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SOME PROPERTIES OF MONOTONE TYPE MULTIVALUED OPERATORS IN BANACH SPACES

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Summary. Some properties of monotone type multivalued operators including accretive operators and the duality mapping are studied in connection with the structure of Banach spaces.

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1. Introduction

Lately, the theory of monotone and accretive operators has been intensively studied in connection with the solvability of nonlinear partial, ordinary differential and the Hammerstein and Uryson nonlinear integral equations.

The properties of monotone type operators were studied by many authors, see for instance Phelps [23]. Fabian [7] proved that if X is a Banach space which admits an equivalent norm on X such that X^* is a rotund and an (H)-space in its dual norm, and $T\colon X \to 2^{X^*}$ is a maximal monotone mapping with $D = \operatorname{int} D(T) \neq \emptyset$, then the set $C(T)$ of all points of D where T is simultaneously singlevalued and norm to norm upper semicontinuous is a dense G_δ subset of D. Using the method of selection, he in fact proved much more, since he characterized the set $C(T)$ by the set of points of continuity of the function of the minimum modulus of T and by the set of points of continuity of an arbitrary selection of T (see also [18] for a simple proof of Fabian's theorem).

Recall that Kenderov [15] proved the following important result: If X is a Banach space which admits an equivalent norm such that its dual norm on X^* is rotund and $T: X \to 2^{X^*}$ is maximal monotone with $D = \operatorname{int} D(T) \neq \emptyset$, then there exists a dense G_δ-subset $C(f)$ of D such that T is singlevalued at the points of $C(f)$, where $C(f)$ is the set of all points of D where the function $f: D \to \mathbf{R}_+$ defined by $f(u) = \min\{\|u^*\|\}: u^* \in T(u)f\}$, $u \in D$, is continuous. The proof depends on the following facts: (i) f is lower semicontinuous on D; (ii) if $u_0 \in C(f)$, then $f(u_0) = \|u^*\|$ for every $u^* \in T(u_0)$; (iii) the convex set $T(u_0)$ has at most one point of the minimum norm (compare also [16]).

We prove (Theorem 1) that if X and X^* are Fréchet smooth, then the set of all points of W where a maximal accretive mapping $A: X \to 2^X$ with $W = \operatorname{int} D(A) \neq \emptyset$ is singlevalued and norm to norm upper semicontinuous is equal to a dense G_δ subset $C(f)$ of W where the function of the minimum modulus of the operator A is continuous. A more general version of some lemmas allows to generalize the result of Theorem 1 [19], as we need not assume that X is uniformly Fréchet smooth. A similar result to that of Theorem 1 is stated in Theorem 2 under rather different conditions. The proofs of Lemmas 1–4 rely on arguments similar to those of (i), (ii) and (iii) (see Kenderov [15] and Fitzpatrick [9]). The third result deals with the topological properties of the duality mapping in nonreflexive Banach spaces having smooth dual X^*. Recall that the concept of the duality mapping is very useful in the theory of monotone type operators, solvability of operator equations, geometry of Banach spaces and the fixed point theory (see [1], [2] and [14]). The fourth assertion is connected with Theorem 3 [22] concerning the problem, when the Gâteaux derivative of a weak* lower semicontinuous function, defined on X^*, is weak* continuous. Using the argument of Davis and Johnson [3] we show that X is reflexive if and only if X admits an equivalent norm such that the weak* and weak topologies coincide on its second dual sphere of X^{**} (compare also Giles, Gregory and Sims [12] and Remark 2).

2. Definitions and notation

Let X be a real normed linear space, X^* and X^{**} its dual and bidual, $\langle\ ,\ \rangle$ the pairing between X and X^*, $B_r(u)$ the closed ball centered at $u \in X$ and with a radius $r > 0$, $S_r(u)$ its sphere. For a given set $M \subset X$, $\operatorname{int} M$ denotes the interior of M. By $\tau: X \to X^{**}$ we mean the canonical mapping while $\hat{A}$ denotes the image of $A \subset X$ under τ in X^{**}. By $\mathbf{R}$, $\mathbf{R}_+$ we denote the set of all real and nonnegative numbers, respectively; by the symbols $\sigma(X, X^*)$ and $\sigma(X^*, X)$ we mean the weak and the weak* topology on X and X^*, respectively. We use the standard notions for rotund (i.e. strictly convex) and uniformly rotund normed linear spaces and the Gâteaux and Fréchet derivatives of functionals (see [1]). Recall that X is said to be (i) smooth (Fréchet smooth), if the norm of X is Gâteaux (Fréchet) differentiable on $S_1(0)$; (ii) uniformly Fréchet smooth, if the norm of X is uniformly Fréchet differentiable on $S_1(0)$; (iii) weakly locally uniformly rotund (WLUR [15]), whenever given $(x_n) \subset S_1(0)$ and $x_0 \in S_1(0)$ for which $\frac{\|x_n+x_0\|}{2} \to 1$, we have $x_n \to x_0$ weakly; (iv) and (H)-space (or X has the Kadec-Klee property), if for every $(u_n) \subset X$ such that $u_n \to u$ weakly in X, $u \in X$, $\|u_n\| \to \|u\|$, we have $u_n \to u$ in the norm of X, (v) a dual Banach space, if there is a Banach space Z such that $X = Z^*$ (in the sense of topology and the norm). We shall say that X^* is a (w^*H)-space, if for every net $(u^*_\alpha) \subset X^*$, $u^*_\alpha \to u^*$ weakly* in X^*, $u^* \in X^*$ and $\|u^*_\alpha\| \to \|u^*\|$ there is $u^*_\alpha \to u^*$ in the norm of X^*. Let E, G, be topological spaces, $A\colon E \to 2^G$ a multivalued mapping, $D(A) = \{u \in E\colon A(u) \neq \emptyset\}$ its domain, $G(A) = \{(u,v) \in E \times G\colon v \in A(u) \text{ for some } u \in D(A)\}$ its graph in the space $E \times G$ and $R(A)$ its image in G. A mapping $A\colon E \to 2^G$ is said to be (i) upper semicontinuous at $u_0 \in D(A)$, if for every open subset W of G such that $A(u_0) \subset W$ there exists an open neighborhood V of u_0 such that $A(u) \subset W$ for every $u \in V \cap D(A)$; (ii) locally bounded at $u_0 \in D(A)$, if there exists a neighborhood U of u_0 such that $A(u) = \bigcup\{A(u)\colon u \in U \cap D(A)\}$ is bounded in X; (iii) closed at $u_0 \in D(A)$, if (u_α) is a net in $D(A)$, $u_\alpha \to u_0$, $v_\alpha \in A(u_\alpha)$, $v_\alpha \to v_0$ implies that $v_0 \in A(u_0)$. Let X be a real normed linear space. The duality mapping $J\colon X \to w^{X^*}$ is defined by $J(u) = \{u^* \in X^*\colon \langle u^*, u\rangle = \|u\|^2,\ \|u^*\| = \|u\|\}$ for every $u \in X$. Recall [23] that J is norm to weak* upper semicontinuous on X and that $J(u)$ is a nonempty convex and weakly* compact subset of X^* for every $u \in X$. Moreover, X is smooth (Fréchet smooth) if and only if J is singlevalued (continuous) on X (see [2]).

By J^* and J^{**}, we denote the duality mappings on X^* and X^{**}, respectively. Recall that a mapping $A\colon X \to 2^X$ is said to be (i) accretive [14], if $I + \lambda A$, where I is the identity mapping in X, is expansive for every $\lambda > 0$, i.e. for every u, $v \in D(A)$ and every $x \in A(u)$, $y \in A(v)$, we have $\|(u-v) + \lambda(x-y)\| \geqslant \|u-v\|$ for every $\lambda > 0$, or equivalently, if for every u, $v \in D(A)$ and every $x \in A(u)$, $y \in A(v)$ there exists $x^* \in J(u-v)$ such that $\langle x-y, x^*\rangle \geqslant 0$; (ii) maximal accretive ([14]), if A is accretive and if given an element $(u,x) \in X \times X$ such that for every $v \in D(A)$ and $y \in A(v)$ there exists a point $x^* \in J(u-v)$ such that $\langle x-y, x^*\rangle \geqslant 0$, then $u \in D(A)$ and $x \in A(u)$.

Let X be a normed linear space, Y a dual Banach space (i.e. there exists a normed linear space Z such that $Y = Z^*$). We shall say that a mapping $T\colon X \to 2^Y$ has the property (P) at $u_0 \in D(T)$ ([18]), if the following condition is satisfied: If $(u_\alpha) \subset D(T)$ is a net and $u_\alpha \to u_0$ in the norm of X, $y_\alpha \in T(u_\alpha)$ is such that $\|y_\alpha\| \leqslant C$ for some constant $C > 0$, then there exists a subnet (y_{α_j}) of (y_α) with the $\sigma(Z^*, Z)$-limit point y_0 such that $y_0 \in T(u_0)$.

3. Multivalued monotone type operators and the structure of Banach spaces

Let X be a normed linear space, $A\colon X \to 2^X$ a mapping such that $W = \operatorname{int} D(A) \neq \emptyset$. Define a function $f\colon W \to \mathbf{R}_+ \cup \{\infty\}$ by $f(u) = \operatorname{int}\{\|x\|: x \in A(u)\}$, $u \in W$.

Lemma 1. *Let X be a dual Banach space (i.e. $X = Z^*$ for some Banach space Z), $A\colon X \to 2^X$ a mapping such that $W = \operatorname{int} D(A) \neq \emptyset$. Suppose that A is norm to weak* closed on W and $A(u)$ is bounded for every $u \in W$.*

Then f is lower semicontinuous and finite on W. Moreover, there exists a dense G_δ-subset $C(f)$ of W such that f is continuous at the points of $C(f)$.

Proof. Assume that $u_0 \in W$, $(u_n) \in W$, $u_n \to u_0$ and let $\liminf\limits_{n\to\infty} f(u_n) < f(u_0)$. Without loss of generality one can assume that there exist $x_n \in A(u_n)$ and $\alpha > 0$ such that $\|x_n\| < f(u_0) - \alpha$. Since $A(u)$ is bounded and weakly* closed for every $u \in W$, $A(u)$ is weakly* compact. As the norm $\|.\|$ of X^* is weak* lower semicontinuous, we have that f is finite on W, in particular, at u_0. Hence there exists a subnet (x_{n_α}) of (x_n) such that $x_{n_\alpha} \to x_0$ weakly* (i.e. in the $\sigma(Z^*, Z)$-topology of Z^*) in X for some $x_0 \in X$. Since A is norm to weak* closed in W, we conclude that $x_o \in A(u_0)$. We have $\|x_0\| \leqslant \liminf\limits_{\alpha} \|x_{n_\alpha}\| \leqslant f(u_0) - \alpha$, a contradiction. Since f is lower semicontinuous and finite on W, there exists a dense G_δ-subset $C(f)$ of W such that f is continuous at the points of $C(f)$. □

Lemma 2. *Let X be a reflexive smooth Banach space, $A\colon X \to 2^X$ an accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$. Assume that A is norm-to-weak closed at the points of W and that $A(u)$ is bounded for every $u \in W$.*

Then for every (fixed) $u \in C(f)$ there is $f(u) = \|x\|$ for every $x \in A(u)$.

Proof. By Lemma 1, f is continuous at the points of dense G_δ-subset $C(f)$ of W. Suppose that $u \in C(f)$, f is fixed. We have $f(u) \leqslant \|y\|$ for every $y \in A(u)$. Assume that there exists a point $x_0 \in A(u)$ such that $\|x_0\| > f(u)$. We choose $z^* \in X^*$ such that $\|z^*\| = 1$ and $\langle z^*, x_0\rangle = \|x_0\|$. Let $\varepsilon > 0$ be such that $\langle z^*, x_0\rangle > f(u) + \varepsilon$. Since X is smooth and reflexive, the duality mapping $J\colon X \to X^*$ is singlevalued and surjective. Hence there is a point $u_0 \in X$ such that $z^* = J(u_0)$ and $\|u_0\| = 1$. Since $u \in C(f)$, for a given ε there exists a constant $\delta > 0$ such that $u + \delta u_0 \in W$ and $|f(u + \delta u_o) - f(u)| < \varepsilon$. As the norm on X is weak lower semicontinuous and the values of A are weakly compact on W, we choose a point $x_1 \in A(u + \delta u_0)$ such that $f(u + \delta u_0) = \|x_1\|$. Then $\|x_1\| < f(u) + \varepsilon < \langle z^*, x_0\rangle$. By accretivity of A we get

$$0 \leqslant \langle J(u + \delta u_0 - u), x_1 - x_0\rangle = \delta\,\langle J(u_0), x_1 - x_0\rangle = \delta\,\langle z^*, x_1 - x_0\rangle\,.$$

Hence $\langle z^*, x_1\rangle \geqslant \langle z^*, x_0\rangle$ and $\|x_1\| < f(u)+\varepsilon < \langle z^*, x_1\rangle \leqslant \|z^*\|\cdot\|x_1\|$, which implies that $\|z^*\| > 1$, a contradiction.

S. Fitzpatrick [9] proved the following result: Let X be a Banach space, $T\colon X \to 2^{X^*}$ a maximal monotone mapping with $D_0 = \operatorname{int} D(T) \neq \emptyset$. Suppose that $u_0 \in D_0$ is a point of continuity of the function $\varphi\colon D_0 \to \mathbf{R}_+$ defined by $\varphi(u) = \min\{\|u^*\|\colon u^* \in T(u)\}$, $u \in D_0$. If $u_n \in D_0$, $u_n \to u_0$, $u_n^* \in T(u_n)$, $u_0^* \in T(u_0)$, then $\|u_n^*\| \to \|u_0^*\|$. □

The next lemma extends the corresponding Lemma 7 [19], where it is assumed that X is a uniformly Fréchet smooth Banach space and its dual X^* is Fréchet smooth. However, the proof is rather different.

Lemma 3. *Let X be a reflexive smooth Banach space, $A\colon X \to 2^X$ and accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$. Suppose that $A(u)$ is bounded for every $u \in W$ and that A is norm to weak closed at the points of W. If $u_0 \in C(f)$, $u_n \in W$, $u_n \to u_0$ in the norm of X, $x_n \in A(u_n)$ and $x_0 \in A(u_0)$, then $\|x_n\| \to \|x_0\|$.*

Proof. Suppose that $u_n \to u_0$, $u_0 \in C(f)$, $u_n \in W, x_n \in A(u_n)$, $x_0 \in A(u_0)$. First of all, show that $\|x_0\| < \liminf\limits_{n\to\infty}\|x_n\|$. Supposing the contrary, one can assume without loss of generality that $\|x_n\| < \|x_0\| - \alpha$ for some $\alpha > 0$ and infinitely many indexes n. Since X is reflexive, there exists a subsequence of (x_n), say (x_n), such that $x_n \to \bar{x}$ weakly in X. As A is norm to weak closed at the points of W, we get that $\bar{x} \in A(u_0)$ and $\|\bar{x}\| \leqslant \liminf\limits_{n\to\infty}\|x_n\|$. According to Lemma 2, we have $\|\bar{x}\| = \|x_0\|$ and hence $\|x_0\| \leqslant \liminf\limits_{n\to\infty}\|x_n\| \leqslant \|x_0\| - \alpha$, a contradiction.

According to Lemma 1, the set $C(f)$ of all points of W where f is continuous is a dense G_δ-subset of W. Assume that $\|x_0\| < \limsup\limits_{n\to\infty}\|x_n\|$. Without loss of generality one can assume that $\|x_n\| > \|x_0\| + \alpha$ for infinitely many indexes n and some $\alpha > 0$. Let $z_n^* \in X^*$ be such that $\|z_n^*\| = 1$ and $\langle z_n^*, x_n\rangle > \|x_n\| - 1/n$ for every $n \in N$. As X is smooth and reflexive, J is singlevalued and surjective. Hence there exists $z_n \in X$ such that $z_n^* = J(z_n)$ and $\|z_n\| = 1$ for every $n \in N$. We have $u_n + n^{-1}z_n \in W$ for sufficiently large n. Choose $y_n \in A(u_n + n^{-1}z_n)$ such that $\|y_n\| = f(u_n + n^{-1}z_n)$. As $u_n \to u_0$, $u_0 \in C(f)$, $x_0 \in A(u_0)$, we have by Lemma 2 that $\|y_n\| \to f(u_0) = \|x_0\|$. Since A is accretive, we conclude that

$$\begin{aligned} 0 &\leqslant \left\langle y_n - x_n, J(u_n + n^{-1}z_n - u_n)\right\rangle = n^{-1}\left\langle y_n - x_n, J(z_n)\right\rangle \\ &= n^{-1}\left\langle y_n - x_n, z_n^*\right\rangle. \end{aligned}$$

Hence

$$\begin{aligned}\langle z_n^*, y_n\rangle &\geqslant \langle z_n^*, x_n\rangle > \|x_n\| - \frac{1}{n} > \|x_0\| + \alpha - \frac{1}{n}\\ &= f(u_0) + \alpha - \frac{1}{n}.\end{aligned}$$

On the other hand, $\langle z_n^*, y_n\rangle \leqslant \|y_n\| \leqslant f(u_n + n^{-1}z_n)$ and therefore $\lim_{n\to\infty} f(u_n + n^{-1}z_n) \geqslant f(u_0) + \alpha$, which contradicts the fact that f is continuous at u_0. This proves the assertion. □

Lemma 4. *Let X be a reflexive smooth and rotund Banach space, $A\colon X \to 2^X$ an accretive mapping with $W = \operatorname{int} D(A) \neq \emptyset$. Suppose that A is norm to weak closed at the points of W and that $A(u)$ is a convex bounded set for every $u \in W$.*

Then A is singlevalued at the points of the dense G_δ-subset $C(f)$ of W.

Proof. It relies on Lemmas 1, 2 and the fact that a convex set in a rotund normed linear space has at most one point with the minimum norm. □

Lemma 5. *Let X, Y, be normed linear spaces, Y a dual Banach space, $A\colon X \to 2^Y$ a mapping with $D(A) \subseteq X$. Suppose that A is locally bounded at $u_0 \in D(A)$ and possesses the property (P) (in particular, A is norm to weak* closed) at u_0.*

Then A is norm to weak upper semicontinuous at u_0*

Proof. See [19, Lemma 1]. □

Lemma 6. *Let X be a Fréchet smooth normed linear space, $A\colon X \to 2^X$ a maximal accretive mapping with $D(A) \subseteq X$. If A is locally bounded at $u_0 \in D(A)$, then A is norm to weak closed at u_0.*

Proof. Since X is Fréchet smooth, the duality mapping J is singlevalued and norm to norm continuous on X. Let $(u_\alpha, x_\alpha) \in G(A)$ be a convergent net in $(X, \|\cdot\|) \times (X, \sigma(X, X^*))$ such that $u_\alpha \to u_0$ in the norm of X and $x_\alpha \to x_0$ weakly in X. The accretiveness of A implies that $\langle x_\alpha - x, J(u_\alpha - v)\rangle \geqslant 0$ for every fixed $(v, x) \in G(A)$ and every α. Since (x_α) is bounded and J is norm continuous on X, we get that $\langle x_0 - x, J(u_0 - v)\rangle \geqslant 0$ for every $(v, x) \in G(A)$. Consequently, $x_0 \in A(u_0)$ by the maximality of A. □

Lemma 7. *Let X be a reflexive Fréchet smooth Banach spaces, $A\colon X \to 2^X$ a maximal accretive mapping such that A is locally bounded at $u_0 \in D(A)$.*

Then A is norm to weak upper semicontinuous at u_0.

Proof. The assertion follows at once from Lemmas 5 and 6. □

Lemma 8. *Let X be a Banach space, $A\colon X \to 2^X$ an accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$. Let one of the following three conditions be satisfied:*

(i) ([10]) *X^* is uniformly rotund (i.e. X is uniformly Fréchet smooth);*

(ii) ([24]) *X is uniformly rotund;*

(iii) ([17]) *both X and X^* are Fréchet smooth.*

Then A is locally bounded at every point of W.

Theorem 1. *Let X be a Fréchet smooth Banach space such that X^* is Fréchet smooth, let $A\colon X \to 2^X$ be a maximal accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$.*

Then the set $C(A)$ of all points of W where A is singlevalued and norm to norm upper semicontinuous, is equal to the dense G_δ-subset $C(f)$ of all points of W where the function f of the minimum modulus of A is continuous.

Proof. Since X^* is Fréchet smooth, we have that X is a reflexive, rotund and (H)-space. As X is smooth, we conclude that $A(u)$ is convex for every $u \in D(A)$ (see [1] and [14]). By Lemma 8 (iii), A is locally bounded on W and hence $A(u)$ is bounded, $u \in W$. By Lemma 6, A is norm to weak closed on W. In view of Lemma 1, the function f is lower semicontinuous and finite on W and therefore there exists a dense G_δ-subset $C(f)$ of W such that f is continuous at the points of $C(f)$. Let $u_0 \in C(f)$ be arbitrary; we will show that $u_0 \in C(A)$. According to Lemma 4, A is singlevalued at the points of $C(f)$ and by Lemma 5, A is norm to weak upper semicontinuous on W. Suppose that $(u_n) \subset W$, $u_n \to u_0$, $x_n \in A(u_n)$. Then $x_n \to x_0 = A(u_0)$, weakly in X. By Lemma 3 we have that $\|x_n\| \to \|x_0\|$ and since X is an (H)-space, we conclude that $x_n \to x_0$ in the norm of X. Hence $u_0 \in C(A)$ and we have proved that $C(f) \subseteq C(A)$. Suppose now that $u_0 \in C(A)$. Since f is lower semicontinuous on W, it is sufficient to prove that f is upper semicontinuous at u_0. Assume that $(u_n) \subset W$, $u_n \to u_0$, $x_n \in A(u_n)$. Since $u_0 \in C(A)$, for a given $\varepsilon > 0$ there exists an integer n_0 such that $\|x_n - A(u_0)\| \leqslant \varepsilon$ for every $n \geqslant n_0$. By the definition of the function f and the fact A is singlevalued at u_0 we conclude that $f(u_0) = \|A(u_0)\|$ and $f(u_n) \leqslant \|x_n\|$. Therefore $f(u_n) \leqslant \|x_n\| \leqslant \|A(u_0)\| + \varepsilon = f(u_0)+\varepsilon$ for every $n \geqslant n_0$, which proves that $u_0 \in C(f)$ and therefore $C(A) \subseteq C(f)$. Hence $C(A) = C(f)$ and Theorem 1 is proved. □

Analyzing the proof of Theorem 1 and the previous lemmas, we get

Theorem 2. *Let X he a smooth Banach space such that X^* is Fréchet smooth, let $A\colon X \to 2^X$ be an accretive mapping such that $W = \operatorname{int} D(A) \neq \emptyset$. Assume that A is locally bounded and norm to weak closed at the points of W and that $A(u)$ is convex for every $u \in W$.*

Then the set $C(A)$ of all points of W where A is singlevalued and norm to norm upper semicontinuous is equal to the dense G_δ-subset $C(f)$ of all points of W where the function f of the minimum modulus of A is continuous.

Remark 1. It is known (see Phelps [23], §6]) that if X is a normed linear space, $M \subset X$ an open nonvoid subset, $T\colon M \to 2^{X^*}$ a monotone and norm to weak* upper semicontinuous mapping on M and $T(u)$ is nonempty convex and weak* closed for all $u \in M$, then T is maximal monotone on M.

Using similar arguments as in [23, §6] one can prove the following result: Suppose that X is a reflexive smooth and rotund Banach space, $M \subset X$ an open nonvoid subset, $A\colon M \to 2^X$ an accretive and norm to weak upper semicontinuous mapping on M. If $T(u)$ is nonempty convex and closed for every $u \in M$, then A is maximal accretive on M.

If X is a uniformly Fréchet smooth Banach space and $A\colon X \to 2^X$ is a maximal accretive mapping with $W = \operatorname{int} D(A) \neq \emptyset$, then there exists a dense G_δ-subset W_0 of W such that A is singlevalued and norm to norm upper semicontinuous at the points of W_0 (compare [20]). This result was extended to a reflexive Fréchet smooth Banach space in [26], while the resolvents and selections of accretive and maximal accretive multivalued mappings were studied in [21].

Theorem 3. *Let X be a Banach space, J and J^* the duality mapping on X and X^*, respectively. Then*

(i) *if X is nonreflexive and X^* is smooth, then the graph $G(J^*)$ of J^* is not closet in $\big(X^*, \sigma(X^*, X)\big) \times \big(X^{**}, \sigma(X^{**}, X^*)\big)$;*

(ii) *if X^* is smooth, then J^* is weak* to weak* continuous on $R(J)$ at $u_0^* \in R(J)$ if and only if J^{-1} is weak* to weak continuous at u_0^*.*

Proof. First of all, if X^* is smooth, then $\hat{X} = J^*\big(R(J)\big)$. If X is arbitrary, we always have $\hat{X} \subseteq J^*\big(R(J)\big)$. Indeed, if $u_0^{**} = \hat{u}_0 \in \hat{X}$ for some $u_0 \in X$; we choose $u_0^* \in JU(u_0)$. Then $\hat{u}_0 \in J^*(u_0^*) \subseteq J^*\big(J(u_0)\big) \subseteq J^*\big(R(J)\big)$. Hence $\hat{X} \subseteq J^*\big(R(J)\big)$. Suppose now that X^* is smooth and $v_0^{**} \in J^*\big(R(J)\big)$. Then there exists a point $z_0^* \in R(J)$ and $v_0 \in X$ such that $z_0^* \in J(v_0)$ and $v_0^{**} = J^*(z_0^*)$ in view of the smoothness of X^*. Hence $v_0^{**} = \hat{v}_0$, which proves that $J^*\big(R(J)\big) \subset \hat{X}$.

Assume on the contrary that $G(J^*)$ is closed in $\big(X^*, \sigma(X^*, X)\big) \times \big(X^{**}, \sigma(X^{**}, X^*)\big)$. We find nets $(v_\beta) \subset X$, $(v_\beta^*) \subset X^*$ and points $u_0^* \in X^*$ and $u_0^{**} \in X^{**}$ such that $v_\beta^* \to u_0^*$ weakly* in X^*, $\hat{v}_\beta = J^*(v_\beta^*)$ and $\hat{v}_\beta \to u_0^{**}$ weakly* in X^{**} and $u_0^{**} = J^*(u_0^*)$.

Since X is nonreflexive, we have $X^{**} \neq R(J^*)$. Indeed, if $R(J^*) = X^{**}$, then using the James theorem we see that X^* is reflexive and hence X would be reflexive, a contradiction. Hence there exists a point $u_0^{**} \in X^{**}$ such that $u_0^{**} \notin R(J^*)$. By the Goldstine theorem there exists a net $(u_\alpha) \subset X$ such that $\|u_\alpha\| \leqslant \|u_0^{**}\|$ and $\hat{u}_\alpha \to u_0^{**}$ in the $\sigma(X^{**}, X^*)$-topology of X^{**}. Since $\hat{X} = J^*(R(J))$, there exists $u_\alpha^* \in R(J)$ such that $\hat{u}_\alpha = J^*(u_\alpha^*)$. Now the boundedness of $(\hat{u}_\alpha)$ implies that (u_α^*) is bounded in X^* and hence there exists a subnet $(u_{\alpha_\beta}^*) \equiv (v_\beta^*)$ of (u_α^*) such that $u_{\alpha_\beta}^* \to u_0^*$ in the $\sigma(X^*, X)$-topology of X^*. Now, if $G(J^*)$ were closed in $(X^*, \sigma(X^*, X)) \times (X^{**}, \sigma(X^{**}, X^*))$, then $v_\beta^* \to u_0^*$ weakly*, $\hat{v}_\beta \equiv J^*(v_\beta^*) \to u_0^{**}$ weakly* would imply, that $u_0^{**} = J^*(u_0^*)$. Hence $u_0^{**} \in R(J^*)$, a contradiction

(ii) Suppose that X^* is smooth. Then X is rotund and $\hat{X} = J^*(R(J))$. Hence there exists J^{-1} and it is singlevalued and norm to weak continuous from $R(J)$ into X. Moreover, J^{-1} is weak* to weak continuous at u_0^* if and only if J^* is weak* to weak* continuous on $R(J)$ at $u_0^* \in R(J)$. This follows at once from the facts that $J^{-1} = \tau^{-1} J^* | R(J)$ (compare [4]) and that τ is a homeomorphism from $(X, \sigma(X, X^*))$ into $(X^{**}, \sigma(X^{**}, X^*))$, which concludes the proof. □

Remark 2. Note that the duality mapping $J \colon \ell_p \to \ell_q (p \in (1, \infty))$ is sequentially weak to weak continuous, while the duality mapping on L_p, $p \in (1, \infty)$, $p \neq 2$, is not sequentially weak to weak continuous. If X is a Banach space such that there is a selection of the duality mapping $J \colon X \to 2^{X^*}$ which is sequentially weak to weak* continuous, then X has the Opial property, i.e. if $x_n \to x_0$ weakly in X, then $\liminf_{n \to \infty} \|x_n - x\| > \liminf_{n \to \infty} \|x_n - x_0\|$ for every $x \neq x_0$, $x \in X$. If X has the Opial property, then X has the Brodskij-Milman property (see [13]).

Note that if X^* is a smooth Banach space, then J^{-1} is singlevalued and norm to weak continuous from $R(J)$ into X.

Using the higher dual technique, Giles, Gregory and Sims [12] have proved the following result. Let X be a Banach space which can be equivalently renormed so that there exists a constant k $(0 < k < 1)$ such that for every $x \in S_1(0)$ and $x_x^* \in J(x)$ and $\hat{x}_x^* + x^\perp \in J^{**}(\hat{x})$, where $x^\perp \in X^\perp$, we have $\|x^\perp\| \leqslant k$, then X is an Asplund space. In particular, if X can be equivalently renormed such that the weak* and weak topologies coincide on $J(S_1(0))$, then X is an Asplund space. It is remarked that its proof shows that given a Banach space X whose dual X^* satisfies the condition of the above assertion or its consequence, then X is reflexive ([12]).

The proof of the next assertion is based on the Eberlein-Šmulian theorem, on the argument due to Davis and Johnson [3].

Theorem 4. *Let X be a Banach space.*

Then X is reflexive if and only if X admits an equivalent norm such that the weak* and weak topologies coincide on its second dual unit sphere of X^{**}.

Proof. First of all assume that the condition is satisfied and X is not reflexive. Then the unit ball $B_1^*(0)$ of X^* is $\sigma(X^*, X)$-compact but not $\sigma(X^*, X^{**})$-countably compact. Indeed, if $B_1^*(0)$ were $\sigma(X^*, X^{**})$-countably compact, then $B_1^*(0)$ would be also $\sigma(X^*, X^{**})$-compact by the Eberlein-Šmulian theorem, which is impossible, because the $\sigma(X^*, X^{**})$- and the $\sigma(X^*, X)$-topologies agree on X^* if and only if X is reflexive. As $B_1^*(0)$ is not $\sigma(X^*, X^{**})$-countably compact, there is a sequence $(u_n^*) \subset B_1^*(0)$ having no $\sigma(X^*, X^{**})$-convergent subnet. Since $B_1^*(0)$ is $\sigma(X^*, X)$-compact, there is a subnet (u_α^*) of (u_n^*) and a point $u^* \in B_1^*(0)$ such that $u_\alpha^* \to u^*$ in the $\sigma(X^*, X)$-topology. Put $W = \overline{\operatorname{span}}\{(u_n^*) \cup \{u^*\}\}$. Then W is a norm-closed separable subspace of X^*, and for every fixed $u \in X$ we have $\langle \hat{u}, u_\alpha^* \rangle \to \langle \hat{u}, u^* \rangle$. Since (u_α^*) is a subnet of (u_n^*) and (u_n^*) contains no $\sigma(X^*, X^{**})$-convergent subnet, we conclude that (u_α^*) does not converge to u^* in the $\sigma(X^*, X^{**})$-topology of X^*. Since $\sigma(X^*, X^{**})|W = \sigma(W, W^*)$ by the Hahn-Banach theorem, there exists a point $u_0^{**} \in W^*$ such that the net $\big(\langle u_0^{**}, u_\alpha^* \rangle\big)$ fails to converge to $\langle u_0^{**}, u^* \rangle$. Hence u_0^{**} is not $\sigma(X^*, X)$-continuous, i.e. $u_0^{**} \notin \hat{X}$. By the Goldstine theorem, we have

$$u_0^{**} \in X^{**} = \overline{\hat{X}}^{\sigma(X^{**}, X^*)}.$$

There exists a net (u_β) in X such that $\hat{u}_\beta \to u_0^{**}$ in the $\sigma(X^{**}, X^*)$-topology of X^{**} and $\|u_\beta\| \leqslant \|u_0^{**}\|$, where $\|.\|$ denotes the second dual norm of X^{**} associated with the equivalent norm $\|.\|$ on X. Now we have

$$\|u_0^{**}\| \leqslant \liminf_\beta \|\hat{u}_\beta\| = \liminf_\beta \|u_\beta\| \leqslant \limsup_\beta \|u_\beta\| \leqslant \|u_0^{**}\|.$$

Hence $\|u_\beta\| = \|\hat{u}_\beta\| \to \|u_0^{**}\|$. Since $u_0^{**} \notin \hat{X}$, there exists an index β_0 such that $u_\beta \neq 0$ for every $\beta \geqslant \beta_0$. Put $\hat{v}_\beta = \hat{u}_\beta \|u_\beta\|^{-1}$, $v_0^{**} = u_0^{**}\|u_0^{**}\|^{-1}$. Then $\|\hat{v}_\beta\| = \|v_0^{**}\| = 1$ and $\hat{v}_\beta \to v_0^{**}$ in the $\sigma(X^{**}, X^*)$-topology of X^{**}. Since the weak* and weak topologies agree on the unit sphere $S_{\|.\|}^{**}(0) = \{u^{**} \in X^{**}, \|u^{**}\| = 1\}$ of X^{**}, we have that $\hat{v}_\beta \to v_0^{**}$ weakly in X^{**}. As $\hat{v}_\beta \in \hat{X}$ and $\hat{X}$ is weakly closed in X^{**}, we have $v_0^{**} \in \hat{X}$ and therefore $u_0^{**} = \|u_0^{**}\| v_0^{**} \in \hat{X}$, a contradiction which proves that X is reflexive. Let X be reflexive. Since X^* is also reflexive, we have that the $\sigma(X^{**}, X^*)$-topology and the $\sigma(X^{**}, X^{***})$-topology coincide on X^{**}. Hence, if $\|.\|$ is an arbitrary equivalent norm in X, then the weak* and weak topologies coincide in its second dual unit sphere of X^{**}, which concludes the proof. □

Corollary 1. *Let X be a Banach space such that X admits an equivalent norm whose second dual norm on X^{**} has the (w^*H)-property in nets. Then X is reflexive.*

G. Emmanuele [6] proved that if X is a weakly compactly generated Banach space, then X can be equivalently renormed in such a way that X^* is weak* locally uniformly rotund. As a consequence of this result (see also [6, Theorem 2]) he stated the following assertion: For a Banach space X, the following are equivalent: (i) X is reflexive, (ii) X^{**} can be renormed in an equivalent weakly locally uniformly rotund manner.

Note that if X^* is (WLUR) and $(x_n^*) \subset S_1^*(0)$, $x_0^* \in S_1^*(0)$ are such that $x_n^* \to x_0^*$ weakly* in X^*, then $x_n^* \to x_0^*$ weakly (see [5, §2]). Hence if X^* is (WLUR), then the weak* and weak convergences of sequences agree on $S_1^*(0)$. Theorem 4 is connected with the following result (see [22, Theorem 3]): Let X be a dual Banach space (i.e. $X = Z^*$ for a Banach space Z), $M \subseteq X^*$ a convex open subset, $\hat{u}_0 \in M$, where $\hat{u}_0 \in M$, where $\hat{u}_0$ is a canonical image of $u_0 \in Z$ in X^*. Let $f\colon M \to \mathbf{R}$ be a weak* lower semicontinuous convex functional having the Gâteaux derivative $f'(\hat{u}_0)$ at $\hat{u}_0$. Then $f'(\hat{u}_0) \in \hat{X}$, i.e. $f'(\hat{u}_0)$ is a weak* continuous linear functional on X^*.

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Souhrn

SOME PROPERTIES OF MONOTONE TYPE MULTIVALUED OPERATORS IN BANACH SPACES

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Jsou vyšetřeny některé vlastnosti mnohoznačných zobrazení monotónního typu (akretivní operátory, zobrazení duality) v souvislosti s geometrickou strukturou Banachových prostorů.

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