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INTEGRATION OF SOME VERY ELEMENTARY FUNCTIONS

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Summary. Let m be a natural number. Let f, g, and Q be real polynomials such that $\{\deg f, \deg g\} \subset \{1,2\}$, $\deg Q < m \deg f$, g is not a square and f has imaginary roots, if it is not linear. Effective methods for the integration of $Q/(f^m\sqrt{g})$ are exhibited.

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0. INTRODUCTION

This note deals with the integration of functions $S(x, \sqrt{g(x)})$, where S is a rational function of two variables and g is a polynomial of degree one or two. It is well-known that there is always a simple substitution that converts such an integration to the integration of a rational function. It is also well-known that this procedure sometimes leads to complications and in some textbooks (see, e.g., [1]) we can find various more effective methods. I believe, however, that there still is ample room for improvement. The purpose of this note is to find explicit formulas for primitives of functions $P/\sqrt{g}$ and $Q/(f^m\sqrt{g})$, where f, g, P, Q are real polynomials, m is a natural number, $\{\deg f, \deg g\} \subset \{1,2\}$, g is not a square, $\deg Q < m \deg f$ and f has imaginary roots, if it is not linear. Some of the results hold even for an f with two real roots. The corresponding procedures do not involve any imaginary numbers.

The letter $\mathbf{R}$ denotes the real line. Our polynomials and rational functions have, by definition, real coefficients, unless something else is obvious from the context. Now suppose that a function $S(x, \sqrt{g(x)})$ is given, where S and g have the properties mentioned above. Let $G = \{x \in \mathbf{R};\ g(x) > 0\}$ and let A be the set of all points $x \in G$ for which $S(x, \sqrt{g(x)})$ is meaningful. It is easy to see that there is a finite set B such that $S(x, \sqrt{g(x)})$ can be expressed on $A \setminus B$ as $S_1(x) + S_2(x)/\sqrt{g(x)}$, where S_1 and S_2 are rational functions. If we decompose S_2 in the usual way, then our integration is reduced to the integration of S_1 and of the previously mentioned functions $P/\sqrt{g}$ and $Q/(f^m\sqrt{g})$.

If g is a multiple of f, then $Q/(f^m\sqrt{g})$ has an algebraic primitive; the same holds, obviously, for $P/\sqrt{g}$, if g is linear. We will investigate these cases in Section 1. Sections 2 and 3 deal with functions $P/\sqrt{g}$, where $\deg g = 2$, and $Q/(f^m\sqrt{g})$, where f is any polynomial of degree 1 or 2 relatively prime to g. (We always suppose that g is not a square.) If P is constant, then $P/\sqrt{g}$ has a "purely transcendental" primitive; the same is true about $Q/(f\sqrt{g})$, unless f is a square. This will be shown in Section 2. In Section 3 the general case will be "reduced" to these special cases.

1. Algebraic integration

In this section, we will find primitives of functions $P/\sqrt{g}$, where g is linear, and $Q/(f^m\sqrt{g})$, where g is divisible by f. Our primitives will be algebraic functions.

The results in 1.6–1.8 are trivial and are listed only for the sake of completeness. I believe, however, that theorems 1.3 and 1.4 give a useful simplification of what can be found in textbooks.

1.1 Notation. For each polynomial p let $D_p = (p')^2 - 2pp''$.

1.2 Remark. We will often apply without explanation the following assertions whose proof is left to the reader:

If $a, b, c \in \mathbf{R}$ and $p(x) = ax^2 + bx + c$ $(x \in \mathbf{R})$, then $D_p = b^2 - 4ac$.

If n is a natural number and $\deg p \vee \deg q \leqslant n$, then $\deg(p'q - pq') \leqslant 2n - 2$. (We will need this only for $n \leqslant 2$.)

Let p, q be polynomials. Then $p'q = pq'$ if and only if p, q are linearly dependent.

1.3 Theorem. *Let $\deg h \leqslant \deg f = 1$. Suppose that f, h are relatively prime. Set $M = h'f - hf'$, $g = fh$, $G = \{x \in \mathbf{R};\, g(x) > 0\}$. Let j be a natural number and let Z be a polynomial such that, setting $F(t) = tZ(t^2)$, we have $F'(t) = (h' - f't^2)^{j-1}$ $(t \in \mathbf{R})$. (It is easy to see that such a Z exists.) Then M is a nonzero number and*

$$\frac{1}{f^j\sqrt{g}} = \left(\frac{2\sqrt{g}}{M^j f} Z\left(\frac{h}{f}\right)\right)' \quad \text{on } G. \tag{1}$$

Proof. Set $Z_1 = (\operatorname{sgn} f)(\sqrt{g}/f)Z(h/f)$. Clearly $Z_1 = \sqrt{h/f}Z(h/f) = F(\sqrt{h/f})$ so that

$$Z_1' = \left(h' - f'\frac{h}{f}\right)^{j-1} \cdot \frac{1}{2}\sqrt{\frac{f}{h}} \cdot \frac{M}{f^2}$$
$$= \frac{M^{j-1}}{f^{j-1}} \cdot \frac{1}{2}\sqrt{\frac{f}{h}} \cdot \frac{M \operatorname{sgn} f}{|f|f} = \frac{M^j}{2f^j} \cdot \frac{\operatorname{sgn} f}{\sqrt{g}}.$$

This easily implies (1). □

1.4 Theorem. *Let* $\deg f = 2$, $D_f \neq 0$. *Set* $G = \{x \in \mathbf{R};\ f(x) > 0\}$. *Let* j *be a natural number and let* F *be a polynomial such that* $F'(t) = (t^2 - 2f'')^{j-1}$ $(t \in \mathbf{R})$. *Then*

$$\frac{1}{f^{j+\frac{1}{2}}} = \left(\frac{-2}{D_f^j}F\left(\frac{f'}{\sqrt{f}}\right)\right)' \quad \text{on } G. \tag{2}$$

Proof. We have $(F(\frac{f'}{\sqrt{f}}))' = (\frac{f'^2}{f} - 2f'')^{j-1} \cdot \frac{f''f - \frac{1}{2}f'^2}{f\sqrt{f}} = \frac{D_f^{j-1}}{2f^{j-1}} \cdot \frac{-D_f}{f\sqrt{f}}$ from which (2) follows at once. □

1.5 Remark. Now we are able to find primitives of functions mentioned at the beginning of this section.

1.6 Functions $P/\sqrt{g}$, where $\deg P = n$ and $\deg g = 1$.

Let $G = \{x \in \mathbf{R};\ g(x) > 0\}$. Find numbers $a_0, \dots a_n$ such that $P = \sum_{j=0}^{n} a_j g^j$. Then

$$\frac{P}{\sqrt{g}} = \left(\frac{\sqrt{g}}{g'}\sum_{j=0}^{n}\frac{2a_j g^j}{2j+1}\right)' \quad \text{on } G. \tag{3}$$

1.7 Functions $Q/(f^m\sqrt{g})$, where m is a natural number, $\deg f = 1$, $\deg g \leqslant 2$, $D_g \neq 0$, g is divisible by f and $\deg Q < m$.

Let $H = \{x \in \mathbf{R};\ (f(x))^2 g(x) > 0\}$. Find numbers $b_1, \dots, b_m$ such that $Q = \sum_{j=1}^{m} b_j f^{m-j}$. There is an h such that $\deg h \leqslant 1$ and $g = fh$. Using 1.3 find functions T_j with $T_j' = 1/(f^j\sqrt{g})$. Then

$$Q/(f^m\sqrt{g}) = \left(\sum_{j=1}^{m} b_j T_j\right)' \quad \text{on } H. \tag{4}$$

1.8 Functions $Q/(f^m\sqrt{g})$, where m is a natural number, $\deg f = \deg g = 2$, g is divisible by f and $\deg Q < 2m$.

Let $G = \{x \in \mathbf{R};\ g(x) > 0\}$. We may suppose that $f > 0$ on G. Find numbers α_j, β_j such that $Q/(f^m\sqrt{g}) = \sum_{j=1}^{m}(\alpha_j f' + \beta_j)/f^{j+\frac{1}{2}}$. Using 1.4 find functions V_j such that $V_j' = 1/f^{j+\frac{1}{2}}$. Then

$$\frac{Q}{f^m\sqrt{g}} = \Big(\sum_{j=1}^{m}\Big(\beta_j V_j - \frac{2\alpha_j\sqrt{f}}{(2j-1)f^j}\Big)\Big)' \quad \text{on } G. \tag{5}$$

The easy proofs of (3), (4) and (5) are left to the reader.

Remark. If $D_f < 0$ and if f, g have a common root, then g is divisible by f and we may apply 1.8.

1.9 Example. Find a primitive of $1/((x+1)^3\sqrt{1-x^2})$ ($|x| < 1$). We apply 1.3 with $f(x) = x+1$, $h(x) = 1-x$, $j = 3$. Then $M = -2$, $(h' - f't^2)^2 = t^4 + 2t^2 + 1$ so that $Z(t) = \frac{1}{5}t^2 + \frac{2}{3}t + 1 = (3t^2 + 10t + 15)/15$ and, by (1), we have the primitive

$$\begin{aligned}\frac{2}{(-2)^3}\cdot\frac{1}{15}\cdot\frac{\sqrt{1-x^2}}{x+1}&\Big(3\Big(\frac{x-1}{x+1}\Big)^2 + 10\frac{1-x}{x+1} + 15\Big)\\ &= -\frac{1}{15}\cdot\frac{\sqrt{1-x^2}}{(x+1)^3}(2x^2+6x+7) \qquad (|x| < 1).\end{aligned}$$

1.10 Example. Find a primitive of $x/(x^2+x+1)^{5/2}(x \in \mathbf{R})$. Set $f(x) = x^2+x+1, v(x) = x/(f(x))^{5/2}$. Clearly $v = \frac{1}{2}(f'f^{-5/2} - f^{-5/2})$. To integrate $f^{-5/2}$ we apply 1.4 with $j = 2$. Since $f'' = 2$ and $D_f = -3$, we may choose $F(t) = \frac{1}{3}t^3 - 4t$ and, by (2), the function

$$V(x) = -\frac{2}{9}\cdot\frac{1}{3}\cdot\frac{2x+1}{\sqrt{f(x)}}\Big(\frac{(2x+1)^2}{f(x)} - 12\Big)$$

fulfills $V' = f^{-5/2}$. Thus we get the primitive $\frac{1}{2}(-\frac{2}{3}f^{-3/2} - V)$ which means

$$-\frac{9+(2x+1)(8x^2+8x+11)}{27(f(x))^{3/2}} = -\frac{2}{27}\cdot\frac{8x^3+12x^2+15x+10}{(x^2+x+1)^{3/2}} \qquad (x \in \mathbf{R}).$$

2. Transcendental integration

In this section we find primitives of functions $1/\sqrt{g}$, where $\deg g = 2$ and $D_g \neq 0$, and of functions $Q/(f\sqrt{g})$, where f, g and Q fulfill the usual assumptions and f, g are relatively prime. The first case is simple; one of the functions $\sqrt{2/g''}\ln|g' + \sqrt{2g''g}|$, $\sqrt{-2/g''}\arcsin\left(-g'/\sqrt{D_g}\right)$ is a primitive of $1/\sqrt{g}$ on the set where $g > 0$. The second case is much more complicated. To see it let us assume that $\deg f = 2$ and $D_f < 0$. If we try to integrate $Q/(f\sqrt{g})$ using some of the substitutions that help us to get a rational function of the new variable, then in place of f we get a polynomial of degree 4 that should be decomposed. For this purpose we need first a decomposition of f into (imaginary) linear factors f_1, f_2 and then a decomposition of the (imaginary) quadratic polynomials obtained in place of f_1 and f_2. Even if we have the patience to go through all this trouble, we may get an "unpractical" primitive. (There is, of course, no uniqueness in the representation of a function by, for example, inverse trigonometric functions—not to speak about the "integration constants.") At this occasion it is, perhaps, worth mentioning that we encounter such difficulties even if $\deg g = 1$. Let us try, for example, to integrate the function $(x+1)/\left((x^2 - x + 1)\sqrt{x}\right)$ $(x > 0)$. The substitution $x = t^2$ leads to the integration of $(2t^2 + 2)/(t^4 - t^2 + 1)$. After we decompose the denominator, find partial fractions, integrate them and substitute $t = \sqrt{x}$, we get the primitive

$$2\left(\arctan\left(2\sqrt{x} + \sqrt{3}\right) + \arctan\left(2\sqrt{x} - \sqrt{3}\right)\right).$$

However, it is easy to see that the function $2\arctan((x-1)/\sqrt{x})$ is also a primitive. The reader can verify that we get this simple function, if we apply 2.11 and (12).

The main results of this section are Theorems 2.11 and 2.14. In 2.14 we get a primitive of $1/(f\sqrt{g})$, if $\deg f = 1$. (The reason to write there ℓ instead of f is apparent from the proof of 2.13.) Formula (16) in 2.11 gives a primitive of $Q/(f\sqrt{g})$, if $D_f < 0$. It should be noted, however, that (16) is applicable (and useful) also in other cases. To see it suppose that $1 \leqslant \deg g \leqslant \deg f = 2$, $\deg Q \leqslant 1$, $D_g \neq 0$ and that f and g are relatively prime. Before applying (16) we need a decomposition of the polynomial $W = f'g - fg'$ (whose degree is at most 2) into linear factors; thus we must have $D_W > 0$. (Under our assumptions $D_W \neq 0$.) Theorem 2.5 gives a simple expression for D_W. We see that $D_W > 0$, if $D_f \leqslant 0$; if $D_f > 0$, then $D_W > 0$ if and only if g has an even number of roots between the roots of f.

If $D_f > 0$, we may, of course, decompose Q/f into partial fractions and apply (twice) 2.14. It may happen, however, that f and g have integer coefficients, f has irrational roots while W has rational roots. Then we may decompose W into linear factors with integer coefficients and it is easier to apply 2.11 than to decompose Q/f and apply 2.14.

If $D_f = 0$, then the decomposition of W is somehow "automatic"; this case is covered by 2.13.

Theorem 2.14 is, of course, applicable also in the case when $\deg f = \deg g = 1$. We get, however, a simpler result, if we apply the following assertion whose easy proof is left to the reader:

Let $\deg f \leqslant \deg g = 1$. Set $H = \{x \in \mathbf{R};\ (f(x))^2 g(x) > 0\}$, $W = f'g - fg'$. (Then $W \in \mathbf{R}$.) Let F be a function such that $F'(t) = 2/(f't^2 - W)$ for each t for which $f't^2 \neq W$. Then $1/(f\sqrt{g}) = \left(F(\sqrt{g})\right)'$ on H.

Before we go to the proofs of 2.11 and 2.14 we need some new notation and several auxiliary assertions.

2.1 Convention, notation. Throughout this section, f and g are linearly independent polynomials with $\deg f \vee \deg g \leqslant 2$. We set $W = f'g - fg'$. For any polynomials p, q let $K_{p,q} = 2p'q - pq'$.

Remark. The reader might have expected the requirement that f, g be relatively prime; this, of course, will be needed in 2.11 and 2.14. It is easy to see that relatively prime polynomials p, q with $\deg p \vee \deg q > 0$ are linearly independent. It may interest the reader that some of our results are valid even for f, g with a common root. However, our results are based on the decomposition of W into linear factors and for f, g linearly dependent we cannot get in this section anything "reasonable".

2.2 Lemma. *Let $\deg p \leqslant 1$, $\deg q \leqslant 2$. Let $s = K_{p,q}$. Then $\deg s \leqslant 1$ and $K_{s,q} = pD_q$. If α, β, a, b, c are numbers such that $p(x) = \alpha x + \beta$, $q(x) = ax^2 + bx + c$, then*

$$s(x) = \begin{vmatrix} \alpha, & \beta \\ 2a, & b \end{vmatrix} x + \begin{vmatrix} \alpha, & \beta \\ b, & 2c \end{vmatrix} \qquad (x \in \mathbf{R}). \tag{6}$$

Proof. We have $s' = p'q' - pq''$, $s'' = p'q'' - p'q'' = 0$ whence $\deg s \leqslant 1$. Further $K_{s,q} = 2(p'q' - pq'')q - (2p'q - pq')q' = pD_q$. The proof of (6) is left to the reader. □

2.3 Lemma. *We have*

$$W' = f''g - fg'', \qquad W'' = f''g' - f'g'' \tag{7}$$

and $\deg W \leqslant 2$. We have $\deg W = 0$ if and only if $\deg f \vee \deg g \leqslant 1$. If A_0, B_0, C_0, A, B, C are numbers such that $f(x) = A_0x^2 + B_0x + C_0$, $g(x) = Ax^2 + Bx + C$, then

$$W(x) = \begin{vmatrix} 1, & -2x, & x^2 \\ A_0, & B_0, & C_0 \\ A, & B, & C \end{vmatrix} \qquad (x \in \mathbf{R}). \tag{8}$$

Proof. It is obvious that (7) holds and that $\deg W \leqslant 2$. Since f, g are linearly independent, we have $W' = 0$ if and only if $f'' = g'' = 0$. The proof of (8) is left to the reader. □

2.4 Lemma. *Let $y \in \{f, g\}$. Then*

$$yW'' - y'W' + y''W = 0. \tag{9}$$

Proof. It follows from (7) that the left-hand side in (9) is the determinant with rows (y, y', y''), (g, g', g''), (f, f', f''). □

2.5 Theorem. *Let α, β be complex numbers. Let $\gamma \in \mathbf{R}$, $f(x) = \gamma(x-\alpha)(x-\beta)$ $(x \in \mathbf{R})$. Then $D_W = (f'')^2 g(\alpha)g(\beta)$.*

Proof. We may suppose that $\gamma = 1$ (so that $f'' = 2$). Let $g(x) = Ax^2 + Bx + C$. Setting $A_0 = 1$, $B_0 = -\alpha - \beta$, $C_0 = \alpha\beta$ in (8) we get

$$W(x) = x^2(B + A(\alpha+\beta)) + 2x(C - A\alpha\beta) - (C(\alpha+\beta) + B\alpha\beta).$$

It is easy to verify that

$$(C - A\alpha\beta)^2 + (B + A(\alpha+\beta))(C(\alpha+\beta) + B\alpha\beta) = g(\alpha)g(\beta).$$

□

2.6 Lemma. *We have $D_W = 0$ if and only if f, g are both linear or have a common root. If f, α, β are as in 2.5, then $D_W \geqslant 0$ if and only if $g(\alpha)g(\beta) \geqslant 0$. If, in particular, $D_f < 0$ or $D_g < 0$, then $D_W > 0$.*

(This follows easily from 2.5.)

2.7 Convention, notation. For the remainder of this section we suppose that $D_W \geqslant 0$. Further we suppose that φ, ψ are linear functions such that $W = \varphi\psi$.

Remark. The existence of φ, ψ follows from the assumption that $D_W \geqslant 0$. Since f, g are linearly independent, neither φ nor ψ is identically zero.

2.8 Lemma. *Let $y \in \{f, g\}$. Then there is an $\alpha \in \mathbf{R}$ such that $K_{\varphi,y} = \alpha\psi$.*

Proof. Set $K = K_{\varphi,y}$. Then $K\psi' - K'\psi = (2\varphi'y - \varphi y')\psi' - (\varphi'y' - \varphi y'')\psi = y2\varphi'\psi' - y'(\varphi'\psi + \varphi\psi') + y''\varphi\psi = yW'' - y'W' + y''W = 0$ by (9). It follows that K, ψ are linearly dependent which proves our assertion. □

2.9 Proposition. *Let a, b, c, d be numbers such that $K_{\varphi,g} = a\psi$, $K_{\varphi,f} = b\psi$, $K_{\psi,g} = c\varphi$, $K_{\psi,f} = d\varphi$ (such numbers exist by 2.8). Then*

$$\varphi^2 = af - bg, \qquad \psi^2 = cf - dg, \tag{10}$$

$$ac = D_g, \qquad bd = D_f. \tag{11}$$

Proof. Clearly $a\psi f - b\psi g = K_{\varphi,g}f - K_{\varphi,f}g = (2\varphi' g - \varphi g')f - (2\varphi' f - \varphi f')g = \varphi W = \varphi^2\psi$ whence $af - bg = \varphi^2$. Setting $p = \varphi$, $q = g$ in 2.2 we have $s = a\psi$, $ac\varphi = aK_{\psi,g} = K_{s,g} = \varphi D_g$ whence $ac = D_g$. The rest can be proved similarly. □

2.10 Theorem. *Let $a, b \in \mathbf{R}$, $a \neq 0$, $\varphi^2 = af - bg$. Let $H = \{x \in \mathbf{R};\ (f(x))^2 g(x) > 0\}$. Let F be a function such that $F'(t) = 2/(t^2 + b)$ for each $t \in \mathbf{R}$ for which $t^2 + b \neq 0$. Then*

$$\left(F(\varphi/\sqrt{g})\right)' = \psi/(f\sqrt{g}) \quad \text{on } H.$$

Proof. Since f, g are linearly independent, the numbers a, b are uniquely determined by $\varphi^2 = af - bg$. By 2.9 we have $a\psi = K_{\varphi,g}$. It is easy to see that $F(\varphi/\sqrt{g})$ is differentiable on H and that its derivative is

$$\frac{2}{\varphi^2/g + b} \cdot \frac{\varphi' g - \frac{1}{2}\varphi g'}{g\sqrt{g}} = \frac{K_{\varphi,g}}{(\varphi^2 + bg)\sqrt{g}} = \frac{a\psi}{af\sqrt{g}} = \frac{\psi}{f\sqrt{g}}.$$

□

Remark. If $b > 0$, we may choose $F(t) = (2/\sqrt{b})\arctan(t/\sqrt{b})$ $(t \in \mathbf{R})$. Then

$$F(\varphi/\sqrt{g}) = (2/\sqrt{b})\arctan(\varphi/\sqrt{bg}). \tag{12}$$

Since $\arctan x = \arcsin\frac{x}{\sqrt{1+x^2}}$ for each $x \in \mathbf{R}$, we have also

$$F(\varphi/\sqrt{g}) = (2/\sqrt{b})\arcsin(\varphi/\sqrt{af}). \tag{13}$$

Now let $b < 0$. If we choose $F(t) = (1/\sqrt{|b|})\ln\left(|t - \sqrt{|b|}|/|t + \sqrt{|b|}|\right)$ $(t^2 \neq b)$, we have

$$F(\varphi/\sqrt{g}) = (1/\sqrt{|b|})\ln\left(|\varphi - \sqrt{|b|g}|/|\varphi + \sqrt{|b|g}|\right). \tag{14}$$

Choosing $F(t) = (1/\sqrt{|b|})\ln\left(|a|(t - \sqrt{|b|})^2/|t^2 + b|\right)$, we get

$$\begin{aligned} F(\varphi/\sqrt{g}) &= (1/\sqrt{|b|})\ln\left((\varphi - \sqrt{|b|g})^2/|f|\right) \\ &= (2/\sqrt{|b|})\ln\left(|\varphi - \sqrt{|b|g}|/\sqrt{|f|}\right). \end{aligned} \tag{15}$$

2.11 Theorem. *Let* $\deg g \leqslant \deg f = 2$, $D_g \neq 0$, $\deg Q \leqslant 1$. *Let* f, g *be relatively prime. Let* φ, ψ *be as in 2.7. Let* b *and* d *be numbers such that* $K_{\varphi,f} = b\psi$, $K_{\psi,f} = d\varphi$. *Then* φ, ψ *are linearly independent. For* $z \in \{b, d\}$ *find functions* F_z *such that* $F_z'(t) = 2/(t^2 + z)$ *for each* t *for which* $t^2 + z \neq 0$. *Find numbers* α, β *such that* $\alpha\psi + \beta\varphi = Q$ *and set* $H = \{x \in \mathbf{R};\, (f(x))^2 g(x) > 0\}$, $V_\varphi = F_b\bigl(\varphi/\sqrt{g}\bigr)$, $V_\psi = F_d\bigl(\psi/\sqrt{g}\bigr)$. *Then*

$$Q/\bigl(f\sqrt{g}\bigr) = (\alpha V_\varphi + \beta V_\psi)' \quad \text{on } H. \tag{16}$$

Proof. There are numbers a, c such that (10) holds; by (11), $ac \neq 0$. Since $\deg f = 2$, we have $\deg W \geqslant 1$. If φ, ψ were linearly dependent, then W would have a double root and, by 2.6, f and g would have a common root against our assumption. By 2.10 we have $V_\varphi' = \psi/\bigl(f\sqrt{g}\bigr)$; similarly $V_\psi' = \varphi/\bigl(f\sqrt{g}\bigr)$ whence (16) follows at once. □

Remark. If $D_f \neq 0$, we may find b and d as follows: Choose φ and ψ in such a way that $\psi' \neq 0$. (One of the functions φ, ψ may be constant, but not both.) Then determine b by $K_{\varphi,f}' = b\psi'$ and d by $bd = D_f$. If $\varphi(x) = \gamma x + \delta$ and $f(x) = Ax^2 + Bx + C$, then, by (6),

$$b\psi' = \begin{vmatrix} \gamma, & \delta \\ 2A, & B \end{vmatrix}. \tag{17}$$

If $D_f = 0$, we apply 2.13.

2.12 Remark 1. The reader may be interested in constructing examples of functions $Q/\bigl(f\sqrt{g}\bigr)$ (as in 2.11) whose integration "runs smoothly". In particular, we would like to have f and g with integer coefficients for which W has rational roots. It is easy to see that it would be somehow unpractical to start with f and g. We start rather with relatively prime f, φ with integer coefficients and (to get something interesting) $D_f \neq 0$. Then we find integers a, b such that the polynomial $g = (af - \varphi^2)/b$ has integer coefficients and fulfills $D_g \neq 0$. For such f, g we have $W = \varphi\psi$ with $\psi = K_{\varphi,f}/b$.

Remark 2. The reader may compare 2.11 with the procedure described in [1] on pp. 74–76. On p. 28 in [2] this procedure is called the standard method.

2.13 Theorem. *Let* $\deg Q \leqslant \deg \ell = 1$. *Let* ℓ, g *be relatively prime and let* $D_g \neq 0$. *Let* $\varrho \in \mathbf{R}$, $\ell(\varrho) = 0$. *Set* $H = \{x \in \mathbf{R};\, (\ell(x))^2 g(x) > 0\}$, $z = -4(\ell')^2 g(\varrho)$, $\varphi = K_{\ell,g}$. *Then* φ, ℓ *are linearly independent. Find numbers* α, β *such that* $\alpha\ell + \beta\varphi =$

Q. Find a function F such that $F'(t) = 2/(t^2+z)$ for each $t \in \mathbf{R}$ for which $t^2+z \neq 0$. Then

$$Q/(\ell^2\sqrt{g}) = \bigl(\alpha F(\varphi/\sqrt{g}) - 2\beta\sqrt{g}/\ell\bigr)' \quad \text{on } H. \tag{18}$$

Proof. Set $f = \ell^2$, $\psi = \ell$. It is easy to see that $W = \varphi\psi$. Let a, b, c, d be as in (10). Clearly $c = 1$, $d = 0$. Since $\varphi(\varrho) = 2\ell' g(\varrho)$, we have $4(\ell' g(\varrho))^2 = -bg(\varrho)$ so that $b = z$. Now we apply 2.11 with $F_b = F$ and $F_d(t) = -2/t$. □

Remark. Since $c = 1$, we have $a = D_g$. Using (13) and (15) we see that in (18) we may take for $F(\varphi/\sqrt{g})$ one of the functions

$$\frac{1}{|\ell'|\sqrt{-g(\varrho)}} \arcsin \frac{K_{\ell,g}}{|\ell|\sqrt{D_g}}, \tag{19}$$

$$\frac{1}{|\ell'|\sqrt{g(\varrho)}} \ln \frac{\bigl|K_{\ell,g} - 2|\ell'|\sqrt{g(\varrho)g}\bigr|}{|\ell|}, \tag{20}$$

where $\ell(\varrho) = 0$. (If $H \neq \emptyset$ and $g(\varrho) < 0$, then, clearly, $D_g > 0$.)

2.14 Theorem. *Let ℓ, g, H, φ, F be as in 2.13. Then*

$$1/(\ell\sqrt{g}) = \bigl(F(\varphi/\sqrt{g})\bigr)' \quad \text{on } H. \tag{21}$$

Proof. Taking $Q = \ell$ in 2.13 we get $\alpha = 1$, $\beta = 0$ and (18) becomes (21). □

2.15 Example. Find a primitive of

$$\frac{x}{(x^2-3x+3)\sqrt{2x^2-x+4}} \qquad (x \in \mathbf{R}).$$

By (8) (the meaning of f and g being obvious)

$$W(x) = \begin{vmatrix} 1, & -2x, & x^2 \\ 1, & -3, & 3 \\ 2, & -1, & 4 \end{vmatrix} = 5x^2 - 4x - 9 = \varphi(x)\psi(x),$$

where $\varphi(x) = 5x - 9$, $\psi(x) = x + 1$. By (17) and (11) we have $b = \begin{vmatrix} 5, & -9 \\ 2, & -3 \end{vmatrix} = 3$, $d = D_f/b = -1$. By (12) and (15) we may choose $V_\varphi = (2/\sqrt{3}) \arctan\bigl(\varphi/\sqrt{3g}\bigr)$, $V_\psi = \ln\bigl((\psi - \sqrt{g})^2/f\bigr)$. Since $x = \frac{1}{14}(9\psi(x) + \varphi(x))$, we have the primitive $\frac{1}{14}(9V_\varphi + V_\psi)$ which means

$$\frac{1}{14}\Bigl(6\sqrt{3} \arctan \frac{5x-9}{\sqrt{3(2x^2-x+4)}} + \ln \frac{(x+1-\sqrt{2x^2-x+4})^2}{x^2-3x+3}\Bigr) \qquad (x \in \mathbf{R}).$$

2.16 Example. Find a primitive of $1/\big((x^2-5x+1)\sqrt{-x^2+4x-1}\big)$. (The meaning of f and g is again obvious.) Set $G=\{x\in\mathbf{R};\, g(x)>0\}$. It is easy to see that $G=\big(2-\sqrt{3},2+\sqrt{3}\big)$. If $x\in G$, then $x>0$ and $x^2<4x-1<5x-1$ whence $f(x)\neq 0$. Thus our expression is meaningful on G. By (8) we have

$$W(x)=\begin{vmatrix}1, & -2x, & x^2\\ 1, & -5, & 1\\ -1, & 4, & -1\end{vmatrix}=-x^2+1=\varphi(x)\psi(x),$$

where $\varphi(x)=-x+1$, $\psi(x)=x+1$. By (17) we have $b=\begin{vmatrix}-1, & 1\\ 2, & -5\end{vmatrix}=3$; thus $d=D_f/3=7$. Since $1=(\varphi+\psi)/2$, we have the primitive

$$\frac{1}{\sqrt{3}}\arctan\frac{1-x}{\sqrt{3(-x^2+4x-1)}}+\frac{1}{\sqrt{7}}\arctan\frac{1+x}{\sqrt{7(-x^2+4x-1)}}\qquad (x\in G).$$

Remark. Here f has real roots so that we may decompose f and use 2.14 and (19). In this way we get the primitive

$$\sqrt{\frac{5-\sqrt{21}}{42}}\arcsin\frac{8+2\sqrt{21}-x(1+\sqrt{21})}{(5+\sqrt{21}-2x)\sqrt{3}}\\ -\sqrt{\frac{5+\sqrt{21}}{42}}\arcsin\frac{8-2\sqrt{21}+x(-1+\sqrt{21})}{(2x-5+\sqrt{21})\sqrt{3}}.$$

The computation is left to the reader.

2.17 Remark. Our next goal is Theorem 2.21 which is less effective than 2.11. The only reason to state (and prove) it is that it corrects an assertion appearing on pages 29 and 30 in Hardy's book [2].

2.18 Notation. For any polynomials p, q let $D_{p,q}=p'q'-p''q-pq''$. (Clearly $D_{p,p}=D_p$.)

2.19 Lemma. *Let* $\deg p\vee\deg q\leqslant 2$. *Then* $D_{p,q}$ *is constant.*

Proof. $D'_{p,q}=p''q'+p'q''-p''q'-p'q''=0$. □

2.20 Lemma. *Let* p, q *be polynomials and let* $a,b\in\mathbf{R}$. *Then*

$$D_{ap-bq}=a^2D_p-2abD_{p,q}+b^2D_q.$$

(The easy proof is left to the reader.)

2.21 Theorem. *Let $\deg g \leqslant \deg f = 2$, $f > 0$ on $\mathbf{R}$, $D_g \neq 0$, $\deg Q \leqslant 1$. Define $\Phi(t) = t^2 D_f - 2tD_{f,g} + D_g$ $(t \in \mathbf{R})$. Then $D_\Phi > 0$. Let m_1, m_2 be the roots of Φ, $m_1 > m_2$. Then $m_1 > 0$ and there are linear functions r_1, r_2 such that $r_1^2 = m_1 f - g$, $r_2^2 = -m_2 f + g$ and*

$$r_1 r_2 \sqrt{|D_f|} = -W. \tag{22}$$

Let F_1, F_2 be functions such that $F_1'(t) = 1/\sqrt{m_1 - t^2}$ $(t^2 < m_1)$, $F_2'(t) = 1/\sqrt{t^2 - m_2}$ $(t^2 > m_2)$. There are α, $\beta \in \mathbf{R}$ such that

$$\alpha r_2 + \beta r_1 = Q. \tag{23}$$

Set $H = \{x \in \mathbf{R};\, g(x)(W(x))^2 > 0\}$. Then

$$Q/(f\sqrt{g}) = \left(2/\sqrt{|D_f|}\right)\left(\alpha(\operatorname{sgn} r_1)F_1\left(\sqrt{g/f}\right) + \beta(\operatorname{sgn} r_2)F_2\left(\sqrt{g/f}\right)\right)' \quad \text{on } H. \tag{24}$$

P r o o f. Let W, φ, ψ be as before. Let a, b, c, d be numbers such that $\varphi^2 = af - bg$, $\psi^2 = cf - dg$ (see 2.9). Since $bd = D_f < 0$, we may assume that $b > 0 > d$. Then $\varphi^2/b = (a/b)f - g$, $\psi^2/|d| = (c/|d|)f + g$, $(a/b - c/d)f = \varphi^2/b + \psi^2/|d| > 0$; hence $a/b > c/d$. By 2.20 we have $a^2 D_f - 2abD_{f,g} + b^2 D_g = D_{\varphi^2} = 0$. Hence $\Phi(a/b) = 0$; similarly $\Phi(c/d) = 0$. It follows that $m_1 = a/b$, $m_2 = c/d$. Set $r_1 = \varphi/\sqrt{b}$, $r_2 = -\psi/\sqrt{|d|}$. Then (22) holds. As in the proof of 2.11 we find that r_1, r_2 are linearly independent which shows the existence of numbers α, β fulfilling (23). Clearly $m_2 < g/f < m_1$ on H so that the functions $S_j = \left(2(\operatorname{sgn} r_j)/\sqrt{|D_f|}\right)F_j\left(\sqrt{g/f}\right)$ are differentiable on H. We have

$$S_1' = \frac{2}{\sqrt{|D_f|}} \cdot \frac{\operatorname{sgn} r_1}{\sqrt{m_1 - g/f}} \cdot \frac{1}{2}\sqrt{\frac{f}{g}} \cdot \frac{-W}{f^2} = \frac{|r_1| r_2}{|r_1| f\sqrt{g}} = \frac{r_2}{f\sqrt{g}},$$
$$S_2' = \frac{2}{\sqrt{|D_f|}} \cdot \frac{\operatorname{sgn} r_2}{\sqrt{g/f - m_2}} \cdot \frac{1}{2}\sqrt{\frac{f}{g}} \cdot \frac{-W}{f^2} = \frac{|r_2| r_1}{|r_2| f\sqrt{g}} = \frac{r_1}{f\sqrt{g}}$$

from which (24) follows at once. □

R e m a r k 1. It may happen that the set $G = \{x \in \mathbf{R};\, g(x) > 0\}$ contains one or two roots of r_1, r_2 (i.e. of W); in this case formula (24) gives a primitive of $Q/(f\sqrt{g})$ only "piecewise". To get a primitive on G we need to choose F_1 and F_2 in such a way that we get a function continuously extendable to the whole of G.

R e m a r k 2. The relation (24) is a corrected version of the formula appearing on lines 2 and 3 on p. 30 in [2]. We get that formula, if we drop the factors sgn r_j in (24) and change the notation. Hardy's explanation on page 29 is not clear. He does not even say on what set the formula should hold, which creates the impression that it is valid on G; the formula, however, is meaningless at the roots of W. I would like to illustrate the matter on an example. Let us try to find a primitive of the function $v(x) = 1/((x^2+1)\sqrt{x^2+2})$ $(x \in \mathbf{R})$. Applying the mentioned formula we get something like $\gamma \arcsin\sqrt{(x^2+2)/(2x^2+2)}$, where γ is a nonzero constant. It is obvious that this function is continuous on $\mathbf{R}$; it is, perhaps, a little less obvious that it is not differentiable at 0. (It is clear, however, that a primitive of v cannot be an even function.) Applying 2.21 we get easily $\Phi(t) = -4(t-1)(t-2)$, hence $m_1 = 2$, $m_2 = 1$, and we may choose $F_1(t) = \arcsin\bigl(t/\sqrt{2}\bigr)$, $r_1(x) = -x$, $r_2(x) = 1$, $\alpha = 1$, $\beta = 0$. Set

$$V(x) = -(\operatorname{sgn} x)\arcsin\sqrt{(x^2+2)/(2x^2+2)} \qquad (x \in \mathbf{R}).$$

Then $V = (\operatorname{sgn} r_1)F_1\bigl(\sqrt{g/f}\bigr)$ and, by (24), $V' = v$ on $H = \mathbf{R}\setminus\{0\}$. It is obvious that V is discontinuous at 0. Choosing appropriate "integration constants" we get the function $T(x) = (\operatorname{sgn} x)\arccos\sqrt{(x^2+2)/(2x^2+2)}$ for which $T' = v$ on $\mathbf{R}$. (From 2.11 and (12) we get the primitive $\arctan\bigl(x/\sqrt{x^2+2}\bigr)$.)

Hardy arrived to the mentioned (incorrect) formula using the substitution $t = \sqrt{f/g}$, which he calls Sir G. Greenhill's substitution, and comments: "This method is very elegant .... On the other hand it is somewhat artificial, and it is open to the logical objection that it introduces the root $\sqrt{f}$, which, in virtue of Laplace's principle, cannot be involved in the final result." Let us note at this occasion that $\sqrt{f}$ does not appear in 2.11.

3. General case

Now it remains to find primitives of functions $P/\sqrt{g}$, where $\deg g = 2$ and $\deg P > 0$, and of functions $Q/\bigl(f^m\sqrt{g}\bigr)$, where $m > 1$, $\deg Q < m \deg f$ and f, g are relatively prime. We will see that the first case is not difficult; the second is similar if $\deg f = 1$. If $\deg f = 2$ then the procedure is more complicated.

In this section we will actually get the algebraic part (which is always nontrivial) of the primitive. With the help of 2.11 or 2.14 we can then get the transcendental part.

3.1 Functions $P/\sqrt{g}$, where $\deg g = 2$ and $\deg P = n > 0$. Set $G = \{x \in \mathbf{R};\ g(x) > 0\}$. Choose a polynomial f of degree 1 (for example the identity mapping), find numbers a_j, α, β, γ with $P = \sum_{j=0}^{n} a_j f^{n-j}$, $g = \alpha f^2 + \beta f + \gamma$, set $b_{-1} = b_0 = 0$ and determine numbers $b_1, \ldots, b_n$ and c by

$$\alpha b_{j+1}(n-j) + \beta b_j(n-j+\tfrac{1}{2}) + \gamma b_{j-1}(n-j+1) = a_j/f' \quad (j = 0, \ldots, n-1),$$
$$c + f'(\tfrac{1}{2}\beta b_n + \gamma b_{n-1}) = a_n.$$

(This is possible, because $\alpha \neq 0$.) Then

$$P/\sqrt{g} = \Big(\sqrt{g}\sum_{j=1}^{n} b_j f^{n-j}\Big)' + c/\sqrt{g} \quad \text{on } G. \tag{25}$$

3.2 Functions $Q/(f^{k+1}\sqrt{g})$, where $\deg f = 1$, $\deg g \in \{1,2\}$, g is not divisible by f, k is a natural number and $\deg Q \leqslant k$. Set $H = \{x \in \mathbf{R};\ (f(x))^2 g(x) > 0\}$. Find numbers a_j, α, β, γ with $Q = \sum_{j=0}^{k} a_j f^j$, $g = \alpha f^2 + \beta f + \gamma$, set $b_{-2} = b_{-1} = 0$ and determine numbers $b_0, \ldots, b_{k-1}$ and c by

$$\gamma b_j(k-j) + \beta b_{j-1}(k-j+\tfrac{1}{2}) + \alpha b_{j-2}(k-j+1) = -a_j/f' \quad (j = 0, \ldots, k-1),$$
$$c - f'(\tfrac{1}{2}\beta b_{k-1} + \alpha b_{k-2}) = a_k.$$

(This is possible, because $\gamma \neq 0$.) Then

$$Q/(f^{k+1}\sqrt{g}) = \Big(\sqrt{g}\sum_{j=0}^{k-1} b_j f^{j-k}\Big)' + c/(f\sqrt{g}) \quad \text{on } H. \tag{26}$$

Remark 1. If $k = 1$, we may apply 2.13 instead of 3.2.

Remark 2. Formulas (25) and (26) can be verified by a mechanical computation which is left to the reader.

3.3 Example. Find a primitive of $(x^2 + x + 4)/(x^3\sqrt{x^2 + x + 1})$ $(x \neq 0)$.

We apply 3.2 with $f(x) = x$ etc. We have $k = 2$, $\alpha = \beta = \gamma = 1$ and the equations $2b_0 = -a_0 = -4$, $b_1 + \frac{3}{2}b_0 = -a_1 = -1$, $c - \frac{1}{2}b_1 - b_0 = a_2 = 1$. We get $b_0 = -2$, $b_1 = 2$, $c = 0$ and the primitive $(2(x-1)/x^2)\sqrt{x^2 + x + 1}$ $(x \neq 0)$.

3.4 Theorem. *Let f, g be relatively prime polynomials, $D_f \neq 0$, $\deg g \leqslant \deg f = 2$. Set $H = \{x \in \mathbf{R};\ (f(x))^2 g(x) > 0\}$. Let γ, δ be polynomials such that $\gamma f - \delta g = 1$. Let k be a natural number and let Q be a polynomial with $\deg Q \leqslant 2k+1$. Let v be a number such that $f'(v) = 0$. (Then $f(v) \neq 0$.) Define polynomials Q_j $(j = k,$*

$k-1, \ldots, 1, 0$) as follows: Set $Q_k = Q$. If $1 \leqslant j \leqslant k$ and if Q_j is given, find polynomials Z_j, B_j such that $Q_j\delta = Z_j f + B_j$ and $\deg B_j \leqslant 1$, set $E_j = B_j(v)/f(v)$, $A_j = Q_j\gamma - Z_j g$ and define

$$Q_{j-1} = A_j - (2jf'')^{-1}(B_j'g' + E_j((2j-2)f''g + \tfrac{1}{2}K_{f',g})) \tag{27}$$

(see 2.1). Then

$$\begin{aligned} &A_j f = Q_j + B_j g \quad \textit{for } j = k, \ldots, 1, \\ &\deg Q_j \leqslant (\deg Q_{j+1} - 2) \vee 2 \quad \textit{for } j = k-1, \ldots, 1, \\ &\deg Q_j \leqslant 2j+1 \quad \textit{for } j = k, \ldots, 0 \textit{ (in particular } \deg Q_0 \leqslant 1) \end{aligned} \tag{28}$$

and

$$\frac{Q}{f^{k+1}\sqrt{g}} = \left(\frac{\sqrt{g}}{2f''}\sum_{i=1}^{k}\frac{2B_i' - E_i f'}{if^i}\right)' + \frac{Q_0}{f\sqrt{g}} \quad \textit{on } H. \tag{29}$$

Proof. Set $M_j = 2B_j' - E_j f'$ ($j = k, \ldots, 1$). We will show that

$$\frac{Q}{f^{k+1}\sqrt{g}} = \left(\frac{\sqrt{g}}{2f''}\sum_{i=j+1}^{k}\frac{M_i}{if^i}\right)' + \frac{Q_j}{f^{j+1}\sqrt{g}} \tag{30}$$

for $j = k, \ldots, 0$. The relation (30) is trivially correct, if $j = k$. Now suppose that $1 \leqslant j \leqslant k$ and that (30) holds. Set $C = M_j/(2jf'')$, $V = M_j f'/(2f'') + E_j f$. Clearly $A_j f - B_j g = Q_j\gamma f - Z_j f g - B_j g = Q_j\gamma f - Q_j\delta g = Q_j$; this proves (28). Since $V(v) = E_j f(v) = B_j(v)$ and $V' = (M_j' f' + M_j f'')/(2f'') + E_j f' = -\frac{1}{2}E_j f' + B_j' - \frac{1}{2}E_j f' + E_j f' = B_j'$, we have

$$jCf' + E_j f = V = B_j. \tag{31}$$

Further we have

$$\begin{aligned} &E_j g + C'g + \tfrac{1}{2}Cg' \\ &\quad = (2jf'')^{-1}(2jf''E_j g + M_j' g + \tfrac{1}{2}M_j g') \\ &\quad = (2jf'')^{-1}(2jE_j f''g - E_j f''g + \tfrac{1}{2}(2B_j' - E_j f')g') \\ &\quad = (2jf'')^{-1}((2j-2)f''gE_j + B_j'g' + \tfrac{1}{2}E_j K_{f',g}) \end{aligned}$$

so that

$$Q_{j-1} = A_j - E_j g - C'g - \tfrac{1}{2}Cg'. \tag{32}$$

Clearly $\left(C\sqrt{g}/f^j\right)' + Q_{j-1}/(f^j\sqrt{g}) = S/(f^{j+1}\sqrt{g})$, where, by (32), (31), and (28), $S = C'gf + \frac{1}{2}Cg'f - jCgf' + Q_{j-1}f = f(A_j - E_jg) - jCgf' = A_jf - g(jCf' + E_jf) = A_jf - B_jg = Q_j$. It follows that we may replace j by $j-1$ in (30).

From (28) we see that $\deg A_j \leqslant (\deg Q_j \vee 3) - 2 = (\deg Q_j - 2) \vee 1$. By (27) we have $\deg Q_{j-1} \leqslant \deg A_j \vee 2 \leqslant (\deg Q_j - 2) \vee 2$. Since $Q_0 = A_1 - (2f'')^{-1}(B_1'g' + \frac{1}{2}K_{f',g})$, we have $\deg Q_0 \leqslant 1$. This completes the proof. □

Remark 1. It may be easier to find A_j from (28) than from the relation $A_j = Q_j\gamma - Z_jg$.

Remark 2. Theorem 3.4 is complicated; it should be noted, however, that the corresponding steps are easy even if k is large. (Nothing more difficult than "long division" by f is involved.) It is possible to formulate simpler theorems that lead in a similar way to a primitive of $Q/\left(f^{k+1}\sqrt{g}\right)$ but use more difficult procedures. For example, we see that under the mentioned assumptions there is a polynomial T with $\deg T < 2k$ and a linear function l such that $Q/\left(f^{k+1}\sqrt{g}\right) = \left(T\sqrt{g}/f^k\right)' + l/(f\sqrt{g})$. We may find T and l using the method of unknown coefficients; to get them we must, of course, solve a system of $2k+2$ equations.

3.5 Example. Find a primitive of

$$\frac{x^5}{(x^2 - x + 1)^3\sqrt{2x^2 - x + 1}} \qquad (x \in \mathbf{R}).$$

Let us denote by x the identity mapping. To apply 3.4 set $f = x^2 - x + 1$, $k = 2$ etc. We find easily that $(2x+1)(x^2 - x + 1) = x(2x^2 - x + 1) + 1$. Thus we may choose $\gamma = 2x + 1$, $\delta = x$. Clearly $Q\delta = x^6$. A "long division" yields $x^6 = (x^4 + x^3 - x - 1)(x^2 - x + 1) + 1$; thus $B_2 = 1$. Since $v = \frac{1}{2}$, we have $f(v) = \frac{3}{4}$, $E_2 = \frac{4}{3}$. From (28) and (6) we get $A_2 = (Q_2 + g)/f = x^3 + x^2 + 1$,

$$K_{f',g} = \begin{vmatrix} 2, & -1 \\ 4, & -1 \end{vmatrix} x + \begin{vmatrix} 2, & -1 \\ -1, & 2 \end{vmatrix} = 2x + 3.$$

By (27), $Q_1 = A_2 - \frac{1}{8}\cdot\frac{4}{3}\cdot(4g + x + \frac{3}{2}) = (6x^3 - 2x^2 + 3x + \frac{1}{2})/6$. Another "long division" yields $6x^4 - 2x^3 + 3x^2 + x/2 = (6x^2 + 4x + 1)(x^2 - x + 1) - (5x+2)/2$; hence $B_1 = -(5x+2)/12$, $B_1(v) = -3/8$, $E_1 = -1/2$. Since $Q_1 + B_1g = (2x^3 - 3x^2 + 3x - 1)/12$ and $(2x^3 - 3x^2 + 3x - 1)/(x^2 - x + 1) = 2x - 1$, we have by (28) $A_1 = (2x-1)/12$ and from (27) we obtain

$$Q_0 = \frac{1}{12}(2x - 1) - \frac{1}{4}\left(-\frac{5}{12}(4x - 1) - \frac{1}{2}\frac{2x+3}{2}\right) = \frac{17}{24}x.$$

Setting $M_j = 2B_j' - E_j f'$ $(j = 1,\ 2)$ we get $\sum_{i=1}^{2} M_i/(if^i) = -\frac{2}{3}(2x-1)/f^2 + \frac{1}{3}(3x-4)/f$ and, by (29),

$$Q/(f^3\sqrt{g}) = \left(\sqrt{g}(3x^3 - 7x^2 + 3x - 2)/(12f^2)\right)' + 17x/(24f\sqrt{g}).$$

Now we must integrate $x/(f\sqrt{g})$. By (8) we have

$$W = \begin{vmatrix} 1, & -2x, & x^2 \\ 1, & -1, & 1 \\ 2, & -1, & 1 \end{vmatrix} = x^2 - 2x.$$

Set $\varphi = x - 2$, $\psi = x$. To apply 2.10 we get (see (17)) $b = \begin{vmatrix} 1, & -2 \\ 2, & -1 \end{vmatrix} = 3$ so that $\left((2/\sqrt{3}) \arctan\left((x-2)/\sqrt{3g}\right)\right)' = x/(f\sqrt{g})$. Thus we obtain the primitive

$$\frac{\sqrt{2x^2 - x + 1}}{12(x^2 - x + 1)^2}(3x^3 - 7x^2 + 3x - 2) + \frac{17}{12\sqrt{3}} \arctan \frac{x-2}{\sqrt{3(2x^2 - x + 1)}}.$$

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