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A CONCURRENCY CONTROL PROBLEM IN A TIME-SHARING SYSTEM WITH DIFFERENT JOB TYPES*

HANS DADUNA

We consider a time-sharing system where due to the locking of resources by some jobs other jobs are not admitted to the system. We compute the interruption time distribution and prove ergodicity of the system.

0. INTRODUCTION

The problem which is investigated in this note is from the field of controlling the use of shared data by parallel processes. These processes operate on the data in two different ways: Read operations may be done by several processes concurrently while write operations of one process lock the database for other processes.

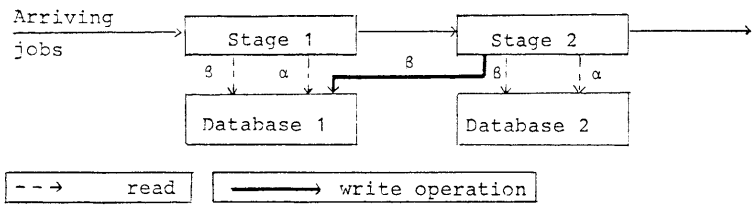

Fig. 1.

Our system is a two-stage tandem, where the second stage is the central processing unit (CPU) which operates in a time-sharing modus while the first stage is a preprocessor which works under a first-come-first-served (FCFS) regime. To each stage a specified database is associated, which has to be accessible for a job to be performed

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at this stage. If the database is not accessible for a job when its service has to be started, then this job is assumed to be lost.

Jobs of different type, say: α, β, resp., arrive at the system, requesting for service at the CPU which can be given to them after being served at the preprocessor. All jobs have to perform read operations at both stages on the data of that stage, and jobs of type β additionally have to update the data of stage 1 while being executed at the CPU. Therefore if a type-β job is present at stage 2 no job can be served at stage 1. More formally: A type-β job leaving stage 1 takes the key for the database of stage 1 with it and gives it back if it leaves stage 2.

We assume the time for passing stage 1 to be negligible compared to the execution time at the CPU. This leads to considering a time sharing service system where two types of jobs are served, and where the arrival process is interrupted while a job of type-β is in the time-sharing system.

The Round-Robin queueing discipline of the time-sharing system is modeled as a FCFS queue with feedback (for a discussion see e.g. [4], [7]).

The main quantity of interest for us is the length of the time during which a type-β job at the CPU locks the system.

Conditioning on the state of the system when the interruption commences, we determine in Section 2 this interruption time distribution for the system which is formally described in Section 1.

In Section 3 we prove ergodicity for the system for any parameter values – in contrast to the classical $M/M/1/\infty$-feedback queue. At least for a thin arrival stream of type-β jobs and large feedback probability this seems to be not intuitively obvious.

In Section 4 we deal with the equilibrium distribution of the system and with the steady state interruption time distribution.

The feedback model under consideration has found some interest in recent research; e.g. by van den Berg, Boxma [1], where a connection to processor sharing systems is pointed out, van den Berg, Boxma Groenendijk [2], Lam, Shankar [9], and the references cited there. For earlier work see [6] and the references cited there. For the interruption time problem in which we are interested the main reference is the work of Takacs [10]. The method used here is adapted from this paper to our problems. A similar evaluation of a different problem was done by Boxma [3].

1. THE MODEL

Customers arrive one by one in a Poisson stream of intensity $\lambda > 0$ at a single server with infinite waiting room. If they are allowed to enter the system they join the tail of the queue if there are any customers waiting or in service. If the server is free their service immediately commences. A customer being served leaves the system with probability $q = 1 - p \in (0, 1]$, with probability p he joins the tail of the queue again requesting for a further service. The service discipline is FCFS.

An arriving customer is of type α with probability $r \in (0, 1)$, with probability $1 - r$ he is of type β.

All interarrival times, service times, type selections and feedback decisions form an independent family.

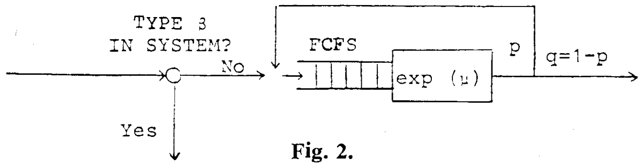

Fig. 2.

The entrance control is as follows: if a type-β customer is present at the server, all arrivals are lost.

2. CONDITIONAL LOCKING TIME DISTRIBUTIONS

We consider the following situation in the model of Section 1:

A type-β customer arrives and is admitted to the system because he finds only type-α customers present. At his arrival the arrival stream is interrupted due to the entrance control. (Because this type-β customer has to update the data at the pre-processor, his exclusive write access prevents other customers from entering the system – which needs read operations on that data.)

We want to compute the conditional locking time of the system given the number of type-α customers the type-β customer finds at his arrival, i.e.: we have to compute the conditional response time of a type-β customer given the queue length just before his arrival.

During the type-β customer's sojourn a suitable state description of the system is given by

$$\boldsymbol{E} = \{(h, u) : h, u \in \mathbb{N}\}$$

where the state (h, u) indicates that the type-β customer sees u other customers before and h other customers behind him, all of type-α. Using $\boldsymbol{E}$ as state space the time development of the system is described by a two-dimensional death process.

We denote by

$$f(h, u; s), \quad s \geqq 0,$$

the conditional Laplace-Stieltjes-transform (LST) of the type-β customer's sojourn time in the system given the actual state of the system is $(h, u) \in \boldsymbol{E}$.

The strong Markov property of the describing process yields the following set of first-entrance equations for the conditional LSTs:

$$f(h, u; s) = \frac{\mu}{\mu + s} \left[1_{(u>0)} p f(h + 1, u - 1; s) + 1_{(u>0)}(1 - p) f(h, u - 1; s) + \right.$$
$$\left. + 1_{(u=0)} p f(0, h; s) + 1_{(u=0)}(1 - p) 1\right], \quad (h, u) \in \boldsymbol{E}. \tag{1}$$

Introducing generating functions and writing $q = 1 - p$ we obtain from (1)

$$\sum_{h=0}^{\infty} \sum_{u=0}^{\infty} x^h y^u f(h, u; s) \left[\mu + s - \mu p \frac{y}{x} - \mu q y\right] =$$

$$= \sum_{h=0}^{\infty} x^h f(0, h; s)\, \mu p - \sum_{u=0}^{\infty} y^u f(0, u; s)\, \mu p \frac{y}{x} + \frac{\mu q}{1 - x},$$

$$|x| < 1, \quad |y| < 1. \tag{2}$$

For $\bar{y} \in [0, 1]$ we set $\bar{x} := \mu p \bar{y}/(\mu(1 - q\bar{y}) + s) \in [0, 1]$, and inserting the pairs $(\bar{y}, \bar{x})$, $y \in [0, 1)$ into (2) we obtain:

$$\sum_{u=0}^{\infty} y^u f(0, u; s)\, [\mu(1 - qy) + s] =$$

$$= \sum_{h=0}^{\infty} \left(\frac{\mu p y}{\mu(1 - qy) + s}\right)^h f(0, h; s)\, \mu p + \mu q \frac{\mu(1 - qy) + s}{\mu(1 - y) + s}, \quad y \in [0, 1). \tag{3}$$

Introducing the transformation (for fixed $s > 0$)

$$T\colon \mathbb{R} \to \mathbb{R}$$

$$y \to \frac{\mu p y}{\mu(1 - qy) + s}$$

equation (3) can be written as

$$\sum_{u=0}^{\infty} y^u f(0, u; s) =$$

$$= \sum_{h=0}^{\infty} (T(y))^h f(0, h; s) \frac{T(y)}{y} + \frac{q}{p} \frac{1}{1 - T(y)} \frac{T(y)}{y}, \quad y \in (0, 1). \tag{4}$$

Because we have $T(0, 1) \subseteq (0, 1)$ we obtain by induction from (4) for any $k \in \mathbb{N}_+$:

$$\sum_{u=0}^{\infty} y^u f(0, u; s) =$$

$$= \sum_{h=0}^{\infty} (T^k(y))^h f(0, h; s) \frac{T^k(y)}{y} + \frac{q}{p} \sum_{i=1}^{k} \frac{T^i(y)}{1 - T^i(y)} \frac{1}{y}. \tag{4'}$$

Applying equation (4) to $T^k(y)$ instead of y we obtain

$$\sum_{u=0}^{\infty} (T^k(y))^u f(0, u; s) =$$

$$= \sum_{h=0}^{\infty} (T(T^k(y)))^h f(0, h; s) \frac{T(T^k(y))}{T^k(y)} + \frac{q}{p} \frac{1}{1 - T(T^k(y))} \frac{T(T^k(y))}{T^k(y)}.$$

Changing the notation $(u \to h)$ this can be inserted into the RHS of (4′), which by some direct manipulations yields (4′) for $k + 1$.

Now for all $y \in [0, 1)$ is the attracting fixpoint 0:

$$T^u(y) \to 0\,, \quad u \to \infty\,, \quad y \in [0, 1)\,.$$

This implies:

$$\sum_{u=0}^{\infty} y^u f(0, u; s) = \frac{q}{py} \sum_{i=1}^{\infty} \frac{T^i(y)}{1 - T^i(y)}\,, \quad y \in (0, 1)\,. \tag{5}$$

It remains to find an explicit expression for the RHS of (5).

Let

$$\mathbf{L}(\eta; \xi; x) = \sum_{k=1}^{\infty} \eta \xi^k \frac{x^k}{1 - x^k}\,, \quad |x| < 1\,, \quad |\alpha| < 1\,, \quad \beta \in \mathbb{R}\,,$$

denote the (special) *Lambert* series (cf. [8], p. 464).

We have the following result:

Theorem 1. Let

$$f(y, s) = \sum_{u=0}^{\infty} y^u f(0, u; s)\,, \quad s > 0\,, \quad y \in [0, 1]\,,$$

be the generating function of the sequence of conditional LST of the response time distribution for type-β customers. Then

$$f(y, s) = \mathbf{L}\left(\frac{q(\mu q + s)}{psy}; \frac{sy}{\mu q(1 - y) + s}; \frac{\mu p}{\mu + s}\right), \quad s \geqq 0\,, \quad y \in [0, 1]\,.$$

Proof. The function T is a linear transformation; the representation matrix of T is

$$\mathbf{T} = \begin{bmatrix} \mu p & 0 \\ -\mu q & \mu + s \end{bmatrix}$$

Let the i-fold iteration T^i of T have the representing matrix power of $\mathbf{T}$

$$\mathbf{T}^i = \begin{bmatrix} a_i & b_i \\ c_i & d_i \end{bmatrix}, \quad i \in \mathbb{N}_+\,.$$

It follows that

$$\frac{T^i(y)}{1 - T^i(y)} = \frac{a_i y + b_i}{(c_i - a_i)\, y + (d_i - b_i)}$$

is a linear transformation with representation matrix

$$\mathbf{S}_i = \begin{bmatrix} a_i & b_i \\ c_i - a_i & d_i - b_i \end{bmatrix}, \quad i \in \mathbb{N}_+\,.$$

It follows by induction:

$$\mathbf{S}_{i+1} = \mathbf{S}_i \,.\, \mathbf{T} = \mathbf{S}_1 \,.\, \mathbf{T}^i\,, \quad i \geqq 1\,,$$

and

$$\mathbf{S}_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 - a_1 & d_1 - b_1 \end{bmatrix} = \begin{bmatrix} \mu p & 0 \\ -\mu & \mu + s \end{bmatrix}.$$

Again by induction we can prove:

$$\mathbf{T}^k = \begin{bmatrix} (\mu p)^k & 0 \\ -\mu q \sum\limits_{\substack{i,j \geqq 0 \\ i+j=k-1}} (\mu p)^j (\mu + s)^i & (\mu + s)^k \end{bmatrix}, \quad k \geqq 1 ,$$

and

$$\mathbf{S}^k = \begin{bmatrix} (\mu p)^k & 0 \\ -(\mu p)^k - \mu q \sum\limits_{\substack{i,j \geqq 0 \\ i+j=k-1}} (\mu p)^j (\mu + s)^i & (\mu + s)^k \end{bmatrix}, \quad k \geqq 1 .$$

Recalling that $\mathbf{S}^k$ is the representation matrix of the linear transformation

$$\frac{T^k(\cdot)}{1 - T^k(\cdot)},$$

we obtain for $y \in (0, 1)$ in the RHS of (5):

$$\frac{q}{py} \sum_{k=1}^{\infty} \frac{T^k(y)}{1 - T^k(y)} = \frac{q}{p} \sum_{k=1}^{\infty} \frac{(\mu p)^k}{\left(-(\mu p)^k - \mu q \sum\limits_{i+j=k-1} (\mu p)^j (\mu + s)^i\right) y + (\mu + s)^k} =$$

$$= \frac{q}{p} \frac{\mu q + s}{sy} \sum_{m=1}^{\infty} \left(\frac{sy}{\mu q(1 - y) + s}\right)^m \frac{\left(\dfrac{\mu p}{\mu + s}\right)^m}{1 - \left(\dfrac{\mu p}{\mu + s}\right)^m} . \qquad \square$$

Remarks. a) From

$$\mathbf{L}(\eta; \xi; x) = \sum_{k=1}^{\infty} x^k \Big(\sum_{\substack{m=1 \\ m|k}}^{k} \eta \xi^m \Big),$$

where "$m \mid k$" means "m divides k", (cf. 8, p. 466), we obtain $f(y, s)$ merely as power series in $\mu p/(\mu + s)$.

b) The *Lambert* series $\mathbf{L}(1; 1; x)$ has found considerable interest in number theory, see e.g. [8].

c) Theorem 1 can be interpreted as a proposition on a $G/M/1/\infty$-FCFS queue with Bernoulli feedback in steady state which is perturbated by a break down of the arrival process due to a special customer's arrival.

The number of customers the interrupting customer finds in this system is geometrically distributed with parameter $\gamma \in (0, 1)$. Here γ is the unique solution in $(0, 1)$ of

$$\gamma = \int_{[0,\infty)} e^{-t\mu(1-p)(1-\gamma)} A(dt),$$

where $A(\cdot)$ is the cumulative distribution function of the interarrival distribution, (which is assumed to be non-lattice). The ergodicity condition $\lambda < \mu \cdot q$ guarantees the existence and uniqueness of γ, see [5], p. 653. The response time of the interrupting customer is given by its LST

$$f(s) = \mathbf{L}\left(\frac{q(1-\gamma)}{p\gamma}\left(1 + \frac{\mu q}{s}\right); \frac{\gamma s}{\mu q(1-\gamma) + s}; \frac{\mu p}{\mu + s}\right), \quad s \geqq 0 .$$

d) From the theorem we obtain

$$f(0, u; s) = \left(\frac{\mu q}{\mu q + s}\right)^u \sum_{k=0}^{u} \binom{u}{k} \left(\frac{ps}{q}\right)^k \frac{\mu q}{(\mu + s)^{k+1} - (\mu p)^{k+1}},$$

which directly yields the moments of the conditional interruption times.

3. THE ERGODICITY OF THE SYSTEM

In the classical $M/M/1/\infty$-FCFS system with feedback determining the equilibrium conditions poses no serious problems. Let us assume that only the type-α customers arrive at the system in a Poisson-λr-stream. Then the queue length process is transient iff $\lambda r > \mu q$, recurrent iff $\lambda r \leqq \mu q$, and ergodic iff $\lambda r < \mu q$ holds.

Introducing the additional Poisson-$\lambda(1 - r)$-arrival stream of type-β customers and incorporating the described concurrency control a new problem appears.

Let us first consider the "transient" case: We assume $\lambda r > \mu$, i.e.: for any feedback probability the type-α customers alone would create an infinite backlog of work over time.

What would be the effect of introducing type-β customers and the concurrency control protocol?

For $p = 0$ the answer is obvious:

For any arrival intensity of α- and β-type customers the queue length process is ergodic. For, if a type-β customer is admitted to the system, he clears the system completely. This observation makes the following suggestion reasonable:

For $\lambda r > \mu$ and λr near to μ for sufficiently small feedback probability $p > 0$ the queue length process of the system should be recurrent.

The following theorem states that this intuitive reasoning is correct.

We describe the system's development by a Markov process $X = (X_t : t \geqq 0)$ with state space

$$\boldsymbol{S} = \mathbb{N}^2 \cup \mathbb{N}$$

where the states have the following interpretation:

$X_t = u \in \mathbb{N} \Leftrightarrow u$ type-α customers are present ,

$X_t = (h, u) \in \mathbb{N}^2 \Leftrightarrow$ one type-β customer is present having u type-α customers before and h type-α customers behind him .

We have the following

Theorem 2. If $r \in (0, 1)$, then for any choice of the system's parameter $X = (X(t) : t \geqq 0)$ is ergodic.

Proof. From the irreducibility of X it follows that the limits

$$\lim_{t\to\infty} \mathsf{P}(X(t) = k) =: \pi(k), \quad k \in S,$$

exist and are independent of the initial conditions. We have to show that

$$\pi = (\pi(k): k \in \boldsymbol{S})$$

is a probability measure.

We assume $X(0) \equiv 0$. Let

$$0 \leqq \zeta_1 < \zeta_2 < \dots$$

be the sequence of jump times of X, and

$$Y = (X(\zeta_n) =: Y(n): n = 1, 2 \dots)$$

the embedded jump chain of X. We shall show that Y is positive recurrent.

Then, if $\tilde{\pi} = (\tilde{\pi}(k): k \in \boldsymbol{S})$ is the limiting distribution of Y,

$$\left(\frac{\tilde{\pi}(k)}{q(k)} : k \in \boldsymbol{S}\right)$$

solves the steady-state equations of X, where

$$q(k) = -\lim_{h\downarrow 0} \frac{1}{h} \left(\mathsf{P}(X_{t+h} = k \mid X_t = k) - 1\right), \quad k \in \boldsymbol{S}.$$

Now we have only finitely many different $q(k)$, $k \in \boldsymbol{S}$, which implies that

$$\left(\frac{\tilde{\pi}(k)}{q(k)} : k \in \boldsymbol{S}\right)$$

may be normalized to obtain π. The regularity of X guarantees positive recurrence.

In the following we assume: $p > 0$, and right continuous paths with left limits for X. Let

$$0 \leqq \sigma_1 < \sigma_2 < \dots$$

be the sequence of arrival times of type-β customers which are admitted to the system, and

$$0 < \tau_1 < \tau_2 < \dots$$

be the sequence of departure times of type-β customers from the system after being served there. The (ζ_n), (σ_i), (τ_i) are Markov times for X.

Let N_i denote the number of service times the ith type-β customer who is admitted to the system requests for. N_i is geometrically distributed on $\{1, 2, \dots\}$. For $k \in \boldsymbol{S}$ let

$$L(k) = \begin{cases} u & \text{if} \quad k = u \\ h + u + 1 & \text{if} \quad k = (h, u) \end{cases}$$

denote the queue length (number in system) of the system. We shall prove in a first step that the sequence of expectations

$$\mathsf{E}[L(X(\zeta_n))], \quad n \geqq 1,$$

is bounded by a constant $C > 0$.

We have for $i > 1$:

$$\mathsf{E}[L(X(\tau_i))] =$$

$$= \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \mathsf{E}[L(X(\tau_i)) \mid L(X(\sigma_i)) = j, N_i = l] \, \mathsf{P}(N_i = l, L(X(\sigma_i)) = j).$$

Now, given $N_i = l$ and $L(X(\sigma_i)) = j$, the type-α customers staying in the system when the type-β customers departs are selected by $j - 1$ independent Bernoulli experiments with success parameter p^l, so the number of type-α customers still in the system when the type-β customer departs has a binomial distribution with expectation $(j - 1) p^l$, which yields:

$$\mathsf{E}[L(X(\tau_i))] = \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} [(j - 1) p^l] \, p^{l-1} q \, \mathsf{P}(L(X(\sigma_i) = j)) =$$

$$= \sum_{j=1}^{\infty} (j - 1) \, \mathsf{P}(L(X(\sigma_i)) = j) \frac{p}{1 + p} = (\mathsf{E}[L(X(\sigma_i))] - 1) \frac{p}{1 + p}.$$

From

$$\mathsf{E}[L(X(\sigma_1))] \leqq \frac{1}{\lambda(1 - r)} \lambda r + 1,$$

and

$$\mathsf{E}[L(X(\sigma_i))] \leqq \mathsf{E}[L(X(\tau_{i-1}))] + r/(1 - r) + 1, \quad i \geqq 2,$$

we obtain

$$\mathsf{E}[L(X(\tau_1))] \leqq \frac{r}{1 - r} \frac{p}{1 + p},$$

$$\mathsf{E}[L(X(\tau_{i+1}))] \leqq \left\{\mathsf{E}[L(X(\tau_i))] + \frac{r}{1 - r}\right\} \frac{p}{1 + p}, \quad i \geqq 1,$$

which finally yields:

$$\mathsf{E}[L(X(\tau_i))] \leqq \frac{rp}{1 - r}\left(1 - \left(\frac{p}{1 + p}\right)^i\right) \leqq \frac{rp}{1 - r}, \quad i \geqq 1.$$

Substituting this bound, we obtain

$$\mathsf{E}[L(X(\sigma_i))] \leqq \frac{rp}{1 - r} + \frac{r}{1 - r} + 1 = \frac{1 + rp}{1 - r}.$$

Now for any $\zeta_n \in [\tau_i, \sigma_{i+1}]$ we have

$$\mathsf{E}[L(X(\zeta_n))] \leqq \mathsf{E}[L(X(\tau_i)) + r/(1 - r) + 1], \quad n \in \mathbb{N},$$

which yields

$$C := \frac{1 + rp}{1 - r}$$

as a uniform bound for the queue length expectations. This will be used to show that the sequence $(L(X(\zeta_n)): n \geqq 1)$ is a tight sequence of probability measures. This implies weak convergence to a probability measure on $\boldsymbol{S}$, which is just $\tilde{\pi}$.

For tightness we shall show that for each $\varepsilon > 0$ there is a finite subset $A_\varepsilon \subseteq \boldsymbol{S}$, such that

$$\mathsf{P}(X(\zeta_n) \in A_\varepsilon) > 1 - \varepsilon\,, \quad n \in \{1, 2, \ldots\}\,,$$

holds. For $\varepsilon > 0$ let $M(\varepsilon) := \min\,(m \in \mathbb{N}: C/m \leqq \varepsilon)$, and

$$A_\varepsilon := \{k \in \boldsymbol{S}: L(k) < M(\varepsilon)\}\,.$$

Then from

$$C \geqq \mathsf{E}[L(X(\zeta_n))] \geqq \sum_{j=M(\varepsilon)}^{\infty} j\,\mathsf{P}(L(X(\zeta_n)) = j) \geqq M(\varepsilon)\,\mathsf{P}(L(X(\zeta_n)) \geqq M(\varepsilon))$$

we obtain for all $n \geqq 1$:

$$\varepsilon \geqq \frac{C}{M(\varepsilon)} \geqq \mathsf{P}(L(X(\zeta)) \geqq M(\varepsilon)) = \mathsf{P}(X(\zeta_n) \in A_\varepsilon^c)\,.$$

This completes the proof. □

Remark. The methods used in the proof can be used to construct an increasing sequence of lower bounds for the

$$\mathsf{E}[L(X(\zeta_n))], \quad n \geqq 1\,.$$

4. STEADY STATE DISTRIBUTIONS

We consider the model of Sections 1 and 3 and want to compute the steady state interruption time distribution as well as the equilibrium distribution for X.

The first is obtained by conditioning via the results of Section 2 if the second is known:

From the Poisson assumption it follows that any arriving customer sees the other customers distributed according to the steady state distribution, irrespective of whether he is lost or allowed to joint the queue.

Conditioning on the event "a type-β customer arrives and joins the queue" we obtain our result on the distribution of the α-type customers when an interruption time commences.

We have the following

Theorem 3. In the system of Section 1 and 3 the equilibrium distribution $\pi = (\pi(k): k \in \boldsymbol{S})$ is given by its generating function in formulas (11), (12) below.

Proof. The equilibrium flow equations which in our case determine π uniquely if it exists are

(A1) $\pi(0)\,\lambda = \pi(0,0)\,\mu q + \pi(1)\,\mu q$

(A2) $\pi(u)\,(\lambda + \mu q) = \pi(u-1)\,\lambda r + \pi(u+1)\,\mu q + \pi(u,0)\,\mu q\,, \quad u \geqq 1$

(B1) $\pi(0,0)\,\mu q = \pi(0)\,\lambda(1-r) + \pi(0,1)\,\mu q$

(B2) $\pi(0,u)\,\mu = \pi(u)\,\lambda(1-r) + \pi(0,u+1)\,\mu q + \pi(u,0)\,\mu p\,, \quad u \geqq 1$

(B3) $\pi(h,u)\,\mu = \pi(h-1,u+1)\,\mu p + \pi(h,u+1)\,\mu q\,, \quad h \geqq 1\,, \quad u \geqq 0$

Introducing

$$H(y) = \sum_{u=0}^{\infty} y^u\,\pi(u)\,, \quad G(x,y) = \sum_{h=0}^{\infty}\sum_{u=0}^{\infty} x^h y^u\,\pi(h,u)\,, \quad |x| \leqq 1\,, \quad |y| \leqq 1\,,$$

we obtain from (A)

$$H(y)\,[\lambda(1-ry) + \mu q(1-1/y)] = G(y,0)\,\mu q + \pi(0)\,\mu q(1-1/y)\,, \quad |y| \leqq 1\,, \tag{6}$$

and from (B)

$$\begin{aligned} &G(x,y)\,\mu[1-(px+q)/y] = \\ &= -G(x,0)\,\mu(px+q)/y + G(y,0)\,\mu p + H(y)\,\lambda(1-r)\,, \\ &|x| \leqq 1\,, \quad |y| \leqq 1\,. \end{aligned} \tag{7}$$

Inserting (6) into (7) yields

$$\begin{aligned} &G(x,y)\,[1-(px+q)/y] = \\ &= -G(x,0)\,(px+q)/y + G(y,0)\left[p + \frac{q\lambda(1-r)}{\lambda(1-ry) + \mu q(1-1/y)}\right] + \\ &+ \pi(0)\,\frac{q\lambda(1-r)\,(1-1/y)}{\lambda(1-ry) + \mu q(1-1/y)}\,, \quad |x| < 1\,, \quad |y| < 1\,. \end{aligned} \tag{8}$$

Introducing the transformation

$$S\colon \mathbb{R} \to \mathbb{R}$$
$$y \to py + q$$

we obtain from (8):

$$\begin{aligned} G(x,0) = {}& G(S(x),0)\left[p + \frac{q\lambda(1-r)}{\lambda(1-rS(x)) + \mu q\,(1-1/S(x))}\right] + \\ &+ \pi(0)\,\frac{q\lambda(1-r)\,(1-1/S(x))}{\lambda(1-r\,S(x)) + \mu q(1-1/S(x))}\,, \quad |x| \leqq 1\,. \end{aligned} \tag{9}$$

Iterating (9) similarly to the iteration of (4) in the proof of Theorem 1 we obtain with

$$W(y) = p + \frac{q\lambda(1-r)}{\lambda(1-ry) + \mu q(1-1/y)}$$

and

$$V(y) = \frac{q\lambda(1-r)(1-1/y)}{\lambda(1-ry) + \mu q(1-1/y)},$$

$$G(x,0) = \pi(0)\,[\sum_{i=1}^{\infty} V(S^i(x))\prod_{k=1}^{i-1} W(S^i(x))] + G(1,0)\,[\prod_{k=1}^{\infty} W(S^k(x))],$$

$$|x| \leqq 1, \tag{10}$$

where we used, that 1 is the only fixed point of S in $\mathbb{R}$ which is attracting for $[-1, 1]$. Denoting by η the smaller root of $\lambda r y^2 - (\lambda + \mu q)\, y + \mu q = 0$ which is the only one in $[-1, 1]$, we obtain from (6)

$$G(\eta, 0) = \pi(0)\,(\eta^{-1} - 1)$$

and, using this, from (10)

$$G(1,0) = \pi(0)\, R(\eta),$$

where

$$R(\eta) = (\eta^{-1} - 1)\,[\sum_{i=1}^{\infty} V(S^i(\eta))\prod_{k=1}^{i-1} W(S^k(\eta))]\,[\prod_{k=1}^{\infty} W(S^k(\eta))]^{-1}.$$

Inserting this again into (10) we obtain

$$G(x,0) = \pi(0)\,\{[\sum_{i=1}^{\infty} V(S^i(x))\prod_{k=1}^{i-1} W(S^k(x))] + [\prod_{k=1}^{\infty} W(S^k(x))]\, R(\eta)\}, \quad |x| \geqq 1.$$

From (8) it follows that

$$G(x,y) = \pi(0)\,(1 - S(x)/y)^{-1}\,.$$
$$.\,\{W(y)\,([\sum_{i=1}^{\infty} V(S^i(y))\prod_{k=1}^{i-1} W(S^k(y))] + [\prod_{k=1}^{\infty} W(S^k(y))]\, R(\eta)) -$$
$$- S(x)/y\,([\sum_{i=1}^{\infty} V(S^i(x))\prod_{k=1}^{i-1} W(S^k(x))] + [\prod_{k=1}^{\infty} W(S^k(x))]\, R(\eta))\},$$

$$|x|, |y| < 1, \tag{11}$$

and from (6) we have

$$H(y) = \pi(0)\,\frac{\mu q}{\lambda(1-ry) + \mu q(1-y^{-1})}\,.$$
$$.\,\{(1 - 1/y) + [\sum_{i=1}^{\infty} V(S^i(y))\prod_{k=1}^{i-1} W(S^k(y))] + [\prod_{k=1}^{\infty} W(S^k(y))]\, R(\eta)\},$$

$$|y| < 1. \tag{12}$$

$\pi(0)$ finally is obtained by normalization. □

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REFERENCES

[1] J. L. van den Berg and O. J. Boxma: Sojourn Times in Feedback Queues. CWI-Report OS-R 8710, Amsterdam 1987.

[2] J. L. van den Berg, O. J. Boxma and W. P. Groenendijk: Sojourn Times in the M/G/1 Queue with Deterministic Feedback. CWI-Report OS-R 8708, Amsterdam 1987.

[3] O. J. Boxma: A queueing model for finite and infinite source interaction. Oper. Res. Lett. *5* (1986), 245—254.

[4] W. Chang: Queues with feedback for time-sharing computer system analysis. Oper. Res. *16* (1968), 613—627.

[5] J. W. Cohen: The Single Serve Queue. Second edition. North-Holland, Amsterdam 1982.

[6] R. L. Disney and D. König: Queueing networks: A survey of their random processes. SIAM Review *27* (1985), 335—403.

[7] L. Kleinrock: Analysis of a time-shared processor. Naval Res. Log. Quart. *11* (1964), 59—73.

[8] K. Knopp: Theorie und Anwendung der unendlichen Reihen. Springer-Verlag, Berlin—Göttingen—Heidelberg—New York 1964.

[9] S. S. Lam and A. U. Shankar: A derivation of response time distribution for a multi-class feedback queueing system. Performance Eval. *1* (1981), 48—61.

[10] L. Takacs: A single-server queue with feedback. Bell System Techn. J. *42* (1963), 505—519.

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