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Directable Automata and Directly Subdefinite Events

ALICA PIRICKÁ

This work deals with class of the directable automata. The main problem is to characterize the events, which can be represented by a directable automata.

We are also interested in the class of the directly subdefinite events, which contains the class of ultimate-definite events [4]. We want to point out an algorithm for deciding, whether given finite automaton defines directly subdefinite event.

1. DEFINITIONS AND NOTATIONS

It is assumed that the reader is familiar with the basic properties of the regular expressions, regular events, derivatives of the regular expressions and finite automata. Nevertheless, for the sake of completeness we recapitulate here the basic definitions and results significant to that work.

Let X be a finite nonempty alphabet. The elements of X are called symbols. Finite sequences of symbols are called words. The set of all words over X is denoted X^*. The empty word is denoted by λ. Let $l(u)$ be the number of symbols into word u; $l(\lambda) = 0$. If u and v are words, then uv denotes concatenation of u and v. If P, Q are subsets of X^*, then P, Q are called events and $P \cup Q$ and $P \cap Q$ are the set union and set intersection, respectively, of P and Q; $P - Q$ denotes the set of all words which are in P and are not in Q; $P . Q$ is defined as $P . Q = \{xy : x \in P, y \in Q\}$ and $P^* = \bigcup_{l=0}^{\infty} P^l$ is called star event, where P^0 denotes the set whose simple element is λ. The empty set of words is denoted by $\emptyset$. The number of elements of a set S will be denoted by $|S|$.

A finite automaton over the alphabet X is a system $\mathscr{A} = (S, X, s_0, \delta, F)$ where S is a finite nonempty set of internal states, $s_0 \in S$ is the initial state, δ a transition function $S \times X \rightarrow S$ and $F \subseteq S$ a set of final states of $\mathscr{A}$.

The function δ can be extended in the following natural way:

$$(\forall s)\, \delta(s, \lambda) = s\,,$$

$$(\forall s)\,(\forall u)\,(\forall x)\, \delta(s, ux) = \delta(\delta(s, u), x)\,.$$

A word u is accepted by $\mathscr{A}$ if $\delta(s_0, u) \in F$. Then set of all words accepted by $\mathscr{A}$ is denoted by $T(\mathscr{A})$.

Let be $B \subset S$ and $x \in X^*$. We denote by $B_x = \delta(B, x) = \{s \in S : \bar{s} \in B;\ \delta(\bar{s}, x) = s\}$. We define $\mathscr{A}(S)$ for the given finite automaton $\mathscr{A}$:

1. $S \in \mathscr{A}(S)$.
2. If $B \in \mathscr{A}(S)$ then $B_x \in \mathscr{A}(S)$ for all $x \in X$.
3. $\mathscr{A}(S)$ is the least set with the properties 1 and 2.

An event R is called regular if there is some finite automaton such that $R = T(\mathscr{A})$. Let P be an event and $u \in X^*$. The derivative of P with respect to u is defined as

$$\partial_u(P) = \{x : ux \in P\}\,.$$

By $\mathscr{D}(P)$ we denote the derivative closure of event P.

In the following considerations we shall use the simple star events. An event $P = R^*$ is simple star event if there exists a finite automaton $\mathscr{A} = (S, X, s_0, \delta, \{s_0\})$ such that $T(\mathscr{A}) = \mathrm{P}$.

Theorem. *An event P is simple star event iff for every word u it holds*

$$\lambda \in \partial_u P \Leftrightarrow \partial_u P = P\,.$$

Proof. See [1].

Let a connected automaton $\mathscr{A}$ be given, $T(\mathscr{A}) = R$. We point the correspondence between the sets S and $\mathscr{D}(R)$.

Let $s \in S$. There exists a word u such that $\delta(s_0, u) = s$. Then it corresponds $\partial_u R = \{v : \delta(s, v) \in F\}$ to state s. On the other hand to each element $\partial_u R$ there corresponds a state $s_u = \delta(s_0, u)$. Since $\partial_u R = \partial_v R$ does not imply $u = v$, the same element of $\mathscr{D}(R)$ can correspond to different states.

2. DIRECTABLE AUTOMATA

2.1. *Remark.* In this section we admit regular events only.

2.2. Definition. The finite automaton $\mathscr{A} = (S, X, s_0, \delta, F)$ is directable to the initial state s_0 (d.a.) if there exists $u \in X^*$ such that for every $s \in S$ $\delta(s, u) = s_0$. The word u is called the directing word.

2.3. Lemma. *The finite automaton $\mathscr{A}$ is directable to the initial state iff $\{s_0\} \in \in \mathscr{A}(S)$.*

Proof follows immediately from the definition of $\mathscr{A}(S)$ and 2.2.

2.4. Example. The finite automaton given in the fig. 1 is directable to the initial state. Its directing word is xyz or $xxyxz$ for example.

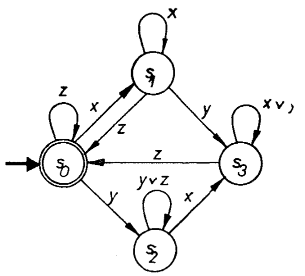

Fig. 1.

The automaton in the fig. 1 can be understand as a model of the automaton for cigarettes, working as follows:

1. Putting the money into it.
2. Turning the knob to the desired kind of cigarettes.
3. Pulling the drawer out and taking the cigarettes.

The set X has three elements $X = \{x, y, z\}$ where x — means putting the money in, y — means the choice of cigarettes, z — taking out the cigarettes.

Table 1.

	s_0	s_1	s_2	s_3
C	n	n	y	y
M	n	y	n	y

The set S has four elements s_0, s_1, s_2, s_3, see the table 1. C — n means the kind of cigarettes not chosen, C — y the kind of cigarettes already chosen, M — n the money is not into the automaton and M — y the money is into it.

2.5. Lemma. *An event R can be represented by a connected d.a. iff*

1. *there exist $u \in X^*$, $R_2 \subset X^*$ and $T \subset X^*$ such that $R = (X^*u \cup T)^* R_2$ and*

2. *the events R, $R_1^* = (X^*u \cup T)^*$ can be represented by the connected finite automata $\mathscr{A}_R = (S, X, s_0, \delta, F)$ and $\mathscr{A}_{R_1^*} = (S, X, s_0, \delta, F_{R_1^*} = \{s_0\})$, respectively.*

Proof. a) $\rightarrow$ Let R be represented by the connected d.a. $\mathscr{A} = (S, X, s_0, \delta, F)$ and u be its directing word. Let us denote $T = \{v \in X^* : \delta(s_0, v) = s_0\}$ and $R_2 = \{v \in X^* : \delta(s_0, v) \in F \;\&\; (\forall v_1, v_2 \in X^*)((v = v_1 v_2 \;\&\; v_1 \neq \lambda) \rightarrow \delta(s_0, v_1) \neq s_0)\}$. The reader can easily prove that $X^*u \subset T$. The finite automaton $\mathscr{A}_{R_1^*} = (S, X, s_0, \delta, \{s_0\})$ represents the event $R_1^* = (X^*u \cup T)^*$.

b) $\leftarrow$ Let there exist $u \in X^*$, $R_2 \subset X^*$ and $T \subset X^*$ with properties 1 and 2. The finite automaton $\mathscr{A}_R$ is d.a. iff $\mathscr{A}_{R_1^*}$ is d.a.

We shall point out that $\mathscr{A}_{R_1^*}$ is d.a. For every $s \in S$ there exists $v \in X^*$ such that $\delta(s_0, v) = s$. Since $vu \in X^*u \subset R_1^*$ we have $\delta(s, u) = \delta(s_0, vu) = s_0$.

2.6. *Remark.* The event R_1^* from 2.5. is simple star event.

2.7. Lemma. *Let R_1^* be simple star event, $R = R_1^* R_2$. The events R_1^*, $R = R_1^* R_2$ can be represented by $\mathscr{A}_{R_1^*} = (S, X, s_0, \delta, \{s_0\})$ and $\mathscr{A}_R = (S, X, s_0, \delta, F)$, respectively, iff for every word u*

$$(1) \qquad \partial_u R_1^* = R_1^* \rightarrow \partial_u R = R .$$

Proof. a) $\leftarrow$ Let the condition (1) hold. We construct automata $\mathscr{A}_{R_1^*}$, $\mathscr{A}_R$ in the following way:

$$S = \{(A, B) : A = \partial_u R_1^* \;\&\; B = \partial_u R\} ,$$

$$s_0 = (R_1^*, R) ,$$

$$\delta((A, B), x) = (\partial_x A, \partial_x B) ,$$

$$F_{R_1^*} = \{(A, B) : \lambda \in A\} ,$$

$$F_R = \{(A, B) : \lambda \in B\} .$$

We show that $F_{R_1^*} = \{(R_1^*, R)\}$. R_1^* is simple star event, therefore for every word u, $\lambda \in \partial_u R_1^* \rightarrow \partial_u R_1^* = R_1^*$ and (1) holds; therefore $F_{R_1^*} = \{(R_1^*, R)\}$.

$T(\mathscr{A}_V) = V$ for $V = R_1^*$ or $V = R$, because $u \in T(\mathscr{A}_V) \Leftrightarrow \lambda \in \partial_u V \Leftrightarrow u \in V$.
$\mathscr{A}_V$ is connected, it ensues from the definition of δ-function.

b) $\rightarrow$ Let us have the connected finite automata $\mathscr{A}_R$, $\mathscr{A}_{R_1^*}$. To every state $s_u \in S$ there is $\partial_u V$ ($V = R_1^*$ or $V = R$) corresponding to the state s_u. Since R_1^* is simple star event, $\partial_u R_1^*$ corresponds to the initial state s_0 for every word u such that $\partial_u R_1^* = R_1^*$.

δ-function is the same for both automata, therefore $\partial_u R_1^* R_2$ corresponds to the initial state of the $\mathscr{A}_R$ for all $u \in X^*$ such that $\partial_u R_1^* = R_1^*$ and therefore (1) holds.

We summarize 2.5, 2.6, 2.7 to the main theorem.

2.8. Theorem. *The event R can be represented by the automaton directable to the initial state by the directing word u iff there are T, $R_2 \subset X^*$ such that*

1. $R = (X^* u \cup T)^* R_2$,
2. $R_1^* = (X^* u \cup T)^*$ *is simple star event,*
3. $(\forall u \in X^*)(\partial_u R_1^* = R_1^* \to \partial_u R = R)$.

2.9. Example. The event $B = (1 \cup 0(1 \cup 0))^* 0$ can be represented by a finite automaton directable to the initial state.

We need to prove the following points:

1. The event B can be expressed in the form $B = (X^* u \cup T)^* B_2$. It is true for $u = 001$, $B_2 = 0$, $T = (1 \cup 0(1 \cup 0))^*$.

2. $B_1 = (X^* u \cup T)^* = (1 \cup 0(1 \cup 0))^*$ is simple star event.

3. For every $u \in X^*$ $\partial_u B_1 = B_1 \to \partial_u B = B$.

We find $\mathscr{D}(B_1)$, $\mathscr{D}(B)$ to show the points 2 and 3:

$$\partial_0 B_1 = (1 \cup 0)(1 \cup 0(1 \cup 0))^*, \qquad \partial_0 B = \lambda \cup (1 \cup 0)(1 \cup 0(1 \cup 0))^* 0,$$
$$\partial_1 B_1 = B_1, \qquad \partial_1 B = B,$$
$$\partial_{01} B_1 = B_1, \qquad \partial_{01} B = B,$$
$$\partial_{00} B_1 = B_1, \qquad \partial_{00} B = B,$$

$\mathscr{D}(B_1) = \{B_1, \partial_0 B_1\}$ and $\mathscr{D}(B) = \{B, \partial_0 B\}$. The point 2 holds because $\lambda \in \partial_u B_1 \to \partial_u B_1 = B_1$. The point 3 is evident. The automaton $\mathscr{A}$ such that $T(\mathscr{A}) = B$ is shown in the fig. 2 ($s_1 \in F$).

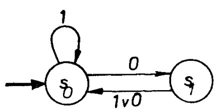

Fig. 2.

2.10. Theorem. *If an event R can be represented by the connected automaton directable to the initial state, then the reduced automaton representing R is directable to the initial state.*

Proof. Let an automaton $\mathscr{A} = (S, X, s_0, \delta, F)$ be directable to the initial state. Let $\bar{\mathscr{A}} = (\bar{S}, X, \bar{s}_0, \bar{\delta}, \bar{F})$ be its reduced automaton. Since $\mathscr{A}$ is d.a. $\delta(s, u) = s_0$ for every $s \in S$ e.g. for every $v \in X^*$ $\partial_u \partial_v R = R$. For reduced automaton there is exactly one state $s \in S$ corresponding to $\partial_v R$, $v \in X^*$. Since the set of states S of the reduced automaton is isomorphic to $\mathscr{D}(R)$ and $\partial_u \partial_v R = R$ holds for every $v \in X^*$, the reduced automaton is directable to the initial state.

2.11. Corollary. *Let the event R be given by a connected automaton $\mathscr{A}$. The event R can be represented by the automaton directable to the initial state iff the reduced automaton to the automaton $\mathscr{A}$ is directable to the initial state.*

2.12. Example. The event $R = [0(1(01^*0)^*\,1)^*\,0 \cup 1\,(1 \cup 0(1(00)^*\,1)^*\,0)^*\,0(1(00)^*\,1)^*\,1(00)^*0]^*$ can not be represented by a directable automaton. The automaton shown in the fig. 3 represents R and it is a reduced automaton. That automaton is not directable to the initial state. (It can be shown by using the quick method from [2].)

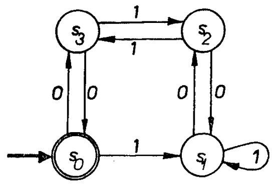

Fig. 3.

3. DIRECTLY SUBEDFINITE EVENTS

We found out that the events, representable by the d.a. have the form

$$R = (X^*u \cup T)^*\,R_2 = X^*uT^*R_2 \cup T^*R_2 = (X^*u)^*\,T^*R_2\,.$$

Let us examine bigger class of that events, the directly subdefinite events.

3.1. Definition. An event R is directly subdefinite event, if there exist an event T and a word u such that $R = (X^*u)^*\,T$. Let us denote the class of directly subdefinite events by $\mathbf{S}$. Let $u\mathbf{S}$ be the class of the directly subdefinite events with given $u \in X^*$.

3.2. *Remark.* The ultimate definite events are directly subdefinite events for every word u, since $X^*R = (X^*u)^*\,X^*R$.

3.3. *Remark.* The event given by an automaton directable to the initial state is directly subdefinite event.

A. Operation properties

3.4. Theorem. a) $R \in \mathbf{S} \Leftrightarrow (\exists u \in X^*)\,(R = (X^*u)^*\,R)$.

b) *The event $R \in \mathbf{S}$ is a star event iff $R = (X^*u)^*\,T^*$.*

Proof. a) Is evident.

b) $\leftarrow$ $R = (X^*u)^*\,T^* = ((X^*u)^*\,T^*)^*$.
$\rightarrow$ It follows from the a)-deal of the theorem.

3.5. Theorem. *Let* $R_\alpha \in u\mathbf{S}$ *for* $\alpha \in A$. *Then* $\bigcup_{\alpha\in A} R_\alpha \in u\mathbf{S}$ *and* $\bigcap_{\alpha\in A} R_\alpha \in u\mathbf{S}$.

Proof. $\bigcup_{\alpha\in A} R_\alpha = \bigcup_{\alpha\in A} (X^*u)^*\,R_\alpha = (X^*u)^*\,(\bigcup_{\alpha\in A} R_\alpha)$.

The proof for $\bigcap_{\alpha\in A} R_\alpha$ can be done in the same way.

3.6. *Remark.* The class **S** is not closed under the a) union, b) intersection and c) complement.

a) Let $R_1 = ((1 \cup 0)^* 11)^* 0$, $R_2 = ((1 \cup 0)^* 0)^* 1$, $R = R_1 \cup R_2 = ((1 \cup 0)^* 11)^* 0 \cup \cup ((1 \cup 0)^* 0)^* 1 = 0 \cup (1 \cup 0)^* 110 \cup 1 \cup (1 \cup 0)^* 01$. Let there exist $u \in X^*$ and $S \subset X$ such that $R = (X^*u)^* S$. Then $0 \in S$ and the word $u0 \in X^*uS \subset R$. So $u0 \in \in (1 \cup 0)^* 110$, therefore there exists the word v_1 such that $u = v_1 11$. Since $1 \in S$ there is $v_2 \in X^*$ such that $u = v_1 11 = v_2 0$. It is not possible therefore $R \notin \mathbf{S}$.

b) Let $R_1 = ((1 \cup 0)^* 11)^* 1$ and $R_2 = ((1 \cup 0)^* 0^* 1$. We can show that $R = = R_1 \cap R_2 \notin \mathbf{S}$. $R = ((1 \cup 0)^* 11)^* 1 \cap ((1 \cup 0)^* 0)^* 1 = (1 \cup (1 \cup 0)^* 111) \cap \cap (1 \cup (1 \cup 0)^* 01) = 1 \notin \mathbf{S}$.

c) Let $S = (1^*1)^* 11$. Then $S^C = \lambda \cup 1 \notin \mathbf{S}$.

B. Directly subdefinite automata

3.7. Definition. A finite automaton $\mathscr{A}$ is directly subdefinite automaton iff $T(\mathscr{A}) \in \mathbf{S}$.

3.8. Example. The finite automaton $\mathscr{A}$ given in the fig. 4 is directly subdefinite automaton. $T(\mathscr{A}) = ((1 \cup 0)^* 1)^* 1$.

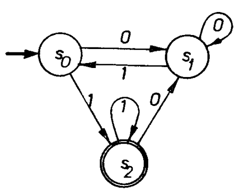

Fig. 4.

3.9. Theorem. *A connected automaton is directly subdefinite automaton iff there is $u \in X^*$, $l(u) \leqq 2^{|S|}$ such that for every word y, $l(y) \leqq |S| (|S| - 1)$ it holds*

$$\delta(s_0, y) \in F \to (\forall s \in S) \quad \delta(s, uy) \in F\,. \tag{2}$$

Proof. a) $\to$ Let $T(\mathscr{A}) = (X^*u)^* T(\mathscr{A})$ and $\delta(s_0, y) \in F$. Since the automaton $\mathscr{A}$ is a connected one, for every $s \in S$ there is $x \in X^*$ such that $\delta(s_0, x) = s$ and $xuy \in \in T(\mathscr{A})$. It means $\delta(s, uy) = \delta(s_0, xuy) \in F$.

We prove that there is a word u, $l(u) \leqq 2^{|S|}$. Let $l(u) > 2^{|S|}$, $u = x_1 x_2 \ldots x_k$. There are $i, j \leqq k$, $i < j$, such that $\delta(S, x_1 x_2 \ldots x_i) = \delta(S, x_1 x_2 \ldots x_j)$. For $u_0 = = x_1 x_2 \ldots x_i x_{j+1} \ldots x_k$ we have $l(u_0) < l(u)$ and $\delta(s, uy) = \delta(s, u_0 y)$.

b) $\leftarrow$ Let $\mathscr{A}$ be not the directly subdefinite automaton. Then for every $u \in X^*$ there are $x, y \in X^*$ such that $\delta(s_0, x) \in F$ and $\delta(s_0, xuy) \notin F$. For $\bar{s} = \delta(s_0, x)$ the relation $\delta(\bar{s}, uy) \notin F$ holds.

We need to show that there is $y \in X^*$ with properties $l(y) \leqq |S|\,(|S| - 1)$. Let $y_0 \in X^*$ and $k = l(y_0) \geqq |S|\,(|S| - 1)$. There exist i, j $1 \leqq i < j \leqq k$ such that $\delta(s_0, x_1 \ldots x_i) = \delta(s_0, x_1 \ldots x_j)$ and $\delta(\bar{s}, x_1 \ldots x_i) = \delta(\bar{s}, x_1 \ldots x_j)$, because there are at most $|S|\,(|S| - 1)$ different pairs $\langle \delta(s_0, x_1 \ldots x_h), \delta(\bar{s}, x_1 \ldots x_h) \rangle$. Let $y = x_1 \ldots x_i x_{j+1} \ldots x_k$ then we have $l(y) < l(y_0)$ and $\delta(\bar{s}, uy) \notin F$.

3.10. Corollary. *The directly subdefinite property for finite automaton can be effectively checked in a finite number of steps.*

3.11. Example. Let us have the finite automaton given in the fig. 5. For $u = 1001$ we have $T(\mathscr{A}) \in S$. $S_u = \{g\}$ by the definition. We must point out that $(\forall x \in X^*)\ \delta(a, x) \in F \rightarrow \delta(g, x) \in F$. (In fig. 5 $e \xrightarrow{1} g$.)

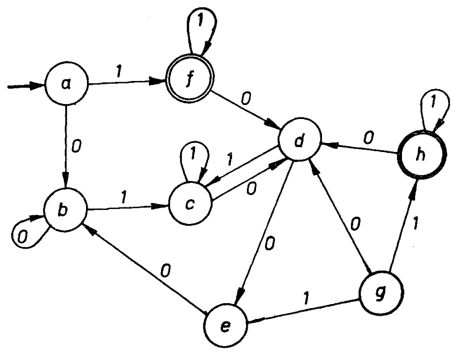

Fig. 5.

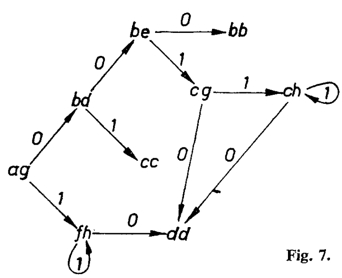

Fig. 6.

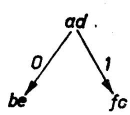

Fig. 7.

It is obvious from the tree shown in the fig. 6. (The pairs in the tree are ordered e.g. $(b, e) \xrightarrow{1} (c, g)$ means $b \xrightarrow{1} c$ and $e \xrightarrow{1} g$.) If the first state of pair belongs to F then the second state of it belongs to F, too.

For $u = 10$, $T(\mathscr{A}) \notin u\mathbf{S}$ because $S_u = \{d\}$ and it follows from the tree shown in the fig. 7 that $1 \in T(\mathscr{A})$ and $101 \notin T(\mathscr{A})$.

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REFERENCES

[1] Brzozowski J. A.: Roots of Star Events. JACM *14* (1967), 467—470.
[2] Černý J.: Poznámka k homogénnym experimentom s konečnými automatmi. Mat.-fyz. čas. SAV (1964), 208—215.
[3] Havel I. M.: The Theory of Regular Events II. Kybernetika *5* (1969), 520—544.
[4] Paz A., Peleg B.: Ultimate-Definite and Symmetric-Definite Events and Automata. JACM *12* (1965), 399—410.

VÝŤAH

Usmernitelné automaty a priamo subdefinitné javy

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Konečný automat $\mathscr{A} = (S, X, s_0, \delta, F)$ je usmernitelný do počiatočného stavu, ak existuje slovo u s vlastnosťou $\delta(s, u) = s_0$ pre každý stav $s \in S$.

Úlohou článku je charakterizovať javy, ktoré je možné predstaviť týmito automatmi.

a) Nech je jav R daný svojím regulárnym výrazom. Jav R môžeme reprezentovať automatom usmernitelným do počiatočného stavu práve vtedy, keď existuje slovo u a javy R_2, T také, že $R = (X^*u \cup T)^* R_2$ a $R_1^* = (X^*u \cup T)^*$ je jednoduchý iteračný jav, pričom pre každé slovo u platí $\partial_u R_1^* = R_1^* \rightarrow \partial_u R = R$.

b) Nech je jav R daný konečným automatom $\mathscr{A}$. (Obecne nemusí byť usmernitelný do počiatočného stavu.) Jav R je možno reprezentovať automatom usmernitelným do počiatočného stavu práve keď minimálny automat k automatu $\mathscr{A}$ je usmernitelný do počiatočného stavu.

Javy, ktoré je možno reprezentovať automatmi usmernitelnými do počiatočného stavu patria do triedy priamo subdefinitných javov, tj. javov typu $(X^*u)^* T$. Trieda priamo subdefinitných javov nie je uzavretá na operáciu zjednotenia, prieniku a komplementu. Pre pevne dané slovo u je táto trieda uzavretá na zjednotenia a prieniky.

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