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# THE HAUSDORFF DIMENSION OF SOME SPECIAL PLANE SETS

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*Summary.* A compact set $T \subset \mathbb{R}^2$ is constructed such that each horizontal or vertical line intersects $T$ in at most one point while the $\alpha$-dimensional measure of $T$ is infinite for every $\alpha \in (0, 2)$.

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Let $T$ be a simple arc in the Cartesian plane. For each real $t$ let $p(t)$ $[q(t)]$ be the number of points in $T$ whose first [second] coordinate is $t$. It follows easily from Banach's theorem (see, e.g., [1], 6.4, p. 280) that the length of $T$ is finite, if and only if $\int_{-\infty}^{\infty} (p(t) + q(t))\,\mathrm{d}t < \infty$. The main purpose of this paper is to show that this assertion fails, if we replace the assumption that $T$ is a simple arc by the assumption that $T$ is compact. We construct a plane compact set $T$ such that each horizontal or vertical line intersects $T$ in at most one point and that the $\alpha$-dimensional measure of $T$ is infinite for each (positive) $\alpha < 2$. (See theorems 9 and 20.)

We write $\mathbb{N} = \{1, 2, \ldots\}$, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. If $a = \langle a_1, a_2 \rangle$, $b = \langle b_1, b_2 \rangle \in \mathbb{R}^2$ and if $A, B \subset \mathbb{R}^2$, we get

$$\operatorname{dist}(a, b) = \max\{|b_1 - a_1|, |b_2 - a_2|\},$$
$$\operatorname{dist}(A, B) = \inf\{\operatorname{dist}(a, b);\ a \in A,\ b \in B\}.$$

If $\emptyset \neq C \subset \mathbb{R}^2$, we define $\operatorname{diam} C$ (diameter of $C$) as $\sup\{\operatorname{dist}(c, c^*);\ c, c^* \in C\}$; further we set $\operatorname{diam} \emptyset = 0$.

**1.** Let $-\infty < a < b < \infty$, $I = [a, b]$, $0 < \alpha < 1$; let $m$, $j$ be integers, $0 \leqslant j < m$. We define

$$\gamma(m, j, \alpha, I) = \bigl[a + j(b - a)/m,\ a + (j + \alpha)(b - a)/m\bigr].$$

Notice that $\gamma(m,k,\alpha,I)$ $(k=0,\dots,m-1)$ are pairwise disjoint subintervals of $I$; each of them has length $\alpha(b-a)/m$.

**2.** *Let $a$, $b$, $I$, $\alpha$, $m$, $j$ be as before; let $k$ be an integer, $0 \leqslant k < m$. Let $t \in \gamma(m,j,\alpha,I)$, $v \in \gamma(m,k,\alpha,I)$. Then $|v-t| \geqslant (|k-j|-\alpha)(b-a)/m$.* (Easy.)

**3.** Throughout this paper, $r_n$ are integers and $q_n$ are numbers such that $2 \leqslant r_0 \leqslant r_1 \leqslant \dots$, $q_{-1} > q_0 > q_1 > \dots$, $r_n \to \infty$, $q_n \to 1$ $(n \to \infty)$ and $q_{n-1}(r_n - 1) \geqslant r_n$ $(n = 0,1,\dots)$. We set $R_0 = 1$, $R_n = r_1 \dots r_n$, $\mu_n = r_{n-1}/R_n$ $(n \in \mathbb{N})$.

**4.** *Let $n \in \mathbb{N}$. Then $q_{n-2}(r_{n-1}-1)/R_n \geqslant \mu_n > 1/R_n \geqslant \mu_{n+1}$.* (Easy.)

**5.** Let $I = [0, q_{-1}r_0]$, $J = [0, q_0]$. If $v_j$, $w_j$ are integers such that $0 \leqslant v_j < r_{2j-2}r_{2j-1}$, $0 \leqslant w_j < r_{2j-1}r_{2j}$ $(j \in \mathbb{N})$, we set

$$I(v_1) = \gamma(r_0r_1, v_1, q_1/q_{-1}, I), \quad J(w_1) = \gamma(r_1r_2, w_1, q_2/q_0, J),$$
$$I(v_1,v_2) = \gamma\big(r_2r_3, v_2, q_3/q_1, I(v_1)\big), \quad J(w_1,w_2) = \gamma\big(r_3r_4, w_2, q_4/q_2, J(w_1)\big)$$

etc.; in general,

$$I(v_1,\dots,v_j) = \gamma\big(r_{2j-2}r_{2j-1}, v_j, q_{2j-1}/q_{2j-3}, I(v_1,\dots,v_{j-1})\big),$$
$$J(w_1,\dots,w_j) = \gamma\big(r_{2j-1}r_{2j}, w_j, q_{2j}/q_{2j-2}, J(w_1,\dots,w_{j-1})\big).$$

If $k_n$ are integers such that $0 \leqslant k_n < r_n$ $(n \in \mathbb{N})$, we set $L(k_1) = I(r_0k_1) \times J$, $L(k_1,k_2) = I(r_0k_1) \times J(k_1 + r_1k_2)$,

$$L(k_1,k_2,k_3) = I(r_0k_1, k_2 + r_2k_3) \times J(k_1 + r_1k_2),$$
$$L(k_1,k_2,k_3,k_4) = I(r_0k_1, k_2 + r_2k_3) \times J(k_1 + r_1k_2, k_3 + r_3k_4)$$

etc.; in general, $L(k_1,\dots,k_n) = V \times W$, where

$$V = I(r_0k_1, k_2 + r_2k_3, \dots, k_{n-4} + r_{n-4}k_{n-3}, k_{n-2} + r_{n-2}k_{n-1}),$$
$$W = J(k_1 + r_1k_2, k_3 + r_3k_4, \dots, k_{n-3} + r_{n-3}k_{n-2}, k_{n-1} + r_{n-1}k_n)$$

for $n$ even and

$$V = I(r_0k_1, k_2 + r_2k_3, \dots, k_{n-3} + r_{n-3}k_{n-2}, k_{n-1} + r_{n-1}k_n),$$
$$W = J(k_1 + r_1k_2, k_3 + r_3k_4, \dots, k_{n-4} + r_{n-4}k_{n-3}, k_{n-2} + r_{n-2}k_{n-1})$$

for $n$ odd ($n > 2$).

Notice that $L(k_1) \supset L(k_1, k_2) \supset \dots$.

**6.** *Let $n \in \mathbb{N}$. Let $V, W$ be as in 5, $V = [a,b]$, $W = [c,d]$. Then $b-a = q_{n-1}/R_{n-1}$, $d-c = q_n/R_n$ for $n$ even and $b-a = q_n/R_n$, $d-c = q_{n-1}/R_{n-1}$ for $n$ odd.* (Easy.)

**7.** *Let $n \in \mathbb{N}$. Let $k_j$, $k_j^*$ be integers, $0 \leqslant k_j < r_j$, $0 \leqslant k_j^* < r_j$ $(j = 1, \dots, n)$. Set $K = L(k_1, \dots, k_n)$, $K^* = L(k_1^*, \dots, k_n^*)$. Then either $k_j = k_j^*$ $(j = 1, \dots, n)$ or $\operatorname{dist}(K, K^*) > \mu_n$.*

Proof. (1) Let $n = 1$. Then $K = I(r_0k_1) \times J$, $K^* = I(r_0k_1^*) \times J$, $I(r_0k_1) = \gamma(r_0r_1, r_0k_1, q_1/q_{-1}, I)$, $I(r_0k_1^*) = \gamma(r_0r_1, r_0k_1^*, q_1/q_{-1}, I)$.

We may suppose that $k_1 \neq k_1^*$. Applying 2 and 4 we get $\operatorname{dist}(K, K^*) > (r_0|k_1^* - k_1| - 1)q_{-1}r_0/r_0r_1 \geqslant (r_0 - 1)q_{-1}/r_1 \geqslant \mu_1$.

(2) Let $n > 1$. Suppose that our assertion holds, if $n$ is replaced by $n-1$. Let, e.g., $n$ be odd. Suppose first that $k_1 = k_1^*, \dots, k_{n-1} = k_{n-1}^*$, $k_n \neq k_n^*$. Set

$$U = I(r_0k_1, \dots, k_{n-3} + r_{n-3}k_{n-2}) \quad (= I(r_0k_1) \text{ for } n = 3),$$
$$W = J(k_1 + r_1k_2, \dots, k_{n-2} + r_{n-2}k_{n-1}).$$

Then

$$K = \gamma(r_{n-1}r_n, k_{n-1} + r_{n-1}k_n, q_n/q_{n-2}, U) \times W,$$
$$K^* = \gamma(r_{n-1}r_n, k_{n-1} + r_{n-1}k_n^*, q_n/q_{n-2}, U) \times W.$$

It is easy to see that the length of $U$ is $q_{n-2}/R_{n-2}$. Applying 2 and 4 we get $\operatorname{dist}(K, K^*) > (r_{n-1}|k_n^* - k_n| - 1)q_{n-2}/(R_{n-2}r_{n-1}r_n) \geqslant (r_{n-1} - 1)q_{n-2}/R_n \geqslant \mu_n$.

If $k_j \neq k_j^*$ for some $j < n$, then, by induction assumption and by 4, $\operatorname{dist}(K, K^*) > \mu_{n-1} > \mu_n$.

If $n$ is even, we proceed analogously. □

**8.** For each $n \in \mathbb{N}$ let $\mathfrak{T}_n$ be the system of all rectangles $L(k_1, \dots, k_n)$, where $k_j$ are integers such that $0 \leqslant k_j < r_j$ $(j = 1, \dots, n)$. Let $T_n$ be the union of the system $\mathfrak{T}_n$. Finally set $T = \bigcap_{n=1}^{\infty} T_n$.

Notice that, by 7, $\mathfrak{T}_n$ consists of $R_n$ pairwise disjoint rectangles.

**9.** *Let $t \in \mathbb{R}$. Then there is at most one $x \in \mathbb{R}$ such that $\langle x, t\rangle \in T$ and at most one $y \in \mathbb{R}$ such that $\langle t, y\rangle \in T$.*

Proof. Suppose that there are numbers $x$, $x^*$ such that $x \neq x^*$, $\langle x,t\rangle \in T$, $\langle x^*,t\rangle \in T$. There are even $n > 2$ for which $q_{n-1}/R_{n-1} < |x^* - x|$. There are integers $k_j$, $k_j^*$ such that $0 \leqslant k_j < r_j$, $0 \leqslant k_j^* < r_j$ $(j = 1,\dots,n)$ and that $\langle x,t\rangle \in L(k_1,\dots,k_n)$, $\langle x^*,t\rangle \in L(k_1^*,\dots,k_n^*)$. Then $t \in J(k_1 + r_1k_2,\dots,k_{n-1} + r_{n-1}k_n)$ and also $t \in J(k_1^* + r_1k_2^*,\dots,k_{n-1}^* + r_{n-1}k_n^*)$; thus $k_1 + r_1k_2 = k_1^* + r_1k_2^*,\dots,k_{n-1} + r_{n-1}k_n = k_{n-1}^* + r_{n-1}k_n^*$. This easily implies that $k_1 = k_1^*,\dots,k_n = k_n^*$. Therefore both $x$ and $x^*$ belong to the interval $I(r_0k_1,\dots,k_{n-2} + r_{n-2}k_{n-1})$ whose length is $q_{n-1}/R_{n-1}$. This is a contradiction.

The second part of the theorem can be proved similarly. □

**10.** For each set $M \subset \mathbb{R}^2$ and each $n \in \mathbb{N}$ let $P_n(M)$ be the number of elements of $\mathfrak{T}_n$ intersecting $M$ and let $p_n(M) = P_n(M)/R_n$.

**11.** *Let $M \subset \mathbb{R}^2$ and let $n \in \mathbb{N}$. Then $p_{n+1}(M) \leqslant p_n(M)$.*

Proof. Each element of $\mathfrak{T}_{n+1}$ lies in some element of $\mathfrak{T}_n$ and each element of $\mathfrak{T}_n$ contains $r_{n+1}$ elements of $\mathfrak{T}_{n+1}$. Therefore $p_{n+1}(M) \leqslant r_{n+1}P_n(M)$ whence our assertion follows at once. □

**12.** *Let $n \in \mathbb{N}$ and let $M \subset K \in \mathfrak{T}_n$. Then*

$$P_{n+1}(M) \leqslant 1 + \mu_{n+1}^{-1} \operatorname{diam} M.$$

Proof. Let $v = P_{n+1}(M)$. We may suppose that $v > 1$. Let $K_1,\dots,K_v$ be all elements of $\mathfrak{T}_{n+1}$ intersecting $M$. Let, e.g., $n$ be even. We may write $K = L(k_1,\dots,k_n) = V \times W$, $K_s = \gamma(r_nr_{n+1}, k_n + r_nj_s, q_{n+1}/q_{n-1}, V) \times W$ $(s = 1,\dots,v)$ and we may suppose that $j_1 < j_2 < \dots < j_v$. Choosing $\langle x_s, y_s\rangle \in K_s$ $(s = 1, v)$ we get, by 2, 6 and 4, $\operatorname{diam} M \geqslant x_v - x_1 > \big(r_n(j_v - j_1) - 1\big)q_{n-1}/(R_{n-1}r_nr_{n+1}) \geqslant (v-1)(r_n - 1)q_{n-1}/R_{n+1} \geqslant (v-1)\mu_{n+1}$ whence $v < 1 + \mu_{n+1}^{-1} \operatorname{diam} M$.

If $n$ is odd, we proceed analogously. □

**13.** Let $\omega$ be a continuous increasing function on $[0,\infty)$, $\omega(0) = 0$, and let $M \subset \mathbb{R}^2$. For each $\varepsilon > 0$ define $\Lambda(\omega, M, \varepsilon) = \inf \sum_{n=1}^{\infty} \omega(\operatorname{diam} S_n)$, where $\mathbb{R}^2 \supset \bigcup_{n=1}^{\infty} S_n \supset M$, $\operatorname{diam} S_n < \varepsilon$ $(n \in \mathbb{N})$. If $0 < \varepsilon_1 < \varepsilon_2$, then $\Lambda(\omega, M, \varepsilon_1) \geqslant \Lambda(\omega, M, \varepsilon_2)$. Further we set $\Lambda(\omega, M) = \lim_{\varepsilon \downarrow 0} \Lambda(\omega, M, \varepsilon)$.

For each $\alpha > 0$ write $\Lambda_\alpha(M) = \Lambda(\omega, M)$, where $\omega(t) = t^\alpha$ $(t \geqslant 0)$.

We define the Hausdorff dimension of $M$ (H.d. $M$) as $\inf\{\alpha > 0;\ \Lambda_\alpha(M) = 0\}$. It is easy to prove that $\Lambda_\alpha(M) = 0$, if $\alpha >$ H.d. $M$, and $\Lambda_\alpha(M) = \infty$, if $0 < \alpha <$ H.d. $M$.

In what follows, $\liminf \ldots$ will mean $\liminf\limits_{n\to\infty} \ldots$; similarly for $\limsup$ and $\lim$.

**14.** *Let $\omega$ be as in 13. Then*

$$\Lambda(\omega, T) \leqslant \liminf r_n R_n \omega(q_{n-1}/R_n). \tag{1}$$

P r o o f. Let $n \in \mathbb{N}$. According to 6, each element of $\mathfrak{T}_n$ can be covered by $r_n$ rectangles of diameter $q_{n-1}/R_n$. Therefore $\Lambda(\omega, T, q_{n-2}/R_n) \leqslant r_n R_n \omega(q_{n-1}/R_n)$. This easily implies (1). □

**15.** Throughout the paper, $\varphi$ and $\psi$ will be continuous increasing functions on $[0, \infty)$ such that $\varphi(0) = \psi(0) = 0$, $\varphi(t)\psi(t) = t^2$ and that the function $\varphi(t)/t$ $( = t/\psi(t))$ is non-decreasing $(t > 0)$.

**16.** *Let $n \in \mathbb{N}$ and let $M \subset \mathbb{R}^2$. Let $\mu_{n+1} \leqslant \operatorname{diam} M < \mu_n$. Then*

$$p_{n+2}(M) \leqslant (1 + r_n^{-1}) R_{n-1} \big(\psi(1/R_n) + \psi(\mu_{n+1})\big) \varphi(\operatorname{diam} M). \tag{2}$$

P r o o f. Set $\operatorname{diam} M = \delta$. By 7, $M$ intersects at most one element of $\mathfrak{T}_n$. It follows from 12 that $P_{n+1}(M) \leqslant 1 + \delta/\mu_{n+1}$. Thus (see 11)

$$p_{n+2}(M) \leqslant p_{n+1}(M) \leqslant \frac{1}{R_{n+1}} + \frac{\delta}{r_n}. \tag{3}$$

(1) Let $\delta > 1/R_n$. Since $\varphi(t)/t$ is non-decreasing, we have $1 \leqslant \frac{\varphi(\delta)}{\delta} \cdot \frac{\psi(x)}{x}$ for each $x \in (0, \delta)$. For $x = 1/R_n$ we get $\delta/r_n \leqslant \varphi(\delta) R_{n-1} \psi(1/R_n)$; taking $x = \mu_{n+1}$ and applying the inequality $1/\delta < R_n$ we get

$$\frac{1}{R_{n+1}} \leqslant \frac{\varphi(\delta)}{\delta} \cdot \frac{\psi(\mu_{n+1})}{r_n} < \varphi(\delta) R_{n-1} \psi(\mu_{n+1}).$$

This and (3) implies (2).

(2) Let $\delta \leqslant 1/R_n$. By 12 we have $P_{n+2}(K \cap M) \leqslant 1 + \delta/\mu_{n+2}$ for each $K \in \mathfrak{T}_{n+1}$. Thus $P_{n+2}(M) \leqslant (1 + \delta/\mu_{n+2})(1 + \delta/\mu_{n+1})$ whence

$$\begin{aligned} p_{n+2}(M) &\leqslant \frac{\delta^2}{R_{n+2}\mu_{n+2}\mu_{n+1}} \Big(\frac{\mu_{n+2}}{\delta} + 1\Big)\Big(\frac{\mu_{n+1}}{\delta} + 1\Big) \\ &\leqslant \frac{\varphi(\delta)\psi(\delta) R_{n+1}}{\mu_{n+1} r_n} \Big(\frac{\mu_{n+2}}{\mu_{n+1}} + 1\Big)\Big(\frac{\mu_{n+1}}{\delta} + 1\Big). \end{aligned} \tag{4}$$

Obviously

$$\mu_{n+2}/\mu_{n+1} = \frac{r_{n+1}}{R_{n+2}} \cdot \frac{R_{n+1}}{r_n} \leqslant \frac{1}{r_n};$$

since $\psi(t)$ is increasing and $\psi(t)/t$ is non-increasing, we have $\psi(\delta) \leqslant \psi(1/R_n)$, $\psi(\delta)\delta^{-1}\mu_{n+1} \leqslant \psi(\mu_{n+1})$ whence $\psi(\delta)(\delta^{-1}\mu_{n+1}+1) \leqslant \psi(1/R_n) + \psi(\mu_{n+1})$. This and (4) implies (2). □

**17.** *Set* $A_n = R_{n-1}\psi(1/R_n)$, $B_n = R_{n-1}(\psi(1/R_n) + \psi(\mu_{n+1}))$ $(n \in \mathbb{N})$, $\lambda = \liminf A_n^{-1}$, $\lambda^* = \liminf B_n^{-1}$. *Then*

$$(5) \qquad \lambda/2 \leqslant \lambda^* \leqslant \Lambda(\varphi, T) \leqslant \lambda.$$

*If* $\varphi(t)/t^\beta$ *increases for some* $\beta > 1$, *then* $\Lambda(\varphi, T) = \lambda = \lambda^*$.

Proof. Obviously $r_n R_n \varphi(q_{n-1}/R_n) \leqslant q_{n-1}^2/A_n$. According to 14 we have $\Lambda(\varphi, T) \leqslant \lambda$. Now let $n \in \mathbb{N}$ and let $\mathfrak{G}$ be an open cover of $T$ such that $\operatorname{diam} G < \mu_n$ for each $G \in \mathfrak{G}$. We may suppose that $\mathfrak{G}$ is finite. For each $G \in \mathfrak{G}$ there is an integer $n_G \geqslant n$ such that $\mu_{n_G+1} \leqslant \operatorname{diam} G < \mu_{n_G}$. Let $Q_n = \sup\{(1 + r_k^{-1})B_k;\ k \geqslant n\}$. There is an $m \in \mathbb{N}$ such that $m \geqslant n_G + 2$ for each $G \in \mathfrak{G}$. Then by 11 and 16, $p_m(G) \leqslant p_{n_G+2}(G) \leqslant Q_n\varphi(\operatorname{diam} G)$ $(G \in \mathfrak{G})$. Since each element of $\mathfrak{T}_m$ contains some element of $T$, we have $\sum_{G\in\mathfrak{G}} P_m(G) \geqslant R_m$ whence $\sum_{G\in\mathfrak{G}} p_m(G) \geqslant 1$. It follows that $\sum_{G\in\mathfrak{G}} \varphi(\operatorname{diam} G) \geqslant Q_n^{-1}$, $\Lambda(\varphi, T, \mu_n) \geqslant Q_n^{-1}$, $\Lambda(\varphi, T) \geqslant \lim Q_n^{-1} = \lambda^*$. Since $\mu_{n+1} \leqslant 1/R_n$, we have $B_n \leqslant 2A_n$ whence $\lambda^* \geqslant \lambda/2$. This proves (5).

Let us suppose, finally, that there exists a $\beta > 1$ such that $\varphi(t)/t^\beta$ increases and that $\lambda^* < \lambda$. Then there are numbers $A$, $B$ such that $\limsup A_n < A < B < \limsup B_n$. We can find an $n_0$ such that $A_n < A$ for each integer $n > n_0$. Now let $n > n_0$ and $B_n > B$. Since $\psi(t)t^{\beta-2} = t^\beta/\varphi(t)$ decreases, we have $\psi(\mu_{n+1})\mu_{n+1}^{\beta-2} < \psi(1/R_{n+1})R_{n+1}^{2-\beta}$ whence

$$\frac{B-A}{A} < \frac{B_n - A_n}{A_{n+1}} = \frac{R_{n-1}\psi(\mu_{n+1})}{R_n\psi(1/R_{n+1})} < \frac{1}{r_n}(\mu_{n+1}R_{n+1})^{2-\beta} = r_n^{1-\beta}.$$

Choosing $n$ sufficiently great we get a contradiction. This completes the proof. □

**18.** *Let* $\alpha > 0$. *Then* $\Lambda_\alpha(T) = \liminf r_n R_n^{1-\alpha}$.

Proof. Set $\varrho = \liminf r_n R_n^{1-\alpha}$. Obviously $\varrho = \infty$ for $\alpha \leqslant 1$, $\varrho = 0$ for $\alpha \geqslant 2$.

Taking $\omega(t) = t^\alpha$ we have $r_n R_n \omega(q_{n-1}/R_n) = q_{n-1}^\alpha r_n R_n^{1-\alpha}$ so that, by 14, $\Lambda_\alpha(T) \leqslant \varrho$. This proves our assertion for $\alpha \geqslant 2$. Now let $1 \leqslant \alpha < 2$; set $\varphi(t) = t^\alpha$, $\psi(t) = t^{2-\alpha}$ and let $A_n$ be as in 17. Then $A_n^{-1} = r_n R_n^{1-\alpha}$ and it follows easily from 17 that $\Lambda_\alpha(T) = \varrho$. If $\alpha < 1$, then $\Lambda_\alpha(T) \geqslant \Lambda_1(T) = \infty$ so that $\Lambda_\alpha(T) = \varrho$ again. □

Example. Let $a, b$ be integers, $a, b > 1$, and let $r_n = a^{b^n}$ $(n = 0, 1, \ldots)$. Define $\alpha = 2 - b^{-1}$. Then $r_n R_n^{1-\alpha} = a$ for each $n$ so that $\Lambda_\alpha(T) = a$.

**19.** *We have* $H.d.T = 1 + \liminf(\log r_n / \log R_n)$.

Proof. If $\alpha > 1 + \liminf \ldots$, then $1 > r_n R_n^{1-\alpha}$ for infinitely many numbers $n$ so that, by 18, $\Lambda_\alpha(T) \leqslant 1$; if $\alpha < 1 + \liminf \ldots$, we find similarly that $\Lambda_\alpha(T) \geqslant 1$. This completes the proof. □

**20.** *If* $r_n = 2^{n!}$ $(n = 0, 1, \ldots)$, *then* $H.d.T = 2$.

Proof. Obviously $1! + 2! + \ldots + n! < n! + 2(n-1)!$ so that $\log R_n / \log r_n < 1 + \frac{2}{n}$ $(n \in \mathbb{N})$. Now we apply 19. □

**21.** *Suppose that* $\varphi(t)/t \to 0$ $(t \downarrow 0)$.* *Let* $0 \leqslant \lambda \leqslant \infty$. *Then there are numbers* $r_n, q_n$ *with properties listed in 3 such that the corresponding set* $T$ *fulfills the relation*

$$\lambda/2 \leqslant \Lambda(\varphi, T) \leqslant \lambda. \tag{6}$$

*If* $\varphi(t)/t^\beta$ *increases for some* $\beta > 1$, *then* (6) *may be replaced by* $\Lambda(\varphi, T) = \lambda$.

Proof. Set $\delta(t) = \varphi(t)/t$ $(t > 0)$. Then $\delta$ is non-decreasing, $\delta(0+) = 0$ and

$$k\delta(x/k) = x/\psi(x/k) \to \infty \quad (k \to \infty) \tag{7}$$

for each $x > 0$.

We will find integers $r_n$ such that $2 \leqslant r_0 \leqslant r_1 \leqslant \ldots$, $r_n \to \infty$ and that, setting $R_n = r_1 \ldots r_n$ and $C_n = r_n \delta(1/R_n)$, we have $C_n \to \lambda$ $(n \to \infty)$. It is easy to see that $C_n = A_n^{-1}$, where $A_n$ are as in 17.

(1) Let $\lambda = 0$. Since $\delta(0+) = 0$, there are integers $k_s$ such that $0 = k_0 < k_1 < \ldots$ and that, if we set $r_0 = 2$, $r_n = s + 1$ for $n = k_{s-1} + 1, \ldots, k_s$, we have $\delta(1/R_{k_s}) < \big(s(s+2)\big)^{-1}$ $(s \in \mathbb{N})$. Then $1/s > (s+2)\delta(1/R_{k_s+1}) = C_{k_s+1} \geqslant \ldots \geqslant C_{k_{s+1}}$ so that $C_n \to 0$.

(2) Let $0 < \lambda < \infty$. We find an $m \in \mathbb{N}$ such that $2\delta(2^{-m}) < \lambda$ and set $r_k = 2$ for $k = 0, \ldots, m$. Then $C_m < \lambda$. Now suppose that $n \geqslant m$ and that we have $r_1, \ldots, r_n$ such that $C_n < \lambda$. Let $r$ be the greatest of all integers $k$ for which $k\delta\big((kR_n)^{-1}\big) < \lambda$ (see (7)). Obviously $r \geqslant r_n$ and $r\delta\big((rR_n)^{-1}\big) < \lambda \leqslant (r+1)\delta\Big(\big((r+1)R_n\big)^{-1}\Big) < (r+1)\delta\big((rR_n)^{-1}\big)$; therefore

$$0 < \lambda - r\delta\big((rR_n)^{-1}\big) < \delta\big((rR_n)^{-1}\big) < \lambda/r. \tag{8}$$

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*See 15.

Set $r_{n+1} = r$. Since $(r_{n+1} + 1)\delta(1/R_{n+1}) > \lambda$ and $\delta(0+) = 0$, we have $r_{n+1} \to \infty$ and it follows from (8) that $C_{n+1} \to \lambda$.

(3) The case $\lambda = \infty$ may be left to the reader (see (7)).

Now we apply 17. □

*References*

[1] *S. Saks*: Theory of the integral. Dover Publications, 1964.

## Editorial comments

Professor Jan Mařík died on January 6, 1994. He had no time to take into account the remarks of the referee to this paper.

The main result of the paper is given in Proposition 21 according to which for sufficiently reasonable functions $\varphi$ such that

$$t^2 = o(\varphi(t)), \quad t \to 0+,$$

there is a compact subset $T$ of the plane which intersects lines parallel to the axes in at most one point, nevertheless $\Lambda(\varphi, T) = \infty$ where $\Lambda(\varphi, T)$ is the Hausdorff measure of the set $T$ given by the function $\varphi$.

The paper is based on an interesting and simple construction. Even if the author emphasizes the result for $\alpha$-dimensional measures, i.e. the case $\varphi(t) = t^\alpha$, $\alpha < 0$ the most interesting case is the case of a general $\varphi$. The special result for $\varphi(t) = t^\alpha$ can be easily derived using recent results of P. Matilla from his paper: Hausdorff dimension and capacities of intersections of sets in $n$-space (Acta Math. Uppsala 152 (1984), 77–105). Corollary 6.12 on p. 101 of this paper of Matilla can be used to this reason. This corollary yields also that instead of the two directions parallel to the axes a finite number of directions can be taken into account.

Finally it would be useful to add some fundamental reference concerning Hausdorff measures, for example the book of C. A. Rogers (Hausdorff measures, Cambridge Univ. Press, 1970).

We are publishing this paper of Jan Mařík in its original form. The result is interesting even if, as J. Mařík wrote, the result in this form was ready twenty years ago and in the meantime a lot of things happened in the field.