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# EXACT SOLUTIONS OF CAUCHY PROBLEM FOR PARTIAL DIFFERENTIAL EQUATIONS WITH DOUBLE CHARACTERISTICS AND SINGULAR COEFFICIENTS

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*Summary.* Let $L_{a,b} \equiv (\partial_x - ax^k\partial_t)(\partial_x - bx^k\partial_t) + kbx^{k-1}\partial_t - \frac{k}{x}\partial_x$ be a family of operators with double characteristics and singular coefficients, where $a$, $b$ are reals with $ab \neq 0$ and $a \neq b$, $k > 0$ is an odd integer. Let $\Omega$ be the first quadrant in the plane and $H_+$ the upper half-plane. Consider Cauchy problems

$$(P_1) \qquad \begin{cases} L_{a,b}u = 0 & \text{in } \Omega \text{ or } H_+,\\ u(x,0) = \varphi_0(x),\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \overline{\mathbb{R}_+} \text{ or } x \in \mathbb{R} \end{cases}$$

for $a > 0$, $b > 0$, and initial-boundary value problems

$$(P_2) \qquad \begin{cases} L_{a,b}u = 0 & \text{in } \Omega \text{ or } H_+,\\ u(x,0) = \varphi_0(x),\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \overline{\mathbb{R}_+} \text{ or } x \in \mathbb{R},\\ u(0,t) = \psi_0(t) & \text{for } t \in \overline{\mathbb{R}_+}, \end{cases}$$

$$(P_3) \qquad \begin{cases} L_{a,b}u = 0 & \text{in } \Omega \text{ or } H_+,\\ u(x,0) = \varphi_0(x),\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \overline{\mathbb{R}_+} \text{ or } x \in \mathbb{R},\\ \lim\limits_{\substack{(x,\tau)\to(0,t),x\neq0\\ (x,\tau)\in\Omega \text{ or } H_+}} \dfrac{u_x(x,\tau)}{x^k} = \psi_1(t) & \text{for } t \in \overline{\mathbb{R}_+} \end{cases}$$

for $ab < 0$ and

$$(P_4) \qquad \begin{cases} L_{a,b}u = 0 & \text{in } \Omega \text{ or } H_+,\\ u(x,0) = \varphi_0(x),\ u_t(x,0) = \varphi_1(x) & \text{for } x \in \overline{\mathbb{R}_+} \text{ or } x \in \mathbb{R},\\ u(0,t) = \psi_0(t),\ \lim\limits_{\substack{(x,\tau)\to(0,t),x\neq0\\ (x,\tau)\in\Omega \text{ or } H_+}} \dfrac{u_x(x,\tau)}{x^k} = \psi_1(t) & \text{for } t \in \overline{\mathbb{R}_+} \end{cases}$$

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for $a < 0$, $b < 0$. Under appropriate smoothness conditions on $\varphi_0$, $\varphi_1$, $\psi_0$ and $\psi_1$, we obtain different sufficient and necessary conditions for each problem to have classical solutions. Moreover, we obtain also explicit expressions of solutions in each case.

*Keywords*: exact solutions, Cauchy problem, double characteristics, singular coefficients

*AMS classification*: 35C15, 35L99

## 1. Introduction

Many authors have studied partial differential operators with the principal part $\partial_{xx} - x^2\partial_{tt}$ and some analogues in spaces with higher dimension. These operators have double characteristics at every point of the line $\{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1} \mid x_1 = \cdots = x_n = 0\}$ with $n = 1$ or $n > 1$. For example, Trèves [4] discussed uniqueness of the Cauchy problem for the partial differential equation

$$u_{xx} - x^2 u_{tt} + pu_t = 0 \tag{1.1}$$

with initial conditions being prescribed on the $x$-axis, i.e., $t = 0$, and proved that a necessary and sufficient condition for uniqueness of solutions in a certain class of functions for the problem is that $p \neq 1, 3, 5, \ldots$. (We have changed here the direction of the variable $t$). Among other things, Menikoff [3] generalized Trèves' result to a class of operators

$$P = (\partial_x + ax^k\partial_t)(\partial_x - bx^k\partial_t) - cx^{k-1}\partial_t \tag{1.2}$$

with $a, b \in \mathbb{R}_+, c \in \mathbb{C}$ and $k$ an odd integer, and the corresponding result of his is that $\frac{c}{a+b} - j(k+1) \neq 0$ or $1$ for $j = 0, 1, 2, \ldots$. The author et al. [2] dealt with the Cauchy problem and the Goursat problem for the equation (1.1) in a class of smooth functions and showed that $p \neq 1, 3, 5, \ldots$ is necessary and sufficient for both the uniqueness and existence of the solutions for the problems and that only under some compatibility conditions with some additional data on the line $\{(x, t) \in \mathbb{R}^2 \mid x = 0\}$ when $p \in \{1, 3, 5, \ldots\}$, solutions of the new problems can exist and be unique. The author [1] studied existence in the class of real analytic functions for the Goursat problem of the operator (1.2) with $a = b = 1$ and showed that for any $c \in \mathbb{R}$ when $k > 1$ it is necessary that the Goursat data satisfy compatibility conditions in order to guarantee existence of solutions.

In the present paper the homogeneous equations $L_{a,b}u = 0$ with real $a$, $b$, $ab \neq 0$, $a \neq b$, and an odd integer $k$ will be considered in the upper half-plane $H_+$, and in the first quadrant $\Omega$ as well. We will deal with the Cauchy problems and the Cauchy problems with some additional (boundary) conditions for the equations $L_{a,b}u = 0$ in $D$, where $D = H_+$ or $D = \Omega$, i.e., the problems $(P_1)$ for $a > 0$, $b > 0$, $a \neq b$, $(P_2)$ and $(P_3)$ for $ab < 0$, and $(P_4)$ for $a < 0$, $b < 0$, $a \neq b$.

Our aim is to prove uniqueness, to obtain sufficient and necessary conditions for existence of classical solutions for the problems $(P_1)$–$(P_4)$ and to get explicit expressions of the solutions in terms of the given data $\varphi_0$, $\varphi_1$, $\psi_0$ and $\psi_1$ for the problems $(P_1)$–$(P_4)$, respectively. In §2 we will prove a lemma which is the base of this paper. In §3 we will deal with the main results, namely, we will state and prove four theorems for the problems $(P_1)$–$(P_4)$ and their two corollaries in $H_+$, while the corresponding four theorems and two corollaries for $(P_1)$–$(P_4)$ in $\Omega$ will be just mentioned, of course, without proof.

## 2. A lemma and some notation

It is well known that the operator $L_{a,b}$ has two families of characteristics

$$C_1 : t + \frac{a}{k+1}x^{k+1} = \text{const.},$$

$$C_2 : t + \frac{b}{k+1}x^{k+1} = \text{const.},$$

and that each of the families $C_1$ is tangent to one of the families $C_2$ at a point on the $t$-axis, and the inverse is also true. Before stating and proving the lemmas let us fix some notation for the sake of convenience and brevity. We introduce a series of abbreviations:

$$A = A(x,t) = t + \frac{a}{k+1}x^{k+1},\ B = B(x,t) = t + \frac{b}{k+1}x^{k+1},\ \lambda = \sqrt[k+1]{1 - \frac{b}{a}},$$

$$X^{(a)} = \frac{a}{k+1}x^{k+1},\ X^{(b)} = \frac{b}{k+1}x^{k+1},\ X_0^{(a)} = \frac{a}{k+1}x_0^{k+1},\ X_0^{(b)} = \frac{b}{k+1}x_0^{k+1},$$

$$A_0 = A(x_0,t_0),\ B_0 = B(x_0,t_0),\ T_a = \sqrt[k+1]{\frac{k+1}{a}t},\ T_b = \sqrt[k+1]{\frac{k+1}{b}t},$$

$$\mathcal{A} = \pm\sqrt[k+1]{\frac{k+1}{a}A},\ \mathcal{B} = \pm\sqrt[k+1]{\frac{k+1}{b}B} \text{ for } x > 0 \text{ and } x < 0, \text{ respectively},$$

$$\mathcal{A}_0 = \mathcal{A}(x_0,t_0),\ \mathcal{B}_0 = \mathcal{B}(x_0,t_0),\ \mathcal{C}_0 = \{(x,t) \in \mathbb{R}^2 \mid x = 0,\ t \geqslant 0\},$$

$$\Omega_a^- = \{(x,t) \in \mathbb{R}^2 \mid A(x,t) \leqslant 0,\ t > 0\},\quad \Omega_a^+ = \{(x,t) \in \mathbb{R}^2 \mid A(x,t) > 0\},$$

$$\Omega_b^- = \{(x,t) \in \mathbb{R}^2 \mid B(x,t) \leqslant 0,\ t > 0\},\quad \Omega_b^+ = \{(x,t) \in \mathbb{R}^2 \mid B(x,t) > 0\},$$

$$C_0^{(a)} = \{(x,t) \in \mathbb{R}^2 \mid A(x,t) = 0\},\quad C_0^{(b)} = \{(x,t) \in \mathbb{R}^2 \mid B(x,t) = 0\},$$

and

$$\Phi_1(x) = \varphi_1(x) - \frac{1}{bx^k}\varphi_0'(x), \qquad \Phi_2(x) = \varphi_1(x) - \frac{1}{ax^k}\varphi_0'(x),$$

$$\Psi_1(t) = \psi_0'(t) - \frac{1}{b}\psi_1(t), \qquad \Psi_2(t) = \psi_0'(t) - \frac{1}{a}\psi_1(t).$$

We are now ready to state the lemma.

**Lemma.** *Suppose that a $C^2$-function $u(x,t)$ satisfies $L_{a,b}u = 0$ in a domain $D \subset \mathbb{R}^2$ with the property that $D \cap \{(x,t) \in \mathbb{R}^2 \mid x = 0\} = \emptyset$. Let*

$$v_1(x,t) = u_t(x,t) - \frac{1}{bx^k}u_x(x,t), \tag{2.1}$$

$$v_2(x,t) = u_t(x,t) - \frac{1}{ax^k}u_x(x,t). \tag{2.2}$$

*Then, for $i = 1, 2$, the function $v_i$ is a constant along any connected component in $D$ of each curve of the family $C_i$.*

Proof of Lemma. Take $i = 1$. Along a connected curve $l_1$ of the family $C_1$, which is in $D$ and satisfies the equation $t + \frac{a}{k+1}x^{k+1} = c$ with a real $c$, the function $v_1(x,t)$ becomes

$$\widetilde{v}_1(x) = v_1\Big(x, -\frac{a}{k+1}x^{k+1} + c\Big).$$

A simple computation shows that the derivative of $\widetilde{v}_1(x)$ with respect to $x$ vanishes. In fact,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\widetilde{v}_1(x) &= \left[\frac{\partial v_1}{\partial x} + \frac{\partial v_1}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}x}\right]\Bigg|_{l_1} \\
&= \left[\Big(u_{xt} - \frac{1}{bx^k}u_{xx} + \frac{k}{bx^{k+1}}u_x\Big) + \Big(u_{tt} - \frac{1}{bx^k}u_{xt}\Big)\cdot(-ax^k)\right]\Bigg|_{l_1} \\
&= -\frac{1}{bx^k}\left[u_{xx} - (a+b)x^k u_{xt} + abx^{2k}u_{tt} - \frac{k}{x}u_x\right]\Bigg|_{l_1} \\
&= -\frac{1}{bx^k}\left[(\partial_x - ax^k\partial_t)(\partial_x - bx^k\partial_t)u + \Big(kbx^{k-1}u_t - \frac{k}{x}u_x\Big)\right]\Bigg|_{l_1} \\
&= -\frac{1}{bx^k}L_{a,b}u\Big|_{l_1} = 0.
\end{aligned}$$

Similarly, we have $\frac{\mathrm{d}}{\mathrm{d}x}v_2(x,t) = 0$ along each characteristic curve of the family $C_2$. The lemma follows. □

## 3. Main results

**Theorem 1.** *Suppose that the Cauchy data $\varphi_0$ and $\varphi_1$ satisfy*

$$\varphi_0 \in C^2(\mathbb{R}), \quad \varphi_1 \in C^1(\mathbb{R}). \tag{3.1}$$

*Then, problem $(P_1)$ in $H_+$ with $a > 0$, $b > 0$, $a \neq b$ has a solution $u$ belonging to $C^1(\overline{H_+}) \cap C^2(H_+)$ if and only if*

$$\text{the limit } \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{\varphi_0'(x)}{x^k} \text{ exists and is finite,} \tag{3.2}$$

$$\varphi_0' \text{ is an odd function, } \varphi_1 \text{ is an even function.} \tag{3.3}$$

*Then, the solution is unique and can be expressed for all $(x,t)$ in $H_+$ by the formula*

$$u(x,t) = \frac{ab}{b-a}\left[\int_{\mathcal{B}}^{\mathcal{A}} s^k \varphi_1(s)\,\mathrm{d}s + \frac{1}{a}\varphi_0(\mathcal{B}) - \frac{1}{b}\varphi_0(\mathcal{A})\right]. \tag{3.4}$$

*In addition, we have*

$$\frac{u_x(x,t)}{x^k} \in C^0(\overline{H_+}). \tag{3.5}$$

Remark 1. It is obvious from the definition of the functions $\Phi_1$ and $\Phi_2$ and the oddness of $k$ that the condition (3.3) is equivalent to the two conditions

$$\begin{aligned} &\Phi_1 \text{ is an even function,} \\ &\Phi_2 \text{ is an even function.} \end{aligned} \tag{3.6}$$

Proof of Theorem 1. Suppose that (3.1) holds and that $u \in C^1(\overline{H_+}) \cap C^2(H_+)$ is a solution of Problem $(P_1)$ in $H_+$ with $a > 0$, $b > 0$, and $a \neq b$. For all $x \in \mathbb{R}$ we clearly have

$$u(x,0) = \varphi_0(x),\ u_t(x,0) = \varphi_1(x) \quad \text{and} \quad u_x(x,0) = \varphi_0'(x).$$

Let $(x_0,t_0) \in H_+$ with $x_0 > 0$, i.e., $(x_0,t_0) \in \Omega$. The characteristic curve of the family $C_1$ through the point $(x_0,t_0)$ has the form

$$t + \frac{a}{k+1}x^{k+1} = t_0 + \frac{a}{k+1}x_0^{k+1}, \quad x \geqslant 0,\ t \geqslant 0, \tag{3.7}$$

and intersects the positive $x$-axis at the point $(\mathcal{A}_0, 0)$, and the $t$-axis at the point $(0, \mathcal{A}_0)$. Similarly, the characteristic curve of the family $C_2$ through $(x_0, t_0)$

$$t + \frac{b}{k+1} x^{k+1} = t_0 + \frac{b}{k+1} x_0^{k+1}, \quad x \geqslant 0,\ t \geqslant 0 \tag{3.8}$$

intersects the positive $x$-axis at the point $(\mathcal{B}_0, 0)$, and the $t$-axis at the point $(0, \mathcal{B}_0)$. It is known from Lemma that the functions $v_1$ and $v_2$, defined respectively by the formulas (2.1) and (2.2), are constant along the characteristic curves (3.7) and (3.8) respectively. Thus, we have

$$\begin{aligned}
\Big(u_t - \frac{1}{bx^k} u_x\Big)\Big|_{(x_0,t_0)} &= v_1(x_0, t_0) = v_1(\mathcal{A}(x_0, t_0), 0) \\
&= \Big(u_t - \frac{1}{bx^k} u_x\Big)\Big|_{(\mathcal{A}(x_0,t_0),0)} \\
&= \varphi_1(\mathcal{A}(x_0, t_0)) - \frac{1}{b}\varphi_0'(\mathcal{A}(x_0, t_0))(\mathcal{A}(x_0, t_0))^{-k} \\
&= \Phi_1(\mathcal{A}(x_0, t_0)),
\end{aligned}$$

and

$$\begin{aligned}
\Big(u_t - \frac{1}{ax^k} u_x\Big)\Big|_{(x_0,t_0)} &= \varphi_1(\mathcal{B}(x_0, t_0)) - \frac{1}{a}\varphi_0'(\mathcal{B}(x_0, t_0))(\mathcal{B}(x_0, t_0))^{-k} \\
&= \Phi_2(\mathcal{B}(x_0, t_0)).
\end{aligned}$$

Hence we have

$$\Big(\frac{1}{a} - \frac{1}{b}\Big) u_t(x_0, t_0) = \frac{1}{a}\Phi_1(\mathcal{A}(x_0, t_0)) - \frac{1}{b}\Phi_2(\mathcal{B}(x_0, t_0)).$$

Therefore, integration yields

$$\begin{aligned}
u(x_0, t_0) &= \int_0^{t_0} u_t(x_0, t)\,\mathrm{d}t + u(x_0, 0) \\
&= \frac{ab}{b-a}\int_0^{t_0} \Big[\frac{1}{a}\Phi_1(\mathcal{A}(x_0, t)) - \frac{1}{b}\Phi_2(\mathcal{B}(x_0, t))\Big]\mathrm{d}t + \varphi_0(x_0).
\end{aligned} \tag{3.9}$$

It is easy to show that

$$
\begin{aligned}
\frac{1}{a}\int_0^{t_0} \Phi_1(\mathcal{A}(x_0,t))\,\mathrm{d}t &= \int_{x_0}^{\mathcal{A}_0} s^k \Phi_1(s)\,\mathrm{d}s \\
&= \int_{x_0}^{\mathcal{A}_0} s^k \varphi_1(s)\,\mathrm{d}s - \frac{1}{b}\varphi_0(\mathcal{A}_0) + \frac{1}{b}\varphi_0(x_0), \\
\frac{1}{b}\int_0^{t_0} \Phi_2(\mathcal{B}(x_0,t))\,\mathrm{d}t &= \int_{x_0}^{\mathcal{B}_0} s^k \Phi_2(s)\,\mathrm{d}s \\
&= \int_{x_0}^{\mathcal{B}_0} s^k \varphi_1(s)\,\mathrm{d}s - \frac{1}{a}\varphi_0(\mathcal{B}_0) + \frac{1}{a}\varphi_0(x_0).
\end{aligned}
$$

Substituting these expressions into (3.9) and changing the symbol $(x_0, t_0)$ to $(x, t)$, we obtain in $\Omega$ the formula (3.4).

Similarly, we can get (3.4) for $(x,t) \in H_+ \cap \{(x,t) \in \mathbb{R}^2 \mid x < 0\}$.

Then, a simple computation yields that for $(x,t) \in H_+$ and $x \neq 0$

$$
u_x(x,t) = \frac{ab}{b-a}x^k\Big[\varphi_1(\mathcal{A}) - \varphi_1(\mathcal{B}) + \frac{1}{a}\varphi_0'(\mathcal{B})\mathcal{B}^{-k} - \frac{1}{b}\varphi_0'(\mathcal{A})\mathcal{A}^{-k}\Big], \tag{3.10}
$$

$$
u_t(x,t) = \frac{ab}{b-a}\Big[\frac{1}{a}\varphi_1(\mathcal{A}) - \frac{1}{b}\varphi_1(\mathcal{B}) + \frac{1}{ab}\varphi_0'(\mathcal{B})\mathcal{B}^{-k} - \frac{1}{ab}\varphi_0'(\mathcal{A})\mathcal{A}^{-k}\Big] \tag{3.11}
$$

and

$$
u_{tt}(x,t) = \frac{ab}{b-a}\Big[\frac{1}{a^2}\varphi_1'(\mathcal{A})\mathcal{A}^{-k} - \frac{1}{b^2}\varphi_1'(\mathcal{B})\mathcal{B}^{-k} + \frac{1}{ab^2}r'(\mathcal{B})\mathcal{B}^{-k} - \frac{1}{a^2 b}r'(\mathcal{A})\mathcal{A}^{-k}\Big], \tag{3.12}
$$

where $r(x) = \varphi_0'(x)/x^k$ for $x \neq 0$. Moreover, we can obtain the expressions of $u_{xx}$ and $u_{xt}$ in $H_+ - \mathcal{C}_0$ as well. From (3.11) and since $\varphi_0 \in C^1(\mathbb{R}), \varphi_1 \in C^0(\mathbb{R})$ and $u_t$ is continuous at the origin (recall $u \in C^1(\overline{H_+})$), we get

$$
\lim_{\substack{(x,t)\to(0,0)\\ (x,t)\in H_+,\, x\neq 0}} \left(\varphi_0'(\mathcal{B})\mathcal{B}^{-k} - \varphi_0'(\mathcal{A})\mathcal{A}^{-k}\right) = 0,
$$

which is equivalent to the condition (3.2) because of arbitrariness of the difference $|\mathcal{A} - \mathcal{B}| \neq 0$ when $(x,t) \in H_+ - \mathcal{C}_0$. Hence $u_t$ with expression (3.11) and $u_x/x^k$ where $u_x$ is expressed by (3.10) are both in $C^1(\overline{H_+})$, and therefore so are $v_1$ and $v_2$. It follows that

$$
\begin{aligned}
\Phi_1\left(\sqrt[k+1]{\frac{k+1}{a}\Big(t_0+\frac{a}{k+1}x_0^{k+1}\Big)}\right) &= v_1\Big(0,t_0+\frac{a}{k+1}x_0^{k+1}\Big)\\
&= v_1\Big(0,t_0+\frac{a}{k+1}(-x_0)^{k+1}\Big)\\
&= \Phi_1\left(-\sqrt[k+1]{\frac{k+1}{a}\Big(t_0+\frac{a}{k+1}(-x_0)^{k+1}\Big)}\right),\\
\Phi_2\left(\sqrt[k+1]{\frac{k+1}{b}\Big(t_0+\frac{b}{k+1}x_0^{k+1}\Big)}\right) &= v_2\Big(0,t_0+\frac{b}{k+1}x_0^{k+1}\Big)\\
&= v_2\Big(0,t_0+\frac{b}{k+1}(-x_0)^{k+1}\Big)\\
&= \Phi_2\left(-\sqrt[k+1]{\frac{k+1}{b}\Big(t_0+\frac{b}{k+1}(-x_0)^{k+1}\Big)}\right),
\end{aligned}
$$

i.e., $\Phi_1$ and $\Phi_2$ are even functions because of arbitrariness of $x_0, t_0 \in \mathbb{R}$.

The above shows the necessity of the conditions (3.2), (3.3), and the uniqueness of the solution.

On the other hand, if the functions $\varphi_0$, $\varphi_1$ satisfy the conditions (3.1)–(3.3) and the function $u$ is defined in $H_+$ by the formula (3.4), then it is trivial that $u \in C^0(\overline{H_+})$ and that $u(x,0) = \varphi_0(x)$ for $x \in \mathbb{R}$.

It follows from (3.1), (3.2), (3.10) and (3.11) that both $u_x$ and $u_t$ belong to the class $C^0(\overline{H_+})$, i.e., $u \in C^1(\overline{H_+})$, and that $u_t(x,0) = \varphi_1(x)$ for $x \in \mathbb{R}$.

Differentiating (3.11) again with respect to the variable $t$ we get the expression (3.12) of $u_{tt}$ in $H_+$, and we see easily from (3.1) that $u_{tt} \in C^0(H_+)$. Similarly we can get that $u_{xx} \in C^0(H_+), u_{xt} \in C^0(H_+)$, and combining these with $u \in C^1(\overline{H_+}), u_{tt} \in C^0(H_+)$, we conclude that $u \in C^1(\overline{H_+}) \cap C^2(H_+)$.

Finally, it is not difficult to verify that $L_{a,b}u = 0$ for $(x,t) \in H_+$. It follows by combining all which was obtained above that the function $u$ defined by (3.4) is the solution of Problem $(P_1)$ in $H_+$ under the conditions (3.1)–(3.3). This completes the proof of Theorem 1. □

For $ab < 0$ we have

**Theorem 2.** *Suppose that functions $\varphi_0$, $\varphi_1$ and $\psi_0$ satisfy* (3.1) *and*

$$\psi_0 \in C^2(\overline{\mathbb{R}_+}). \tag{3.13}$$

*Then problem $(P_2)$ in $H_+$ with $a > 0 > b$ has a solution $u$ belonging to $C^1(\overline{H_+}) \cap C^2(H_+)$ and satisfying (3.5) if and only if the compatibility conditions (3.2), (3.6) and*

$$\varphi_0(0) = \psi_0(0), \quad \varphi_1(0) = \psi_0'(0), \tag{3.14}$$

$$\lim_{\substack{x\to+0\\x\neq0}} \frac{\Phi_2'(x)}{x^k} = b\Psi_2'(0) \tag{3.15}$$

*are satisfied, where the function $\psi_1$ occurring in the definition of the function $\Psi_2$ is defined by the formula*

$$\Phi_1(x) = \Psi_1\Big(\frac{a}{k+1}x^{k+1}\Big) \quad \text{for } x \neq 0. \tag{3.16}$$

*Then the solution is unique and can be expressed for $(x,t) \in \overline{\Omega_b^-}$ by the formula (3.4) and for $(x,t) \in \Omega_b^+$ by*

$$\begin{aligned} u(x,t) = \frac{ab}{b-a}\Bigg[\int_0^{\mathcal{A}} s^k\varphi_1(s)\,\mathrm{d}s + \frac{1}{ab}\int_0^{B}\psi_1(s)\,\mathrm{d}s + \frac{1}{a}\varphi_0(0) - \frac{1}{b}\varphi_0(\mathcal{A}) \\ + \frac{1}{b}\psi_0(0) - \frac{1}{b}\psi_0(B)\Bigg]. \end{aligned} \tag{3.17}$$

*In the case of $a < 0 < b$ we have a similar result.*

P r o o f o f T h e o r e m 2. We shall prove the theorem only for $a > 0 > b$; the proof is identical if $a < 0 < b$.

Let $u \in C^1(\overline{H_+}) \cap C^2(H_+)$ be a solution of Problem $(P_2)$ in $H_+$ with $a > 0 > b$ which satisfies (3.5). Then we can obtain the formula (3.4) in $\Omega_b^-$ by using the same method as in the proof of Theorem 1. Therefore the formulas (3.10)–(3.12) for $u_x$, $u_t$ and $u_{tt}$ in $\Omega_b^-$ follow and we can derive (3.2) because $u_t$ is continuous at the origin.

Now we define

$$\psi_1(t) = \lim_{\substack{x\to0\\x\neq0}} \frac{u_x(x,t)}{x^k}, \quad \forall t \geqslant 0,$$

and we have $\psi_1 \in C^0(\overline{\mathbb{R}_+})$ because of (3.5). Let $(x,t) \in \Omega \cap \Omega_b^+$. The characteristic curve of the family $C_1$ through the point $(x,t)$ intersects, respectively, the positive $x$-axis and the $t$-axis at $(\mathcal{A},0)$ and at $(0,A)$, while the characteristic curve of the family $C_2$ through the same point intersects the $t$-axis at $(0,B)$ and the $x$-axis at no points. According to Lemma, we have

$$(3.18)\qquad \begin{aligned} \left(u_t - \frac{1}{bx^k}u_x\right)\bigg|_{(x,t)} &= v_1(x,t) = v_1(\mathcal{A},0) = \Phi_1(\mathcal{A}), \\ \left(u_t - \frac{1}{ax^k}u_x\right)\bigg|_{(x,t)} &= v_2(x,t) = v_2(0,B) = \Psi_2(B), \end{aligned}$$

so

$$u_t(x,t) = \frac{ab}{b-a}\Big[\frac{1}{a}\Phi_1(\mathcal{A}) - \frac{1}{b}\Psi_2(B)\Big].$$

Therefore, integration yields

$$\begin{aligned} u(x,t) &= \int_{-X^{(b)}}^{t} u_t(x,\tau)\,\mathrm{d}\tau + u(x,-X^{(b)}) \\ &= \frac{ab}{b-a}\int_{-X^{(b)}}^{t}\left[\frac{1}{a}\Phi_1\left(\sqrt[k+1]{\frac{k+1}{a}\Big(\tau + \frac{a}{k+1}x^{k+1}\Big)}\right)\right. \\ &\quad \left. - \frac{1}{b}\Psi_2\Big(\tau + \frac{b}{k+1}x^{k+1}\Big)\right]\mathrm{d}\tau \\ &\quad + \frac{ab}{b-a}\left[\int_0^{\lambda x} s^k\varphi_1(s)\,\mathrm{d}s + \frac{1}{a}\varphi_0(0) - \frac{1}{b}\varphi_0(\lambda x)\right] \\ &\quad \text{(here we have used the form given by (3.4) for } u(x,-X^{(b)})) \\ &= \frac{ab}{b-a}\left[\int_0^{\mathcal{A}} s^k\varphi_1(s)\,\mathrm{d}s + \frac{1}{ab}\int_0^{B}\psi_1(s)\,\mathrm{d}s + \frac{1}{a}\varphi_0(0)\right. \\ &\quad \left. - \frac{1}{b}\varphi_0(\mathcal{A}) + \frac{1}{b}\psi_0(0) - \frac{1}{b}\psi_0(B)\right], \end{aligned}$$

i.e., (3.17) holds for $(x,t) \in \Omega \cap \Omega_b^+$.

Now we have $(-x,t) \in (H_+ - \Omega) \cap \Omega_b^+$ when $(x,t) \in \Omega \cap \Omega_b^+$. We obtain as above

$$(3.19)\qquad \begin{aligned} \left(u_t - \frac{1}{bx^k}u_x\right)\bigg|_{(-x,t)} &= v_1(-x,t) = v_1(\mathcal{A},0) = \Phi_1(\mathcal{A}), \\ \left(u_t - \frac{1}{ax^k}u_x\right)\bigg|_{(-x,t)} &= v_2(-x,t) = v_2(0,B) = \Psi_2(B), \end{aligned}$$

so

$$u_t(-x,t) = \frac{ab}{b-a}\Big[\frac{1}{a}\Phi_1(\mathcal{A}) - \frac{1}{b}\Psi_2(B)\Big],$$

where $\mathcal{A} = \mathcal{A}(-x,t) = -\sqrt[k+1]{\frac{k+1}{a}\Big(t+\frac{a}{k+1}(-x)^{k+1}\Big)} = -\mathcal{A}(x,t)$. Thus, it is also verified that (3.17) holds for $(x,t) \in (H_+ - \Omega) \cap \Omega_b^+$, and therefore that (3.17) holds for $(x,t) \in \Omega_b^+$.

It follows from (3.18), (3.19), $v_1 \in C^0(\overline{H_+})$ (because of $u_t \in C^0(\overline{H_+}), u_x/x^k \in C^0(\overline{H_+})$) and Lemma that $v_1(x,t) = v_1(0,A) = v_1(-x,t)$, i.e.,

$$\begin{aligned}\Psi_1\Big(t+\frac{a}{k+1}x^{k+1}\Big) &= \frac{1}{a}\Phi_1\left(\sqrt[k+1]{\frac{k+1}{a}\Big(t+\frac{a}{k+1}x^{k+1}\Big)}\right)\\ &= \frac{1}{a}\Phi_1\left(-\sqrt[k+1]{\frac{k+1}{a}\Big(t+\frac{a}{k+1}(-x)^{k+1}\Big)}\right)\\ &= \frac{1}{a}\Phi_1\left(-\sqrt[k+1]{\frac{k+1}{a}\Big(t+\frac{a}{k+1}x^{k+1}\Big)}\right),\end{aligned}$$

so that (3.6) and (3.16) hold.

It remains only (3.15) to be verified.

Differentiating (3.17) we see that for $(x,t) \in \Omega_b^+$

$$u_x(x,t) = \frac{ab}{b-a}x^k\Big[\varphi_1(\mathcal{A}) + \frac{1}{a}\psi_1(B) - \frac{1}{b}\varphi_0'(\mathcal{A})\mathcal{A}^{-k} - \psi_0'(B)\Big], \tag{3.20}$$

$$u_t(x,t) = \frac{ab}{b-a}\Big[\frac{1}{a}\varphi_1(\mathcal{A}) + \frac{1}{ab}\psi_1(B) - \frac{1}{ab}\varphi_0'(\mathcal{A})\mathcal{A}^{-k} - \frac{1}{b}\psi_0'(B)\Big] \tag{3.21}$$

and

$$u_{tt}(x,t) = \frac{ab}{b-a}\Big[\frac{1}{a^2}\varphi_1'(\mathcal{A})\mathcal{A}^{-k} + \frac{1}{ab}\psi_1'(B) - \frac{1}{a^2b}r'(\mathcal{A})\mathcal{A}^{-k} - \frac{1}{b}\psi_0''(B)\Big], \tag{3.22}$$

where $r(x) = \varphi_0'(x)x^{-k}$. Moreover, we can obtain the expressions of $u_{xx}$ and $u_{xt}$ in $\Omega_b^+$. Then the following series of equivalent relations is obvious (it should be noticed that we use different expressions (3.22) and (3.12) of $u_{tt}$ respectively in the limit process $(x,t) \to +(x_0,t_0)_b$ and $(x,t) \to -(x_0,t_0)_b$ )

$$\begin{aligned}u \in C^2(H_+) &\Longleftrightarrow u_{tt} \in C^0(H_+)\\ &\Longleftrightarrow \lim_{(x,t)\to+(x_0,t_0)_b} u_{tt}(x,t) = \lim_{(x,t)\to-(x_0,t_0)_b} u_{tt}(x,t)\\ &\Longleftrightarrow \lim_{\substack{\mathcal{B}\to 0\\ \mathcal{B}\neq 0}}\Big[\varphi_1'(\mathcal{B}) - \frac{1}{a}r'(\mathcal{B})\Big]\mathcal{B}^{-k} = \lim_{B\to+0} b\Big(\psi_0''(B) - \frac{1}{a}\psi_1'(B)\Big)\\ &\Longleftrightarrow \lim_{\substack{x\to 0\\ x\neq 0}} \frac{\Phi_2'(x)}{x^k} = b\Psi_2'(0),\end{aligned}$$

where $(x_0, t_0) \in C_0^{(b)}$, and the symbols $(x,t) \to \pm(x_0,t_0)_b$ mean respectively $(x,t) \in \Omega_b^+$, $(x,t) \to (x_0,t_0)$ and $(x,t) \in \Omega_b^- - C_0^{(b)}$, $(x,t) \to (x_0,t_0)$. Thus, (3.15) is proved.

Now, we suppose that (3.2), (3.6), (3.14) hold, and that the function $\psi_1$ occurring in the definition of the function $\Psi_2$ is defined by the formula (3.16) and satisfies (3.15). Then it follows from (3.2), (3.6), (3.14) and (3.16) that $\psi_1 \in C^1(\overline{\mathbb{R}_+})$ and

$$\lim_{\substack{x\to 0\\ x\neq 0}} \frac{\varphi_0'(x)}{x^k} = \psi_1(0). \tag{3.23}$$

Let the function $u$ be defined in $\Omega_b^-$ by (3.4) and in $\Omega_b^+$ by (3.17). Then we obtain $u_x, u_t$ and $u_{tt}$ in $\Omega_b^-$ expressed by (3.10), (3.11) and (3.12), and in $\Omega_b^+$ expressed by (3.20), (3.21) and (3.22), respectively. We can also derive the formulas for $u_{xx}$ and $u_{xt}$ in $\Omega_b^-$ and $\Omega_b^+$ from (3.4) and (3.17).Then it is obvious that $u \in C^0(\overline{H_+})$ and $u(x,0) = \varphi_0(x)$ for $x \in \mathbb{R}$, and that for $t \geqslant 0$

$$\begin{aligned}
&u(0,t) = \psi_0(t)\\
\Longleftrightarrow\ &\psi_0(t) = \frac{ab}{b-a}\Big[\int_0^{T_a} s^k\varphi_1(s)\,\mathrm{d}s + \frac{1}{ab}\int_0^t \psi_1(s)\,\mathrm{d}s + \frac{1}{a}\varphi_0(0)\\
&\qquad - \frac{1}{b}\varphi_0(T_a) + \frac{1}{b}\psi_0(0) - \frac{1}{b}\psi_0(t)\Big] \qquad\qquad (*)\\
\Longleftrightarrow\ &\psi_0'(t) = \varphi_1(T_a) + \frac{1}{b}\psi_1(t) - \frac{1}{b}\varphi_0'(T_a)T_a^{-k}
\end{aligned}$$

(since the values at $t = 0$ of both sides of $(*)$ are equal because of $\varphi_0(0) = \psi_0(0)$)

$$\begin{aligned}
\Longleftrightarrow\ &\varphi_1(x) - \frac{1}{b}\varphi_0'(x)x^{-k} = \psi_0'\Big(\frac{a}{k+1}x^{k+1}\Big) - \frac{1}{b}\psi_1\Big(\frac{a}{k+1}x^{k+1}\Big)\\
&\qquad\qquad\qquad\qquad\qquad\qquad \Big(\text{taking } t = \frac{a}{k+1}x^{k+1} \text{ for } x \neq 0\Big)\\
\Longleftrightarrow\ &(3.16).
\end{aligned}$$

Also, it is obvious from the expressions (3.10), (3.20) and (3.11), (3.21) of $u_x$, $u_t$ that $u \in C^1(\Omega_b^- - C_0^{(b)}) \cap C^1(\Omega_b^+)$ and $u_t(x,0) = \varphi_1(x)$ for $x \in \mathbb{R}$.

Moreover, we have the equivalent relations

$$\begin{aligned}
&u \in C^1(\overline{H_+})\\
\Longleftrightarrow\ &\lim_{(x,t)\to+(x_0,t_0)_b} \mathrm{D}u(x,t) = \lim_{(x,t)\to-(x_0,t_0)_b} \mathrm{D}u(x,t)\\
&\qquad\qquad (\text{where } \mathrm{D}u \text{ stands for any one of the derivatives } u_x \text{ and } u_t)\\
\Longleftrightarrow\ &\varphi_1(0) - \frac{1}{a}r(0) = \psi_0'(0) - \frac{1}{a}\psi_1(0),
\end{aligned}$$

and the last equality follows from $\varphi_1(0) = \psi_0'(0)$ and $r(0) = \psi_1(0)$. Thus, we can derive easily from (3.10), (3.15) and (3.16) that

$$\lim_{\substack{(x,\tau)\to(0,t)\\(x,\tau)\in H_+, x\neq 0}} \frac{u_x(x,\tau)}{x^k} = \psi_1(t), \qquad \text{for } t \geqslant 0,$$

so (3.5) holds. As is seen in the first part of the proof, (3.15) guarantees the fact $u \in C^2(H_+)$. Then $u$ is in the class $C^1(\overline{H}_+) \cap C^2(H_+)$ and satisfies $L_{a,b}u = 0$ in $H_+$ as is verified easily. Therefore $u$ is a solution of Problem $(P_2)$ in $H_+$ with $a > 0 > b$. This completes the proof of Theorem 2. □

**Theorem 3.** *Suppose that the functions $\varphi_0$, $\varphi_1$ and $\psi_1$ satisfy* (3.1) *and*

$$\psi_1 \in C^1(\overline{\mathbb{R}_+}). \tag{3.24}$$

*Then problem $(P_3)$ in $H_+$ with $a > 0 > b$ has a solution $u$ belonging to $C^1(\overline{H_+}) \cap C^2(H_+)$ if and only if the compatibility conditions* (3.6), (3.15) *and* (3.23) *are satisfied, where the function $\psi_0$ occurring in the definition of the function $\Psi_2$ is defined by* (3.16) *and the equality*

$$\psi_0(0) = \varphi_0(0). \tag{3.25}$$

*Then the solution is unique and can be expressed by the same formulas as in Theorem 2 with $a > 0 > b$.*

*In the case of $a < 0 < b$ we have a similar result.*

The outline for the proof of Theorem 3 is similar to that of Theorem 2. We only need to point out, for example, the fact that (3.16), (3.23) and (3.25) imply

$$\psi_0'(0) = \varphi_1(0).$$

For $a < 0$, $b < 0$ and $a \neq b$, we have

**Theorem 4.** *Suppose that the functions $\varphi_0$, $\varphi_1$, $\psi_0$ and $\psi_1$ satisfy* (3.1), (3.13) *and* (3.24). *Then problem $(P_4)$ in $H_+$ with $a < b < 0$ has a solution $u$ belonging to $C^1(\overline{H_+}) \cap C^2(H_+)$ if and only if the compatibility conditions* (3.14), (3.15), (3.23) *and*

$$\lim_{\substack{x\to 0\\ x\neq 0}} \Phi_1'(x)x^{-k} = a\Psi_1'(0) \tag{3.26}$$

*hold.*

*Then the solution is unique and can be expressed in $\overline{\Omega_b^-}$ by (3.4), in $\Omega_b^+ \cap \overline{\Omega_a^-}$ by (3.17) and in $\Omega_a^+$ by*

$$\frac{ab}{b-a}\left[\frac{1}{ab}\int_A^B \psi_1(s)\,\mathrm{d}s + \frac{1}{a}\psi_0(A) - \frac{1}{b}\psi_0(B)\right]. \tag{3.27}$$

*In the case of $b < a < 0$ we have a similar result.*

Proof of Theorem 4. It is clear that we only need to prove the theorem with $a < b < 0$. Let $u$ be a solution of Problem $(P_4)$ in $H_+$ with $a < b < 0$ which belongs to $C^1(\overline{H_+}) \cap C^2(H_+)$. Then (3.14) holds because $u \in C^1(\overline{H_+})$. Let $x \neq 0$ and $(x,t) \in C_0^{(b)}$, then Lemma yields

$$u_t(x,t) - \frac{1}{bx^k}u_x(x,t) = v_1(x,t) = \varphi_1(\mathcal{A}) - \frac{1}{b\mathcal{A}^k}\varphi_0'(\mathcal{A}).$$

Letting $(x,t) \to (0,0)$ along $C_0^{(b)}$ and using the definition of $\psi_1$ we see that (3.23) holds.

Moreover, the same procedure as shown in the proof of Theorem 2 yields that $u$ possesses the forms in $H_+$ described in the theorem, i.e., the solution of Problem $(P_4)$ is unique. Therefore, we have in $\Omega_a^+$

$$u_x(x,t) = \frac{ab}{b-a}x^k\left[\frac{1}{a}\psi_1(B) - \frac{1}{b}\psi_1(A) + \psi_0'(A) - \psi_0'(B)\right],$$

$$u_t(x,t) = \frac{ab}{b-a}\left[\frac{1}{ab}\psi_1(B) - \frac{1}{ab}\psi_1(A) + \frac{1}{a}\psi_0'(A) - \frac{1}{b}\psi_0'(B)\right]$$

and

$$u_{tt}(x,t) = \frac{ab}{b-a}\left[\frac{1}{ab}\psi_1'(B) - \frac{1}{ab}\psi_1'(A) + \frac{1}{a}\psi_0''(A) - \frac{1}{b}\psi_0''(B)\right].$$

Thus, checking the continuity of $u_{tt}$ at $C_0^{(b)}$ and at $C_0^{(a)}$ respectively we obtain (3.15) and (3.26).

Conversely, it is not difficult to show under all conditions of the theorem for $a < b < 0$ that the function $u$ expressed respectively by (3.4), (3.17) and (3.27) for the subdomains $\Omega_b^-$, $\Omega_b^+ \cap \Omega_a^-$ and $\Omega_a^+$ of $H_+$ belongs to $C^1(\overline{H_+}) \cap C^2(H_+)$ and that $u$ is a solution of Problem $(P_4)$ in $H_+$ with $a < b < 0$. □

If we study Problems $(P_2)$ or $(P_3)$ in $H_+$ with $a < 0$, $b < 0$ and $a \neq b$, we find from Theorem 4 that their solutions exist if some compatibility conditions are satisfied and they are not unique at that time. In fact, we have

**Corollary 1.** *Suppose that the functions $\varphi_0$, $\varphi_1$ and $\psi_0$ satisfy* (3.1) *and* (3.13). *Then, if the solutions in $C^1(\overline{H_+}) \cap C^2(H_+)$ of problem $(P_2)$ in $H_+$ with $a < 0$, $b < 0$ and $a \neq b$ exist, they must not be unique. Moreover, such solutions exist if and only if the compatibility conditions* (3.2), (3.14) *and*

$$\text{the limits } \lim_{x \to 0} \frac{\varphi_0''(x)}{x^k} \text{ and } \lim_{\substack{x \to 0 \\ x \neq 0}} \Big(\frac{\varphi_1(x)}{x^k}\Big)' x^{-k} \text{ exist,} \tag{3.28}$$

$$\psi_0''(0) = \lim_{\substack{x \to 0 \\ x \neq 0}} \Big(\frac{a+b}{ab}\varphi_0'(x) + \frac{1}{ab}\frac{\varphi_1(x)}{x^k}\Big)' x^{-k} \tag{3.29}$$

*are satisfied. In this case, if a function $\psi_1 \in C^1(\overline{\mathbb{R}_+})$ satisfies* (3.23) *and*

$$\psi_1'(0) = \lim_{\substack{x \to 0 \\ x \neq 0}} \frac{\varphi_0''(x)}{x^k}, \tag{3.30}$$

*then the function $u$ expressed in the conclusion of Theorem 4 is a solution of problem $(P_2)$ in $H_+$ with $a < 0$, $b < 0$ and $a \neq b$.*

For the proof we only need to notice that the compatibility conditions (3.15) and (3.26) in Theorem 4 are equivalent to (3.29) and (3.30), and that there are infinitely many functions $\psi_1 \in C^1(\overline{\mathbb{R}_+})$ satisfying (3.23) and (3.30).

**Corollary 2.** *Suppose that the functions $\varphi_0$, $\varphi_1$ and $\psi_1$ satisfy* (3.1) *and* (3.24). *Then, if the solutions in $C^1(\overline{H_+}) \cap C^2(H_+)$ of problem $(P_3)$ in $H_+$ with $a < 0$, $b < 0$ and $a \neq b$ exist, they must not be unique. Moreover, such solutions exist if and only if the compatibility conditions* (3.23), (3.28) *and* (3.30) *are satisfied. In this case, if a function $\psi_0 \in C^2(\overline{\mathbb{R}_+})$ satisfies* (3.14) *and* (3.29), *then the function $u$ expressed in the conclusion of Theorem 4 is a solution of problem $(P_3)$ in $H_+$ with $a < 0$, $b < 0$ and $a \neq b$.*

One part of the proof follows from Theorem 4 and the fact that the conditions (3.15), (3.26) are equivalent to (3.29) and (3.30), while the other part for non-uniqueness comes from the fact that there are infinitely many functions $\psi_0 \in C^2(\overline{\mathbb{R}_+})$ satisfying (3.14) and (3.29).

For the case in the first quadrant we can also obtain correspondingly four theorems and two corollaries which are almost the same as those for the upper half-plane. What it is sufficient to notice in this case is that the data $\varphi_1$ and $\varphi_2$ satisfy the corresponding conditions on $\overline{\mathbb{R}_+}$ only and that all the limits at the origin are one-sided. Thus, it is clear that the proofs of these unstated theorems and corollaries are just parts of proofs of Theorems 1–4 and Corollaries 1–2.

In the end, we point out that the properties of the operators $L_{a,b}$ studied in this paper are different from those of operators, such as $L \equiv \partial_{xx} - \partial_{tt}$, which are of principal type because all $L_{a,b}$ have double characteristics at any point on the $t$-axis whereas $L$ has just a simple one everywhere. Also, we notice that different shapes of characteristics for pairs $(a, b)$ with various sign yield different well-posed problems, namely, the solution is fully determined by the initial conditions for $a > 0$, $b > 0$, and it is necessary to add one boundary condition for $ab < 0$ and two boundary conditions for $a < 0$, $b < 0$.

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