Mathematica Bohemica

Włodzimierz M. Mikulski

Natural liftings of foliations to the tangent bundle

Mathematica Bohemica, Vol. 117 (1992), No. 4, 409–414

Persistent URL: http://dml.cz/dmlcz/126055

Terms of use:

© Institute of Mathematics AS CR, 1992

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

NATURAL LIFTINGS OF FOLIATIONS TO THE TANGENT BUNDLE

W. M. Mikulski, Kraków

(Received January 3, 1991)

Summary. A classification of natural liftings of foliations to the tangent bundle is given.

Keywords: natural lifting, foliation, tangent bundle

AMS classification: 53C 12, 57 R 30

0. Introduction

All manifolds are assumed to be finite dimensional, Hausdorff, without boundary and C^∞. Mappings are assumed to be C^∞ and foliations are assumed to be C^∞ and without singularities. (For equivalent definitions of foliations see [2].)

From now on we fix two natural numbers n and p such that $p < n$. Suppose that to any p-dimensional foliation F defined on an n-manifold M there corresponds a foliation $L(F)$ on TM projecting (by the tangent bundle projection) onto F. According to the general theory of natural transformations, see [1], we introduce the following definition.

Definition 0.1. A correspondence L as above is called a *natural lifting of foliations* to the tangent bundle iff the following naturality condition is satisfied: for any foliation F of dimension p on an n-manifold M and any diffeomorphism φ from an n-manifold N onto an open subset of M we have $L(\varphi^{-1}F) = (\mathrm{d}\varphi)^{-1}L(F)$, where $\varphi^{-1}F$ is the inverse image of F and $\mathrm{d}\varphi$ denotes the differential of φ.

We have the following examples of natural liftings of foliations to the tangent bundle.

Example 0.1. Let F be a p-dimensional foliation on an n-manifold M. It is well-known that the tangent bundle TM admits canonically defined foliations $L_1(F)$ of dimension $2p$ and $L_2(F)$ of dimension $p+n$ projecting (by the tangent bundle projection $\pi_M : TM \xrightarrow{\bullet} M$) onto the initial foliation F. More precisely, $L_2(F) = \pi_M^{-1}F$, the inverse image, and $L_1(F)$ is defined by a cocycle $(\pi_M^{-1}(U_i), df_i, dg_{ij})$, where (U_i, f_i, g_{ij}) is a cocycle defining F. It is easy to verify that the correspondence $F \to L_i(F)$, $i = 1, 2$, are natural liftings of foliations to the tangent bundle.

The main theorem in this paper is the following one.

Theorem 0.1. *Any natural lifting of foliations to the tangent bundle belongs to the set $\{L_1, L_2\}$ described in Example 0.1.*

1. Notation

From now on we use the following notation. We denote by $\partial_1, \ldots, \partial_n$ the canonical vector fields on $\mathbf{R}^n$, by ∂ the vector $\partial_n|0$, by $\pi: T\mathbf{R}^n \to \mathbf{R}^n$ the tangent bundle projection and by F^p the standard p-dimensional foliation on $\mathbf{R}^n$ spanned by $\partial_1, \ldots, \partial_p$. By e_i we denote the vector $(0, \ldots, 1, \ldots, 0) \in \mathbf{R}^n$, 1 in the i-th position.

From now on we identify a foliation with its tangent distribution.

2. Reducibility Lemma

The following Lemma plays an essential role in the proof of the main theorem.

Lemma 2.1. *Let L_a and L_b be two natural liftings of foliations to the tangent bundle. Suppose that $L_a(F^p)_\partial \subset L_b(F^p)_\partial$. Then $L_a(F) \subset L_b(F)$ for any p-dimensional foliation F on an n-manifold. In particular, if $L_a(F^p)_\partial = L_b(F^p)_\partial$, then $L_a = L_b$.*

Proof. Let F be a p-dimensional foliation on an n-manifold M. Consider $z \in TM \setminus F$. By the Frobenius theorem there exists a diffeomorphism φ from an open subset $U \subset \mathbf{R}^n$ onto an open subset of N such that $\varphi^{-1}F = i^{-1}F^p$ and $d\varphi(\tilde{\partial}) = z$, where $i: U \to \mathbf{R}^n$ is the inclusion and $\tilde{\partial} \in TU$ is the vector such that $di(\tilde{\partial}) = \partial$. Using the naturality condition we see that $L_a(F)_z = d(d\varphi)(L_a(i^{-1}F^p)_{\tilde{\partial}})$ and $d(di)(L_a(i^{-1}F^p)_{\tilde{\partial}}) = L_a(F^p)_z$, and similarly for L_b. Therefore the assumption of the Lemma implies that $L_a(F)_z \subset L_b(F)_z$. Since $TM \setminus F$ is dense in TM, we deduce that $L_a(F) \subset L_b(F)$. □

We introduce the following definition.

Definition 3.1. Let $z \in T_0\mathbf{R}^n$ be a vector. A global diffeomorphism $\varphi\colon \mathbf{R}^n \to \mathbf{R}^n$ is called *z-admissible* iff $\varphi^{-1}F^p = F^p$ and $\mathrm{d}\varphi(z) = z$. A subspace $W \subset T_zT\mathbf{R}^n$ is called z-admissible iff for any z-admissible diffeomorphism φ we have $\mathrm{d}(\mathrm{d}\varphi)(W) = W$ and $\mathrm{d}\pi(W) = F^p_0$.

Using the naturality condition it is easy to verify the following Lemma.

Lemma 3.1. *If L is a natural lifting of foliations to the tangent bundle, then $L(F^p)_z$ is z-admissible for any $z \in T_0\mathbf{R}^n$.*

Therefore, to prove Theorem 0.1 it is sufficient to verify the following proposition.

Proposition 3.1. *Any ∂-admissible subspace contains $L_1(F^p)_\partial$. Any 0-admissible subspace strictly containing $L_1(F^p)_0$ is equal to $L_2(F^p)_0$, where L_1, L_2 are described in Example 0.1.*

From Proposition 3.1 and Lemmas 2.1 and 3.1 we deduce Theorem 0.1 in the following way. Consider a natural lifting $L \neq L_1$. By Lemmas 3.1 and 2.1 and Proposition 3.1 it follows that $L(F^p)_\partial \supsetneq L_1(F^p)_\partial$. Then (by Lemma 2.1 and the dimension argument) $L(F^p)_0 \supsetneq L_1(F^p)_0$ and then (from Proposition 3.1) $L(F^p)_0$ has dimension $p+n$. Therefore $L(F^p)_\partial$ has dimension $p+n$, too. On the other hand, $L(F^p)_\partial \subset (d_\partial\pi)^{-1}(F^p_0) = L_2(F^p)_\partial$. Hence $L(F^p)_\partial = L_2(F^p)_\partial$. Therefore $L = L_2$ because of Lemma 2.1.

4. Transformation rules

We trivialize $T\mathbf{R}^n$ by the diffeomorphism

$$I\colon T\mathbf{R}^n \to \mathbf{R}^{2n}, \quad I\left(\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)_{t=0}\right) = (\gamma(0), \gamma'(0)), \tag{4.1}$$

where $\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t)_{t=0}$ is the vector generated by γ. Denote by $(\overline{\partial}_i)$ the canonical vector fields on $\mathbf{R}^{2n}$. In the vector spaces $T_\partial T\mathbf{R}^n$ and $T_0T\mathbf{R}^n$ ($0 \in T_0\mathbf{R}^n$) we fix the following bases:

$$X_i = \mathrm{d}I^{-1}(\overline{\partial}_i|(0, e_n)), \quad V_i = \mathrm{d}I^{-1}(\overline{\partial}_{i+n}|(0, e_n)) \tag{4.2}$$

and

$$X_i^0 = \mathrm{d}I^{-1}(\overline{\partial}_i|0), \quad V_i^0 = \mathrm{d}I^{-1}(\overline{\partial}_{i+n}|0), \tag{4.3}$$

$i = 1, \ldots, n$. We have the following transformation rules.

Lemma 4.1. *Let $\varphi = (\varphi^1, \ldots, \varphi^n)\colon \mathbf{R}^n \to \mathbf{R}^n$ be a diffeomorphism such that $\mathrm{d}\varphi(\partial) = \partial$. Then for any $i = 1, \ldots, n$ we have*

$$\mathrm{d}(\mathrm{d}\varphi)(V_i) = \partial_i \varphi^j(0) V_j \tag{4.4}$$

and

$$\mathrm{d}(\mathrm{d}\varphi)(X_i) = \partial_i(\partial_n \varphi^j)(0) V_j + \partial_i \varphi^j(0) X_j, \tag{4.5}$$

$j = 1, \ldots, n$ (we use Einstein summation convention). Similarly, if $\psi\colon \mathbf{R}^n \to \mathbf{R}^n$ is a linear isomorphism, then for any $i = 1, \ldots, n$ we have

$$\mathrm{d}(\mathrm{d}\psi)(V_i^0) = \partial_i \psi^j(0) V_j^0, \qquad j = 1, \ldots, n. \tag{4.6}$$

Proof. We prove only formula (4.5). It is obvious that

$$\begin{aligned}
\mathrm{d}I \circ \mathrm{d}(\mathrm{d}\varphi)(X_i) &= \mathrm{d}I \circ \mathrm{d}(\mathrm{d}\varphi)\mathrm{d}I^{-1}\left(\frac{\mathrm{d}}{\mathrm{d}t}(te_i, e_n)_{t=0}\right) \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\left(I \circ \mathrm{d}\varphi\left(\frac{\mathrm{d}}{\mathrm{d}\tau}(te_i + \tau e_n)_{\tau=0}\right)\right)_{t=0} \\
&= \frac{\mathrm{d}}{\mathrm{d}t}\big(\varphi(te_i), \partial_n \varphi(te_i)\big)_{t=0} \\
&= \partial_i \varphi^j(0)\overline{\partial}_j|(0, e_n) + \partial_i \partial_n \varphi^j(0)\overline{\partial}_{j+n}|(0, e_n).
\end{aligned}$$

This implies formula (4.5) immediately. We get (4.4) and (4.6) similarly. □

5. The admissible subspaces given by L_1 and L_2

Let L_1 and L_2 be as in Example 0.1 and let X_i, V_i, X_i^0, V_i^0 be as in Section 4. We see that

$$\mathrm{d}\pi(V_i) = \mathrm{d}\pi(V_i^0) = 0 \quad \text{and} \quad \mathrm{d}\pi(X_i) = \mathrm{d}\pi(X_i^0) = \partial_i|0 \tag{5.1}$$

for $i = 1, \ldots, n$ and

$$\mathrm{d}(\mathrm{d}f)(V_i) = \mathrm{d}(\mathrm{d}f)(V_i^0) = \mathrm{d}(\mathrm{d}f)(X_i) = \mathrm{d}(\mathrm{d}f)(X_i^0) = 0$$

for $i = 1, \ldots, p$, where $f\colon \mathbf{R}^n \to \mathbf{R}^{n-p}$, $f(x) = (x^{p+1}, \ldots, x^n)$. Since F^p is given by the cocycle $(\mathbf{R}^n, f, id)$, $L_1(F^p)$ is given by the cocycle $(T\mathbf{R}^n, \mathrm{d}f, id)$, and then

$$L_1(F^p)_z = (d_z \mathrm{d}f)^{-1}(0)$$

for any $z \in T_0\mathbf{R}^n$. Obviously

$$L_2(F^p)_z = (d_z \pi)^{-1}(F^p_0)$$

for any $z \in T_0\mathbf{R}^n$. The above facts complete the proof of the following formulas:

(5.2) $$L_1(F^p)_\partial = \operatorname{span}(V_1, \ldots, V_p, X_1, \ldots, X_p),$$

(5.3) $$L_1(F^p)_0 = \operatorname{span}(V_1^0, \ldots, V_p^0, X_1^0, \ldots, X_p^0),$$

(5.4) $$L_2(F^p)_\partial = \operatorname{span}(V_1, \ldots, V_n, X_1, \ldots, X_p) \quad \text{and}$$

(5.5) $$L_2(F^p)_0 = \operatorname{span}(V_1^0, \ldots, V_n^0, X_1^0, \ldots, X_p^0)$$

6. Proof of the main theorem

It is sufficient to prove Proposition 3.1. By formulas (5.1) it follows that

(6.1) $$W \subset (d_\partial \pi)^{-1}(F^p_0) = \operatorname{span}(V_1, \ldots, V_n, X_1, \ldots, X_p)$$

for any ∂-admissible subspace W. Similarly, for any 0-admissible subspace W we have

(6.2) $$W \subset \operatorname{span}(V_1^0 \ldots, V_n^0, X_1^0, \ldots, X_p^0)$$

First we prove the second part of Proposition 3.1. Let W be a 0-admissible subspace, such that $W \supsetneq L_1(F^p)_0$. Then formulas (6.2) and (5.3) imply that there exists a vector $Y \in W \setminus \{0\}$ of the form

$$Y = a^{p+1} V_{p+1}^0 + \ldots + a^n V_n^0.$$

Let us consider a number $k \in \{p+1, \ldots, n\}$. There exists a linear isomorphism ψ: $\mathbf{R}^n \to \mathbf{R}^n$ such that $\psi(e_i) = e_i$ for $i = 1, \ldots, p$ and

$$\psi(e_k) = a^{p+1} e_{p+1} + \ldots + a^n e_n.$$

Then ψ^{-1} is 0-admissible and $\mathrm{d}(\mathrm{d}(\psi^{-1}))(Y) = V_k^0$ because of formula (4.6). Since W is 0-admissible and $Y \in W$, we have $V_k^0 \in W$. Hence $W = L_2(F^p)_0$.

It remains to prove the first part of Proposition 3.1. Let W be a ∂-admissible subspace. Then $d\pi(W) = F_0^p$. Therefore formulas (5.1) imply that for any $j \in \{1, \ldots, p\}$ there exist $Z_j \in \text{span}(V_1, \ldots, V_p)$ and $Y_j \in \text{span}(V_{p+1}, \ldots, V_n)$ such that

$$Z_j + Y_j + X_j \in W. \tag{6.3}$$

Hence the first part of Proposition 3.1 is a consequence of the following inclusions:

$$\text{span}(V_1, \ldots, V_p) \subset W \quad \text{and} \tag{6.4}$$

$$W \subset \text{span}(V_1, \ldots, V_p, X_1, \ldots, X_p) \cap W \oplus \text{span}(V_{p+1}, \ldots, V_n) \cap W. \tag{6.5}$$

(In fact, formulas (6.5) and (6.3) yield that $X_j + Z_j \in W$, and then $X_j \in W$ for $j = 1, \ldots, p$ because of formula (6.4). Therefore $L_1(F^p)_\partial \subset W$ as follows from formulas (6.4) and (5.2).) First we prove inclusion (6.5). Let $\varphi \colon \mathbf{R}^n \to \mathbf{R}^n$ be the diffeomorphism given by $\varphi(y^1, \ldots, y^n) = (2y^1, \ldots, 2y^p, y^{p+1}, \ldots, y^n)$. Then φ is ∂-admissible. Consider an arbitrary $Y \in W$. Inclusion (6.1) implies that $Y = Y^1 + Y^2$, where $Y^1 \in \text{span}(V_{p+1}, \ldots, V_n)$ and $Y^2 \in \text{span}(V_1, \ldots, V_p, X_1, \ldots, X_p)$ are vectors. Using Lemma 4.1 we see that $d(d\varphi)(Y) = Y^1 + 2Y^2$. Since $Y \in W$ and W is ∂-admissible we have $Y^1 + 2Y^2 \in W$ and then Y^1, $Y^2 \in W$. Inclusion (6.5) is proved.

Now, we prove inclusion (6.4). Consider a number $k \in \{1, \ldots, p\}$. Let $\Phi \colon \mathbf{R}^n \to \mathbf{R}^n$ be given by

$$\Phi(y^1, \ldots, y^n) = (y^1, \ldots, y^{k-1}, y^k + \tfrac{1}{2} y^k \sin(y^n), y^{k+1}, \ldots y^n).$$

Then Φ is a global diffeomorphism. Evidently Φ is ∂-admissible. Let $X = Z_k + Y_k + X_k \in W$ be as in formula (6.3). It follows from Lemma 4.1 that $d(d\Phi)(X) = X + \frac{1}{2} V_k$. (For $d(d\Phi)(V_i) = V_i$, $i = 1, \ldots, n$, i.e. $d(d\Phi)(Z_k + Y_k) = Z_k + Y_k$, and $d(d\Phi)(X_k) = X_k + \frac{1}{2} V_k$.) Since W is ∂-admissible and $X \in W$, we get that $V_k \in W$. Inclusion (6.4) is proved.

Theorem 0.1 is proved. □

Acknowledgement. I would like to thank R. Wolak for corrections.

References

[1] *I. Kolář, P. Michor, J. Slovák*: Natural operations in differential geometry, in printing.
[2] *R. Wolak*: Geometric structures on foliated manifolds, Universidad de Santiago de Compostella *76* (1989).

Author's address: Institute of Mathematics Jagellonian University Kraków, Reymonta 4, Poland.