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INVARIANT CURVES FROM SYMMETRY

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Summary. We show that certain symmetries of maps imply the existence of their invariant curves.

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1. INTRODUCTION

In this paper we shall investigate the following problem: Does a symmetry of a continuous map imply the existence of invariant curves? An affirmative answer to a similar question for ordinary differential equations was given in [1].

We study a continuous map $F\colon \mathbf{R}^m \to \mathbf{R}^m$, $m \geqslant 2$ equivariant under an orthogonal representation of a compact Lie group. Then assuming some other properties of F we show the existence of invariant curves which lie on spheres. The proof of the theorem of this paper is based on results of the paper [1] and features of orthogonal representations.

2. MAIN RESULT

Consider a continuous map $F\colon \mathbf{R}^m \to \mathbf{R}^m$, $m \geqslant 2$ such that

i) there exist x_1, x_2 satisfying

$$(|F(x_1)| - |x_1|)(|F(x_2)| - |x_2|) < 0;$$

ii) F is equivariant under a linear othogonal representation T of a compact Lie group G, $T = \{T_g : g \in G\}$, i.e.,

$$F(T_g x) = T_g F(x);$$

iii) T is transitive on the unit sphere $S^{m-1} \subset \mathbf{R}^m$, i.e.,

$$\{T_g x : g \in G\} = S^{m-1}$$

for each $x \in S^{m-1}$ (see [1, p. 478]).

Theorem. *Under the above assumptions F has an invariant curve.*

Proof. Since F is equivariant under T and T is transitive, we have $|F(x)| = \varrho(|x|)$ for a continuous function $\varrho\colon [0,\infty) \to [0,\infty)$. We note that the assumption ii) implies $F(0) = 0$. The condition i) implies

$$(\varrho(|x_1|) - |x_1|) \cdot (\varrho(|x_2|) - |x_2|) < 0.$$

Hence there is $h > 0$ such that $\varrho(h) = h$. Thus the sphere $S_h = \{x : |x| = h\}$ is invariant under F. Take $x_0 \in S_h$. Then by [1, Lemma 1 and Lemma 2] there is

$$K_0 \in C(T) = \{K : K \text{ is an } m \times m \text{ matrix, } KT_g = T_g K \text{ for each } g \in G\}$$

such that $F(x_0) = K_0 x_0$.

Further $K_0 T_g x_0 = T_g K_0 x_0 = T_g F(x_0) = F(T_g x_0)$. Thus

$$|K_0 T_g x_0| = |F(T_g x_0)| = |F(x_0)| = h.$$

Since T is transitive and $T_g x_0 \in S_h$, we see that $|K_0 x| = |x|, \forall x \in S_h$. Thus K_0 is orthogonal. Hence the eigenvalues of K_0 lie on the unit circle.

We have $K_0 T_g x_0 = F(T_g x_0)$. Since T is transitive we have

$$K_0 x = F(x)$$

for each $x \in S_h$. Hence $F/S_h = K_0$, i.e., the restriction of F on S_h is the linear map K_0.

Since T is transitive, T is irreducible. Hence the minimal polynomial of K_0 is irreducible as well. Indeed, let the minimal polynomial p be expressed as $p = p_1 p_2$ for p_1, p_2 nonconstant polynomials. Then $\exists\, \alpha \in \mathbf{R}^m$ such that $\beta = p_2(K_0)\alpha \neq 0$. Consider $Y = \ker p_1(K_0)$. Then $\beta \in Y$, since $p_1(K_0)\beta = p_1(K_0) \cdot p_2(K_0)\alpha =$

$p(K_0)\alpha = 0$. Hence $Y \neq \{0\}$. Using the property $K_0 T_g = T_g K_0$, $\forall g \in G$, we have $p_1(K_0)T_g = T_g p_1(K_0)$, $\forall g \in G$. This implies that Y is invariant under T_g, $\forall g \in G$. But T is irreducible, hence $Y = \mathbf{R}^m$ and p_2 has to be constant. This is a contradiction. Thus K_0 satisfies either an equation $K_0 = d \cdot I$, $d \in \mathbf{R}$ or

$$K_0^2 = b \cdot K_0 + c \cdot I \tag{1}$$

for b, $c \in \mathbf{R}$ and $I=$ Identity.

Let the minimal polynomial of K_0 be $y - d$, i.e., $K_0 = d \cdot I$. Since K_0 is orthogonal we have $d = \pm 1$ and the existence of an invariant curve is trivial.

Let $y^2 - by - c$ be the minimal polynomial. Since K_0 has only eigenvalues on the unit circle,

$$y^2 = by + c$$

has only roots with absolute values 1. We have applied the Cayley-Hamilton theorem [2]. This implies

$$|b| \leqslant 2, \quad c = \pm 1.$$

From (1) and $K_0^{\mathsf{T}} = K_0^{-1}$ we have

$$K_0 = b \cdot I \pm K_0^{-1} = b \cdot I \pm K_0^{\mathsf{T}}$$

$$K_0 \mp K_0^{\mathsf{T}} = b \cdot I. \tag{2}$$

First, we consider

$$K_0 - K_0^{\mathsf{T}} = b \cdot I. \tag{3}$$

Then $b = 0$, $K_0 = K_0^{\mathsf{T}}$. In this case the polynomial $y^2 - 1$ is not irreducible. Thus it is not minimal and we arrive at the first case.

Now we consider the second version of (2),

$$K_0 + K_0^{\mathsf{T}} = b \cdot I.$$

Let us take $B = K_0 - \frac{b}{2} \cdot I$. Then $K_0 = \frac{b}{2} \cdot I + B$ and $B^{\mathsf{T}} = -B$. By $K_0 K_0^{\mathsf{T}} = I$ we have

$$I = \left(\frac{b}{2} \cdot I + B\right)\left(\frac{b}{2} \cdot I - B\right)$$

$$= \frac{b^2}{4} \cdot I - B^2,$$

$$B^2 = \left(\frac{b^2}{4} - 1\right) \cdot I. \tag{4}$$

Let $|b| < 2$. Then B is invertible. Consider

$$A = \{c \cdot x + d \cdot Bx \colon c, d \in \mathbf{R}\}, \quad x \neq 0.$$

Note that $B^{\mathsf{T}} = -B$ implies $x \perp Bx$. Then by (4) $K_0 A = A$ and easy computation shows that the matrix $K_0/A = E$ under the basis Bx, x has the form

$$E = \begin{pmatrix} \frac{b}{2} & \frac{b^2}{4} - 1 \\ 1 & \frac{b}{2} \end{pmatrix}.$$

E has eigenvalues $\frac{1}{2}(b \pm \sqrt{b^2 - 4})$. Hence E is equivalent to a rotation. This implies that $C = A \cap S_h$ is an invariant circle of F and F/C is equivalent to a rotation.

Finally, let $b = \pm 2$. Then the polynomial $y^2 \mp 2y + 1 = (y \mp 1)^2$ is not irreducible. Thus we have again arrived at the first case. □

Corollary. *The dynamics of F on an invariant curve predicted by Theorem is equivalent to a rotation.*

P r o o f. The statement follows immediately from the above proof. □

References

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[2] *G. Birkhoff, S. Mac Lane*: A Survey of Modern Algebra, The Macmillan Company, New York, 1965.

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