Kybernetika

Ludvík Prouza

# Appendix to the article “On generalized linear discrete inversion filters”

*Kybernetika*, Vol. 8 (1972), No. 3, (264)--267

Persistent URL: http://dml.cz/dmlcz/125767

KYBERNETIKA — VOLUME 8 (1972), NUMBER 3

# Appendix to the Article "On Generalized Linear Discrete Inversion Filters"

LUDVÍK PROUZA

An expression for the mean square error of the inversion filter is derived and some consequences are shown.

## 1. INTRODUCTION

Let

$$B_j(z) = b_{j0} + b_{j1}z^{-1} + \ldots + b_{jh}z^{-h} \qquad (j = 0, 1, \ldots, m) \tag{1}$$

be Z-transforms of discrete signals, the signal with $j = 0$ being wanted and the other disturbing. The signals are weighted by weights $w_j$, where $w_0 = 1$, $w_j \geqq 0$ for $j = 1, 2, \ldots, m$.

Let us define

$$B(z) = b_0 + b_1 z^{-1} + \ldots + b_s z^{-s}, \quad (s \leqq h) \tag{2}$$

with the aid of the relation

$$|B(z)|^2 = \sum_{j=0}^{m} w_j |B_j(z)|^2, \ z = \exp i\omega. \tag{3}$$

According to the known Fejér-Riesz theorem $B(z)$ exists and is unique if we choose its roots only on and outside the unit circle $C_1$.

Suppose $B(z)$ to have all roots only outside of $C_1$. Then, $B(z)\,B(z^{-1})\,z^s$ is a reciprocal polynomial with roots $\zeta_1, \ldots, \zeta_s$ outside $C_1$ and $\zeta_{s+1}, \ldots, \zeta_{2s}$ inside $C_1$ (the reciprocal values of the roots $\zeta_1, \ldots, \zeta_s$) and

$$B(z)\,B(z^{-1})\,z^s = \frac{b_0 b_s}{(-1)^s \zeta_1 \ldots \zeta_s}(1 - \zeta_1 z) \ldots (1 - \zeta_s z)(z - \zeta_1) \ldots (z - \zeta_s). \tag{4}$$

Define

$$(5) \qquad q_0 + q_1 z + \ldots + q_s z^s = \frac{b_0 b_s}{(-1)^s \zeta_1 \ldots \zeta_s} (z - \zeta_1) \ldots (z - \zeta_s) .$$

Put

$$(6) \qquad C_j(z) = \mathrm{A}(z)\, B_j(z) .$$

In [2] there has been shown that $A(z)$ fulfilling

$$(7) \qquad \frac{1}{2\pi i} \left\{ w_0 \int_{C_1} |z^{-T} - A(z)\, B_0(z)|^2 \,.\, |\mathrm{d}z| + \sum_{j=1}^{m} w_j \,.\, \int_{C_1} |A(z)\, B_j(z)|^2 \,.\, |\mathrm{d}z| \right\} = \min$$

is given by the expression

$$(8) \qquad A(z) = \frac{z^{s-T}(r_0 + r_1 z + \ldots + r_T z^T)}{(1 - \zeta_1 z) \ldots (1 - \zeta_s z)} ,$$

where $r_j$ ($j = 0, 1, \ldots, T$) result from the system of equations

$$(9) \qquad \begin{aligned} q_0 r_0 \qquad\qquad\qquad\qquad &= b_{00} \,, \\ q_1 r_0 + q_0 r_1 \qquad\qquad\qquad &= b_{01} \,, \\ \vdots \qquad\qquad\qquad\qquad & \\ q_T r_0 + q_{T-1} r_1 + \ldots + q_0 r_T &= b_{0T} \,, \end{aligned}$$

where $q_j = 0$ for $j > s$, $b_{0j} = 0$ for $j > h$.

In [1], [2] has been shown that the minimum in (7) is $1 - c_{0T}$, where $c_{0T}$ is the coefficient of $z^{-T}$ in the development of (6), and also that $0 < c_{0T} \leqq 1$.

## 2. AN EXPRESSION FOR $c_{0T}$

From the system (9) it is seen that $r_0, \ldots, r_T$ are coefficients of the development

$$(10) \qquad r_0 + r_1 z^{-1} + \ldots = \frac{b_{00} + b_{01} z^{-1} + \ldots + b_{0h} z^{-h}}{q_0 + q_1 z^{-1} + \ldots + q_s z^{-s}} .$$

The coefficient $c_{0T}$ is given by

$$(11) \qquad c_{0T} = \frac{1}{2\pi i} \int_{C_1} B_0(z)\, A(z)\, z^T \frac{\mathrm{d}z}{z} ,$$

 that is, the zero order term of the development

$$(12)\qquad B_0(z)\,A(z)\,z^T = \frac{b_{00} + \ldots + b_{0h}z^{-h}}{(z^{-1} - \zeta_1)\ldots(z^{-1} - \zeta_s)}(r_0 + \ldots + r_T z^T) =$$

$$= \frac{b_{00} + \ldots + b_{0h}z^{-h}}{\dfrac{(-1)^s\zeta_1\ldots\zeta_s}{b_0 b_s}(q_0 + \ldots + q_s z^{-s})}(r_0 + \ldots + r_T z^T) =$$

$$= \frac{b_0 b_s}{(-1)^s\zeta_1\ldots\zeta_s}(r_0 + r_1 z^{-1} + \ldots)(r_0 + \ldots + r_T z^T).$$

Thus

$$(13)\qquad c_{0T} = \frac{b_0 b_s}{(-1)^s\zeta_1\ldots\zeta_s}(r_0^2 + r_1^2 + \ldots + r_T^2).$$

## 3. SOME SPECIAL CASES

There is seen from (13) that $c_{0T}$ is a nondecreasing function of $T$, thus the minimum of (7), being $1 - c_{0T}$, is a nonincreasing function of $T$. Since $c_{0T} \leqq 1$, there exists the limit of $C_{0T}$ for $T \to \infty$. From (13) with the aid of (12), (4), (5) and the Parseval indetity, there is

(14)

$$\lim_{T\to\infty} c_{0T} = \frac{1}{2\pi i}\int_{C_1} \frac{(b_{00} + \ldots + b_{0h}z^{-h})(b_{00} + \ldots + b_{0h}z^h)}{z^{-s}(1 - \zeta_1 z)\ldots(1 - \zeta_s z)(z - \zeta_1)\ldots(z - \zeta_s)\dfrac{b_0 b_s}{(-1)^s\zeta_1\ldots\zeta_s}}\cdot\frac{dz}{z} =$$

$$= \frac{1}{2\pi i}\int_{C_1}\frac{B_0(z)\,B_0(z^{-1})}{B(z)\,B(z^{-1})}\cdot\frac{dz}{z}.$$

In the case of "pure" inversion, that is $w_0 = 1$, $w_j = 0$ for $j = 1, \ldots, m$, there is $B_0(z) = B(z)$, thus $\lim_{T\to\infty} c_{0T} = 1$.

Suppose further the "pure" inversion and $T = 0$. Then from (9), (5), (13) one gets

$$(15)\qquad r_0 = b_0/q_0 = 1/b_h$$

and since

$$(16)\qquad b_h/b_0 = (-1)^h z_1 \ldots z_h,$$

there is

$$(17)\qquad c_0 = \frac{1}{(-1)^h\zeta_1\ldots\zeta_h}\cdot\frac{b_0}{b_h} = \frac{1}{\zeta_1\ldots\zeta_h z_1\ldots z_h} = \frac{\zeta_1^*\ldots\zeta_h^*}{z_1\ldots z_h},$$

where $z_1, \ldots, z_h$ are the roots of the polynomial $z^h B(z)$ and 

$$\zeta_j^* = \begin{cases} z_j & \text{for } |z_j| < 1 , \\ z_j^{-1} & \text{for } |z_j| > 1 , \end{cases} \quad (j = 1, \ldots, h) . \tag{18}$$

This result has been derived in [1] with the unnecessary restriction that all roots $z_j$ are simple.

The restriction $|z_j| \neq 1$ in (18) is substantial since we know that no stable filter of the form (8) exists if some roots $\zeta_j$ lie on $C_1$.

## 4. CONCLUDING REMARKS

From (9), (5), and (13) one sees that for $c_{0T}$ expressions in the symmetric functions of the roots $\zeta_j$ can be derived, but they will be substantially more complicated than (17).

Furthermore, it is seen from (9), (10) that the sequence $\{r_j\}$ with $j > h$ is solution of a homogeneous linear difference equation with characteristic roots $\zeta_{s+1}, \ldots, \zeta_{2s}$ lying inside $C_1$.

The initial conditions result from (9) or (10). This result may be useful in connection with (14) for computing (13) if $T$ is substantially greater than $h$, especially if a "dominant" root $\zeta_j$ exists.

(Received December 23, 1971.)

REFERENCES

[1] Prouza, L.: On the Inversion of Moving Averages, Linear Discrete Equalizers and "Whitening" Filters, and Series Summability. Kybernetika *6* (1970), 3, 225–240.
[2] Prouza, L.: On Generalized Linear Discrete Inversion Filters. Kybernetika *8* (1972), 1, 30–38.

VÝTAH

# Doplněk k článku „O zobecněných lineárních diskrétních inverzních filtrech“

LUDVÍK PROUZA

V článku se odvozuje výraz pro střední kvadratickou chybu inverzního filtru a ukazují se některé důsledky.

***Dr. Ludvík Prouza, CSc.**, Ústav pro výzkum radiotechniky (Research Institute for Radio Engineering), Opočínek, p. Lány na Důlku.*