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ALGEBRAS WITH n-TRANSFERABLE TOLERANCES

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By a *tolerance* on an algebra A we mean a reflexive and symmetric binary relation on A having the Substitution Property with respect to all operations of A. For some basic results, see e.g. [1]. The set $\operatorname{Tol} A$ of all tolerances of an algebra A forms a complete lattice with respect to set inclusion. Hence, if M, N are non-void subsets of A, denote by $T(M \times N)$ the least tolerance on A containing all pairs $\langle a, b\rangle$ where $a \in M$, $b \in N$. If $M = N$, we write briefly $T(M)$ instead of $T(M \times M)$. For M finite, say $M = \{a_1, \ldots, a_n\}$, we will write only $T(a_1, \ldots, a_n)$. In particular, $T(a, b)$ means $T(M)$ for $M = \{a, b\}$, the so called *principal tolerance* generated by the pair $\langle a, b\rangle$.

If $T \in \operatorname{Tol} A$ and $a \in A$, denote by

$$[a]_T = \{b \in A; \langle a, b\rangle \in T\}$$

the so called *tolerance class of* T.

All the foregoing concepts were investigated by numerous authors, see e.g. [1] and references given there.

Let A be an algebra and n an integer. By an n-ary *algebraic function over* A we mean a mapping $\varphi\colon A^n \to A$ such that $\varphi(x_1, \ldots, x_n) = t(x_1, \ldots, x_n, a_1, \ldots, a_m)$ for some $(n+m)$-ary term t over A and elements $a_1, \ldots, a_m \in A$.

Lemma 1. *Let A be an algebra and $\emptyset \neq M \subseteq A$, a, $b \in A$. Then $\langle a, b\rangle \in T(M)$ if and only if there exist elements $a_1, \ldots, a_n$, $b_1, \ldots, b_n \in M$ and an n-ary algebraic function φ over A with*

$$a = \varphi(a_1, \ldots, a_n), \quad b = \varphi(b_1, \ldots, b_n).$$

For the easy proof, see e.g. Lemma 1.4 in [1].

Transferable principal congruences were introduced in [3] and the concept was generalized for tolerances in [2]. O. M. Mamedov [5] introduced the concept of n-transferable principal congruences and investigated varieties of algebras with such congruences. In our paper, we generalize also this concept for tolerances and prove the basic interrelations between this concept and tolerance regular algebras introduced in [2].

Definition. Let A be an algebra and a, $b \in A$. The principal tolerance $T(a,b)$ is *n-transferable* if for each $c \in A$ there exist elements $d_1, \ldots, d_n$ of A with

$$T(a,b) = T(c, d_1, \ldots, d_n). \qquad (*)$$

An algebra A has n-transferable tolerances if for each a, b, $c \in A$ there exist $d_1, \ldots, d_n \in A$ such that $(*)$ holds.

For $n = 1$, this concept was investigated in [2] with close relations to tolerance regularity:

Definition. An algebra A is *tolerance regular* if $T_1 = T_2$ for each T_1, $T_2 \in \mathrm{Tol}\, A$ whenever $[a]_{T_1} = [a]_{T_2}$ for some $a \in A$.

Example 1. The lattices L_1, L_2 in Fig. 1 are tolerance regular (but they are not tolerance trivial, i.e. there exist $T \in \mathrm{Tol}\, L_i$, $i = 1, 2$, which are not congruences).

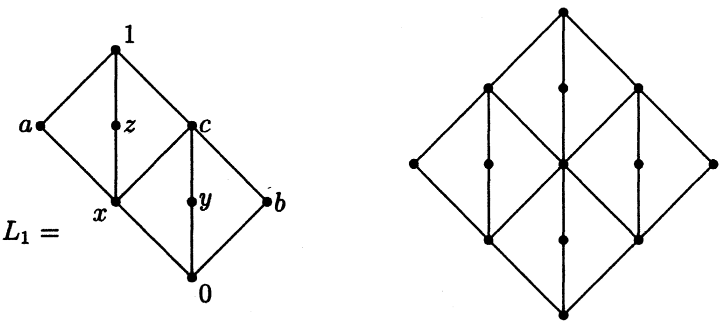
$L_1 =$ 

Fig. 1

Lemma 2. *Let an algebra A have n-transferable tolerances for some $n \geqslant 1$. If $T(a,b)$, $T(c,d) \in \mathrm{Tol}\, A$ have a common tolerance class, then $T(a,b) = T(c,d)$.*

Proof. Suppose $[x]_T$ is a common tolerance class of $T(a,b)$ and $T(c,d)$. Since A has n-transferable tolerances, there exist $y_1, \ldots, y_n$, $z_1, \ldots, z_n \in A$ with

$$T(a,b) = T(x, y_1, \ldots, y_n),$$
$$T(c,d) = T(x, z_1, \ldots, z_n).$$

Hence $T(x, y_1) \subseteq T(a, b)$ and $T(x, z_1) \subseteq T(c, d)$ for each $i = 1, \dots, n$. It implies $y_i \in [x]_T$, $z_i \in [x]_T$ for $i = 1, \dots, n$, thus

$$T(a, b) = T(x, y_1, \dots, y_n) \subseteq T([x]_T),$$
$$T(c, d) = T(x, z_1, \dots, z_n) \subseteq T([x]_T).$$

Since both $T(a, b)$, $T(c, d)$ have a tolerance class $[x]_T$, we have

$$T(a, b) \supseteq T([x]_T), \quad T(c, d) \supseteq T([x]_T),$$

whence $T(a, b) = T([x]_T) = T(c, d)$. □

Lemma 3. *Let A be an algebra, $a \in A$, $\emptyset \neq B \subseteq A$. Suppose $T \in \operatorname{Tol} A$, $T_i \in \operatorname{Tol} A$ for $i \in I$ and let $T_i = T([a]_{T_i})$ for each $i \in I$. If $T = \bigvee\{T_i; i \in I\}$ in $\operatorname{Tol} A$, then*

(a) *if $T = T(\{a\} \times B)$ then $T = T([a]_T)$;*

(b) *$T = T\Big(\{a\} \times \bigcup\{[a]_{T_i}; i \in I\}\Big)$.*

For the proof, see [2] or Lemma 6.4 in [1].

Theorem 1. *If an algebra A has n-transferable tolerances for some $n \geqslant 1$, then A is tolerance regular.*

Proof. Let T_1, $T_2 \in \operatorname{Tol} A$ have a common tolerance class $[x]_{T_1} = [x]_{T_2}$ for some $x \in A$. Evidently (see e.g. Lemma 1.5 in [1]),

$$T_1 = \bigvee\{T(a, b); \langle a, b\rangle \in T_1\},$$
$$T_2 = \bigvee\{T(c, d); \langle c, d\rangle \in T_2\}.$$

By Lemma 2, both $T(a, b)$, $T(c, d)$ are determined by their tolerance classes, i.e.

$$T(a, b) = T([x]_{T(a,b)}) \quad \text{and} \quad T(c, d) = T([x]_{T(c,d)}),$$

thus

$$T_1 = \bigvee\{T([x]_{T(a,b)}); \langle a, b\rangle \in T_1\},$$
$$T_2 = \bigvee\{T([x]_{T(c,d)}); \langle c, d\rangle \in T_2\}.$$

By Lemma 3(b) we have

$$T_1 = T\Big(\{x\} \times \bigcup\{[x]_{T(a,b)}; \langle a, b\rangle \in T_1\}\Big),$$
$$T_2 = T\Big(\{x\} \times \bigcup\{[x]_{T(c,d)}; \langle c, d\rangle \in T_2\}\Big).$$

By Lemma 3(a) we conclude

$$T_1 = T([x]_{T_1}) = T([x]_{T_2}) = T_2.$$

□

Remark. For $n = 1$, Theorem 1 implies the main result of [2], see also Theorem 6.2 in [1]. However, we are now able to state also the converse of the foregoing assertion, namely:

Theorem 2. *Let A be a tolerance regular algebra. Then for every a, b of A there exists an integer n such that $T(a,b)$ is n-transferable.*

Proof. Suppose a, b, $c \in A$ and $T = T(a,b) \in \operatorname{Tol} A$. Since A is tolerance regular, we have

$$T(a,b) = T([c]_T),$$

thus $\langle a, b\rangle \in T([c]_T)$. By Lemma 1 there exists a finite subset $F \subseteq ([c]_T)$ such that

$$\langle a, b\rangle \in T(F).$$

Denote $F = \{d_1, \dots, d_n\}$. Then

$$T(a,b) \subseteq T(F) \subseteq T(F \cup \{c\}) \subseteq T([c]_T) = T(a,b)$$

whence

$$T(a,b) = T(F \cup \{c\}) = T(c, d_1, \dots, d_n).$$

□

Corollary. *Let A be a finite algebra. The following conditions are equivalent:*
(1) *A is tolerance regular;*
(2) *A has n-transferable tolerances for an integer n.*

This follows directly from Theorems 1 and 2 and the fact that if $T(a,b)$ is n-transferable then it is also k-transferable for each integer $k \geqslant n$.

It can be of some interest that for suitable algebras the integer n of Theorem 2 can be "uniform" for all $T(a,b)$. In particular, we prove

Theorem 3. *For a lattice L, the following conditions are equivalent:*
(i) *L is tolerance regular;*
(ii) *L has 2-transferable tolerances.*

P r o o f. By Theorem 1, it is enough to prove (i)$\Rightarrow$(ii). Let L be tolerance regular and a, b, $c \in L$. By Theorem 2 there exists an integer n and elements $d_1, \dots, d_n \in A$ such that

$$T(a,b) = T(c, d_1, \dots, d_n).$$

Put $p_1 = d_1 \wedge \dots \wedge d_n$ and $p_2 = d_1 \vee \dots \vee d_n$ in the lattice L. Then $p_1 \leqslant d_i \leqslant p_2$ for each $i = 1, \dots, n$, thus $\langle d_i, d_j \rangle \in T(c, p_1, p_2)$ for each i, j and

$$\langle d_i, d_i \rangle \in T(c, p_1, p_2), \quad \langle c, p_1 \rangle \in T(c, p_1, p_2), \quad \langle c, p_2 \rangle \in T(c, p_1, p_2)$$

imply

$$\langle c, d_i \rangle = \langle c \wedge (d_i \vee c), p_2 \wedge (d_i \vee p_1) \rangle \in T(c, p_1, p_2),$$

whence $T(a,b) = T(c, d_1, \dots, d_n) \subseteq T(c, p_1, p_2)$.

However,

$$\begin{aligned} \langle c, p_1 \rangle &= \langle c \wedge \dots \wedge c, d_1 \wedge \dots \wedge d_n \rangle \in T(c, d_1, \dots, d_n), \\ \langle c, p_2 \rangle &= \langle c \vee \dots \vee c, d_1 \vee \dots \vee d_n \rangle \in T(c, d_1, \dots, d_n), \\ \langle p_1, p_2 \rangle &= \langle d_1 \wedge \dots \wedge d_n, d_1 \vee \dots \vee d_n \rangle \in T(c, d_1, \dots, d_n). \end{aligned}$$

i.e. $T(c, p_1, p_2) \subseteq T(c, d_1, \dots, d_n) \subseteq T(a, b)$. □

R e m a r k. We are able to show that we cannot replace $n = 2$ by $n = 1$ in Theorem 3, see

E x a m p l e 2. As mentioned in Example 1, the lattice L_1 in Fig. 1 is tolerance regular. Its tolerance lattice Tol L_1 contains only three elements, namely the identity relation ω, the full relation $\iota = L_1 \times L_1$ and the tolerance T given by its tolerance blocks as is shown in Fig. 2 below. Evidently, $T(a,b) = \iota$ (the elements are denoted as in Fig. 1), however, for each $x \in L_1$ either $T(c,x) = T$ (if $x \neq c$) or $T(c,x) = \omega$ (for $x = c$), thus there does not exist $d_1 \in L_1$ with $T(a,b) = T(c, d_1)$.

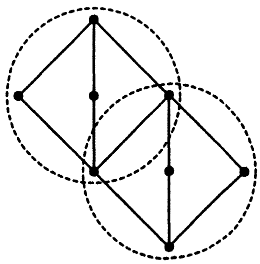


Fig. 2

According to Theorem 3, there exist d_1, d_2 of L_1 with $T(a,b) = T(c, d_1, d_2)$, e.g. $d_1 = b$, $d_2 = 1$.

References

[1] *Chajda I.*: Algebraic Theory of Tolerance Relations. University Olomouc Press, 1991.
[2] *Chajda I.*: Transferable tolerances and weakly tolerance regular lattices. Colloq. Math. Soc. J. Bolyai 43. Szeged (Hungary), North-Holland, 1985, pp. 27–40.
[3] *Chajda I.*: Transferable principal congruences and regular algebras. Math. Slovaca *34* (1984), 97–102.
[4] *J. Duda*: Mal'cev conditions for regular and weakly regular subalgebras of the square. Acta Sci. Math. (Szeged) *46* (1989), 29–34.
[5] *Mamedov O. M.*: Characterizations of varieties with n-transferable principal congruences. VINITI, Akad. Nauk Azerbaidz. SR, Institut Matem. i Mechan. (Baku) (1989), 2–12. (In Russian.)

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