p - c_3 \tag{2.13}$$

with positive constants c_1, c_2, c_3 and $\theta = (0,\dots,0)$.

Then the boundary value problem for the differential equation (0.1) *on V has at least one weak solution for every family of functions f_α, $|\alpha| \leqslant m$, which satisfy* (2.9).

3. Examples and comments

3.1. We will consider the differential operator A appearing in (0.1), i.e.

$$(Au)(x) = \sum_{|\alpha|\leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha\big(x, u(x), \dots, D^m u(x)\big), \tag{3.1}$$

and suppose that the functions $A_\alpha = A_\alpha(x,\xi)$ satisfy the growth conditions from Subsection 2.2. For example, we can suppose that the highest order terms appear in the special simple form

$$A_\alpha(x,\xi) = \nu_\alpha(x)|\xi_\alpha|^{p-2}\xi_\alpha, \quad |\alpha| = m, \tag{3.2}$$

but the particular form of $A_\alpha(x,\xi)$ will not be important. Our aim is to show how we can use Hardy-type inequalities of the form

$$\int_\Omega |v(x)|^p \,\mathrm{d}x \leqslant c \sum_{i=1}^N \int_\Omega \left|\frac{\partial v}{\partial x_i}(x)\right|^p \nu_i(x)\,\mathrm{d}x \tag{3.3}$$

where $v = D^\beta u$, $|\beta| = m-1$, for $u \in V \subset W^{m,p}(\nu,\Omega)$ and $\nu_i(x) = \nu_{\beta(i)}(x)$ with

$$\beta(i) = (\beta_1, \beta_2, \dots, \beta_{i-1}, \beta_i + 1, \beta_{i+1}, \dots, \beta_N), \quad i = 1, 2, \dots, N. \tag{3.4}$$

3.2. Example. For Ω let us choose the *cube*

$$Q = (0,1)^N$$

and write $x = (x', x_N)$ with $x' \in \mathbb{R}^{N-1}$. Then the set

$$M = \{x = (x', x_N) \in \partial Q; x_N = 0\}$$

is the bottom of the cube Q.

Let us consider a boundary value problem on Q with the *Dirichlet data* on M (the boundary conditions on $\partial Q \setminus M$ are not important), and suppose that the weight functions $\nu_\alpha(x)$ have the special form

$$\nu_\alpha(x) = \nu_\alpha(x', x_N) = \nu(x_N) \quad \text{for all } \alpha,\ |\alpha| = m, \tag{3.5}$$

i.e., all weight functions are the same and depend only on x_N, $0 < x_N < 1$. For this special choice, conditions (0.6) have the form

$$\nu \in L^1_{\mathrm{loc}}(0,1), \quad \nu^{\frac{1}{1-p}} \in L^1_{\mathrm{loc}}(0,1) \tag{3.6}$$

and *we will suppose that they are satisfied.*

Since we have Dirichlet boundary conditions on M (which means that the values of $D^\beta u$ on M are prescribed for all β, $|\beta| \leqslant m-1$), we can construct the set $V \subset W^{m,p}(\nu,\Omega)$ by the following procedure: We consider a set $\mathscr{V} \subset C^\infty(\overline{\Omega})$ which satisfies the condition

$$\operatorname{supp} u \cap M = \emptyset \qquad \text{for } u \in \mathscr{V} \tag{3.7}$$

and define V as the *closure* of $\mathscr{V}$ in $W^{m,p}(\nu,\Omega)$. Due to the density, we can consider the inequality (3.3) for $v = D^\beta u$ with $u \in \mathscr{V}$ instead of $u \in V$. But in view of (3.7) we have

$$v = v(x', x_N) = 0 \quad \text{for } x_N = 0. \tag{3.8}$$

For such functions we can use the *onedimensional Hardy inequality*

$$\int_0^1 |v(x',x_N)|^p \, dx_N \leqslant c_0 \int_0^1 \left|\frac{\partial v}{\partial x_N}(x',x_N)\right|^p \nu(x_N)\, dx_N \tag{3.9}$$

which holds, with a constant c_0 *independent of* $x' \in M$, if and only if the function

$$B(t) = (1-t)^{1/p} \left(\int_0^t \nu^{1-p'}(s)\, ds \right)^{1/p'} \tag{3.10}$$

is bounded on $(0,1)$, i.e. if the function

$$\mathscr{N}(t) = \int_0^t \nu^{1-p'}(s)\, ds \tag{3.11}$$

has the property that

$$\mathscr{N}(t) \cdot (1-t)^{p'-1} \leqslant c_1 \quad \text{for } t \in (0,1). \tag{3.12}$$

Moreover, the imbedding expressed by the inequality (3.9) is *compact* provided

$$\mathscr{N}(t) = o\big((1-t)^{1-p'}\big) \quad \text{for } t \to 1. \tag{3.13}$$

(For all these results, see [6], Chapter 1.)

The integration of (3.9) with respect to $x' \in M$ leads to the inequality

$$\int_Q |v(x',x_N)|^p \, dx_N \, dx' \leqslant c_0 \int_Q \left|\frac{\partial v}{\partial x_N}(x',x_N)\right|^p \nu(x_N)\, dx_N \, dx' \tag{3.14}$$

and this inequality immediately yields

$$\int_Q |v(x)|^p \, dx \leqslant c_0 \sum_{i=1}^N \int_Q \left|\frac{\partial v}{\partial x_i}(x)\right|^p \nu(x_N)\, dx \tag{3.15}$$

which holds not only for $v = D^\beta u$ with $u \in \mathscr{V}$, but also with $u \in V$. Inequality (3.15) is the desired inequality (3.3). Since $v = D^\beta u$, $|\beta| = m-1$, we finally have

$$\sum_{|\beta|=m-1} \int_Q |D^\beta u(x)|^p \, dx \leqslant c_1 \sum_{|\alpha|=m} \int_Q |D^\alpha u(x)|^p \nu(x_n)\, dx\,.$$

Since for $u \in V \subset W^{m,p}(\nu,\Omega)$ the left hand side is obviously equivalent to the norm in $W^{m-1,p}(\Omega)$, *we have derived for* $u \in V$ *the inequality*

$$\|u\|_{m-1,p} \leqslant c_2 \|u\|_{m,p,\nu},$$

i.e., the imbedding

$$V \hookrightarrow W^{m-1,p}(\Omega) \tag{3.16}$$

which is compact if ν *satisfies* (3.13).

Let us compare conditions (3.12) and (3.13) with condition (0.10) for s satisfying (0.14). In our case this condition reads

$$\int_0^1 \nu^{-s}(x_N)\,\mathrm{d}x_N < \infty \quad \begin{array}{ll} \text{for } s \geqslant \frac{1}{p-1} & \text{if } p < \dfrac{N}{N-1}, \\ \text{for } s > \frac{N}{p} & \text{if } p \geqslant \dfrac{N}{N-1} \end{array} \tag{3.17}$$

while (3.12) and (3.13) require only the finiteness of the integral

$$\int_0^t \nu^{\frac{1}{1-p}}(x_N)\,\mathrm{d}x_N \quad \text{for } t \in (0,1) \tag{3.18}$$

with a certain "bad behaviour" for $t \to 1$ (notice that $1 - p' = \frac{1}{1-p}$). Thus (3.18) gives us more possibilities in the choice of admissible functions ν, mainly in the case $p \geqslant \frac{N}{N-1}$, since then we have

$$s > \frac{N}{p} \geqslant \frac{1}{p-1}.$$

3.3. Example. In the foregoing example, all weight functions have been the same. If we consider the same boundary value problem but now with weight functions defined by the formulas

$$\begin{array}{ll} \nu_\alpha(x) = \nu_\alpha(x_N) & \text{for all } \alpha \text{ of the form } \alpha = \beta(N),\ |\beta| = m-1, \\ \nu_\alpha(x) \equiv 1 & \text{for all other } \alpha,\ |\alpha| = m \end{array}$$

[see (3.4) for the definition of $\beta(N)$] then we come to the same conclusions as in Example 3.2. Indeed: In this case we have $D^\alpha u = \frac{\partial}{\partial x_N} D^\beta u$ for $\alpha = \beta(N)$ and we need—instead of (3.15)—a Hardy-type inequality of the form

$$\int_Q |v(x)|^p\,\mathrm{d}x \leqslant c_0 \sum_{i=1}^{N-1} \int_Q \left|\frac{\partial v}{\partial x_i}(x)\right|^p \mathrm{d}x + \int_Q \left|\frac{\partial v}{\partial x_N}\right|^p \nu(x_N)\,\mathrm{d}x \tag{3.19}$$

since now $\nu_1(x) = \ldots = \nu_{N-1}(x) = 1$, $\nu_N(x) = \nu(x_N)$. But (3.19) again follows immediately from (3.15).

3.4. Remarks. (i) In the foregoing examples, the essential tool was the onedimensional Hardy inequality (3.9), more precisely, the conditions of its validity and of the compactness of the corresponding imbedding. These conditions have a form which is easy to check and are expressed in terms of the function $\mathcal{N}(t)$ from (3.11). Moreover, the existence of $\mathcal{N}(t)$ at the same time guarantees that also the second condition in (3.6) is satisfied, and consequently, it remains only to check whether $\nu \in L^1_{\mathrm{loc}}(0,1)$.

(ii) If we consider a boundary value problem with Dirichlet data *on the top of our cube* Q, i.e. for $x \in \partial Q$ with $x_N = 1$, and suppose that the weight functions ν_α are given by formula (3.5), we again arrive at the Hardy inequality (3.9), but now for functions $v = v(x', x_N)$ such that $v(x', 1) = 0$. In this case the conditions of its validity and of the compactness are expressed in terms of the function

$$\hat{\mathcal{N}}(t) = \int_t^1 \nu^{1-p'}(t)\,\mathrm{d}t\,, \tag{3.20}$$

and the compactness is guaranteed if

$$\hat{\mathcal{N}}(t) = o(t^{1-p'}) \quad \text{for } t \to 0. \tag{3.21}$$

(See again [6], Chapter 1.)

In order to make the conditions just mentioned more transparent, let us consider a particular function ν.

3.5. Example. If we consider in the foregoing examples

$$\nu(x_N) = x_N^\lambda \tag{3.22}$$

[i.e., if we consider a degeneration of the type $\left(\operatorname{dist}(x, M)\right)^\lambda$, $\lambda > 0$] then we have for $\mathcal{N}(t)$ from (3.11)

$$\mathcal{N}(t) = c \cdot t^{(1-p')\lambda+1} = c \cdot t^{1-\frac{\lambda}{p-1}} \tag{3.23}$$

provided

$$\lambda < p - 1, \tag{3.24}$$

and condition (3.13) which guarantees the compactness is obviously fulfilled. Consequently, for weight functions of the type (3.22) with λ satisfying (3.24) we can use our approach and derive results about the existence of weak solutions with this type of degeneracy in the highest order terms.

Conditions (3.6) are in this case obviously satisfied. On the other hand, condition (3.17) used in [1] allows *less values of* λ than (3.24) if $p > \frac{N}{N-1}$ since it reads

$$\lambda < \frac{1}{s} < \frac{p}{N}.$$

If we consider the boundary value problem from Remark 3.4 (ii), i.e., the Dirichlet data on the top of Q, the situation improves more since condition (3.21) will now be satisfied for

$$\lambda < p, \tag{3.25}$$

i.e., we have more admissible values of λ.

3.6. Singular elliptic operators. All considerations concerning Example 3.5 remain valid if we assume $\lambda < 0$, which means—in view of (3.2)—that *singularities* of the type $(\operatorname{dist}(x, M))^\lambda$, $\lambda < 0$, can occur among the coefficients of our differential operator. Since the conditions (3.24) or (3.25) contain no lower bound for λ, we can immediately conclude that our approach is suitable also for certain singular elliptic differential operators without restriction on the rate of singularity [of course only in the case when the singularity appears on $\partial\Omega$—compare with condition $\nu_\alpha \in L^1_{\mathrm{loc}}(\Omega)$].

3.7. Concerning the growth conditions. As we have shown in the foregoing examples, using our approach we can—at least for certain boundary value problems—weaken the restrictive conditions (0.10), (0.14), and we are allowed to consider differential operators with a stronger degeneration. On the other hand, as was indicated in the introduction, we have in general *more restrictive growth conditions* than in [1]. Indeed: If we consider, e.g., the "coefficient" $A_\alpha(x, \xi)$ for $|\alpha| = m$, then the growth condition in [1] reads

$$|A_\alpha(x,\xi)| \leqslant g(|\eta|)\nu_\alpha^{1/p}(x)\Big[a_\alpha(x) + \sum_{\kappa_1 \leqslant |\beta| \leqslant m-1} |\xi_\beta|^{s_\beta} + \sum_{|\beta|=m} \nu_\beta^{1/p'}(x)|\xi_\beta|^{p-1}\Big] \tag{3.26}$$

where $g_1 = g_1(t)$ is positive and nondecreasing, $\eta = \{\xi_\beta;\ |\beta| < \kappa_1\}$ and

$$s_\beta \leqslant s^*_\beta = \frac{p-1}{p} \cdot \frac{p_1 N}{N - p_1(m - |\beta|)} \quad \text{for } |\beta| > \kappa_1 \tag{3.27}$$

with

$$\kappa_1 = m - \frac{N}{p_1} \quad \text{and} \quad p_1 = p\frac{s}{s+1}, \ s > \frac{N}{p}. \tag{3.28}$$

Comparing (3.26) with (2.5), we can immediately see the following facts:

(i) For the number κ_2 from (2.1) we have

$$\kappa_2 < \kappa_1$$

since $\frac{N}{p_1} < \frac{N}{p} + 1$ due to the definition of p_1 and the condition $s > \frac{N}{p}$. Consequently, the vector η in (3.26) can have *more components* than the vector ξ_0 in (2.5) [see its definition in (2.2)].

(ii) If we compare terms which appear in (3.26) as well as in (2.5), e.g.

$$|\xi_\beta|^{s_\beta} \quad \text{and} \quad |\xi_\beta|^{p_\beta/p'} \quad \text{for } \beta \text{ with } \kappa_2 < |\beta| \leqslant m-1,$$

then an easy calculation shows that the exponent p_β/p' can be smaller than the exponent s_β, since due to (3.27) and (2.3), (2.4)

$$\frac{p_\beta}{p'} \leqslant \frac{q_\beta}{p'} = \frac{p-1}{p} \cdot \frac{Np}{N-(m-1-|\beta|)p} < s^*_\beta = \frac{p-1}{p} \cdot \frac{Np_1}{N-(m-|\beta|)p_1}.$$

Consequently, the growth of the corresponding term $|D^\beta u|$ in the "coefficient" $A_\alpha(x, u(x), \ldots, D^m u(x))$ can be bigger if we use the approach from the paper [1].

This is a disadvantage of our method.

3.8. The last example. The weight function $\nu(x)$ considered in Example 3.5—see (3.22)—describes a degeneration of singularity *on the boundary* $\partial\Omega$ *of* Ω. Therefore, let us consider a weight whose "bad behaviour" is concentrated *in the interior of* Ω.

Let Ω be the square $(-1,1) \times (-1,1)$ and consider the second order differential equation

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big(\nu(x_1,x_2)\Big|\frac{\partial u}{\partial x_i}\Big|^{p-2}\frac{\partial u}{\partial x_i}\Big) + G\big(u(x_1,x_2)\big) = f \tag{3.29}$$

on Ω with $p > 2$ and with the weight function

$$\nu(x_1,x_2) = \begin{cases} 1 & \text{for } x_1 \leqslant 0, \\ x_2^\lambda & \text{for } x_1 > 0,\ x_2 > 0, \\ |x_2|^\mu & \text{for } x_1 > 0,\ x_2 < 0, \end{cases} \tag{3.30}$$

where λ, μ are real numbers.

Here we have $N = 2$, $m - 1$, and conditions (0.6) lead to the restriction

$$\lambda, \mu \in (-1, p-1) \tag{3.31}$$

since the function ν from (3.30) vanishes or becomes singular on the segment $\Gamma = \{(x_1, x_2);\, 0 < x_1 < 1,\; x_2 = 0\} \subset \Omega$.

(i) If we use the approach from [1], we have *still another restriction*

$$\lambda, \mu < \frac{p}{2} \tag{3.32}$$

(note that we take $p > 2$), but the number κ_1 from (3.28) is positive and we can choose the function $G = G(t)$ in (3.29) rather general, for example

$$G(t) = t \cdot e^{t^2} \tag{3.33}$$

(see [1], Example 5.7).

(ii) If we use the approach from this paper, then $\kappa_2 < 0$ and $G(t)$ has to satisfy the growth assumption (2.6), i.e. we can have only

$$|G(t)| \leqslant c|t|^p,$$

which is weaker than (3.33). On the other hand, the Hardy-type inequality can be used and the corresponding imbedding is compact for *all values of λ and μ satisfying* (3.31), which is an improvement of (3.32) (notice that we supposed $p > 2$).

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