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INTEGRAL NETS AND FUZZY RELATIONS IN DETERMINISTIC AUTOMATA

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A net structure is placed on the state set Q of an automaton $A = (X, Q)$ by counting, for any ordered pair $q_1, q_2 \in Q$ the number of elementary inputs $x \in X$ which send q_1 to q_2. For homogeneous automata this structure is used to obtain a net isomorphism theorem between Q and a group quotient, thus, by displaying more information, improving on a previous theorem of the author's in which the action of X simply defines a relation on Q. In fact dividing the number of elementary inputs taking q_1 to q_2 by the cardinality of X transforms the net structure into a fuzzy relation on Q. Composition of fuzzy relations is discussed in this context.

An illustrative example is constructed.

1. INTRODUCTION

1.1. Attempts to formulate a notion of continuity in the theory of finite automata have led to the introduction of a tolerance (reflexive symmetric relation) in the state space (Arbib [1]). This was later refined by Warner [5] into a relation which is not in general symmetric or even reflexive.

1.2. We assume an *automaton* A to be a pair (X, Q) where X, the finite input set, acts on Q, the finite state set, by right translation $q \to q \,.\, x$, $q \in Q$, $x \in X$. This is sometimes called a semi-automaton (Ginzburg [2]), but since we are not concerned here with outputs we shall, for brevity and convenience, simply use the term automaton.

1.3. The *inertial tolerance* ϱ introduced by Arbibs is defined for q, q' in Q by $q \varrho q'$ iff there exists $x \in X$ such that $q' = q \,.\, x$ or $q = q' \,.\, x$. Warner's modification introduces an *inertial relation* v given by $q \, v \, q'$ iff there exists $x \in X$ such that $q' = q \,.\, x$. This is clearly not in general either symmetric or reflexive. The empty (or identity) input is deemed to be in X.

1.4. Both these structures have proved useful in studying automata (Arbib [1], Muir and Warner [4], Warner [5]). However the gains in insight afforded by the imposition of a mathematical structure on the state space are accompanied, inevitably it would seem, by loss of information. More specifically, if it is known that $q \, v \, q'$, $q, q' \in Q$, there is no clue about which input x sends q to q', nor is it known how many inputs have this property. The aim of this paper is to rectify the latter deficiency.

1.5. A function $\Phi : Q \times Q \to \mathbb{Z}^{\geqq 0}$ from ordered pairs of elements of Q to the set of non-negative integers is defined by letting $\Phi(q, q')$ be the number of different elementary inputs (elements x of X) such that $q' = q \, . \, x$. If $\Phi(q, q')$ is divided by the cardinality of X it may be regarded as the probability that the machine in the state q will achieve the state q' by the action of one elementary input selected at random. Finite sets Q with real valued *connectivity functions* $\Phi : Q \times Q \to \mathbb{R}$ were considered by Muir [3] in the context of neural nets. Our input-count function $\Phi : Q \times Q \to \mathbb{Z}^{\geqq 0}$ will therefore be called an *integral connectivity function*, and (Q, Φ) an *integral net*. We shall not at first replace the range by the rationals in $[0, 1]$ by conversion to a probability function.

1.6. The second section is devoted to establishing the necessary tools of net theory to apply to the study of automata. It should be noted that some of the definitions, for example of function space connectivity, are imposed by the context and are neither unique nor necessarily the most suitable in other circumstances. Some alternative definitions will be referred to at the appropriate times.

1.7. In Section 3 we study net structures on groups, in order to equip the group of isomorphisms generated by the permutation inputs with a function space connectivity. Such a group is used to establish in Section 4 a group quotient isomorphism theorem for homogeneous automata. This is the appropriate version of the corresponding theorem for automata with the inertial relation established in Warner [5].

1.8. It will not have escaped attention that the probability function of Section 1.5, $p : (q, q') \to \Phi(q, q')/[X]$, defines a fuzzy subset of $Q \times Q$, or, more relevantly, a fuzzy relation on Q. We have thus simply passed from a relation v on Q to a fuzzy relation p. Section 5 is devoted to a brief discussion of fuzziness, with particular reference to the formulation of a two-stage version of the main theorem in terms of the composition of fuzzy relations.

It should be noted that we are not dealing with a fuzzy automaton in the usual sense (Wee and Fu [6]). In such a case the fuzzy relation is a fuzzy subset of $X \times Q \times Q$ representing the probability that a given input and given state will give rise to a given new state. Thus the next-state function is itself fuzzy, in contrast to our fuzzy relation which arises from a non-fuzzy next-state function in a deterministic automaton.

1.9. We conclude in Section 6 with a simple example of a homogeneous automaton with its connectivities and group quotient isomorphism.

2. INTEGRAL NETS

2.1. Definition. An *integral net* is a finite set Q with a non-negative integer valued function $\Phi : Q \times Q \to \mathbb{Z}^{>0}$ called the *integral connectivity function* of the net.

2.2. Definition. A *homomorphism* from (Q, Φ) to (Q', Φ') is a function $f : Q \to Q'$ such that for all $q_1, q_2 \in Q$

$$\Phi(f(q_1), f(q_2)) \geqq \Phi(q_1, q_2)$$

Thus, in the automaton example of § 1.5 if q_1 and q_2 are linked by n elementary inputs, then $f(q_1)$ and $f(q_2)$ are also linked by at least n elementary inputs.

2.3. Definition. If a homomorphism f is bijective and its inverse function f^{-1} is also a homomorphism then f is an *isomorphism*.

Two nets are therefore isomorphic if corresponding pairs of points have the same connectivity.

2.4. We consider henceforth only integral nets which arise from automata as described in § 1.5, i.e. in $A = (X, Q)$, $\Phi(q, q')$ is the number of different elementary inputs x such that $q' = q \,.\, x$.

Let $A = (X, Q)$, $A' = (X', Q')$ be two automata.

Definition. A *homomorphism* from A to A' is a pair (β, γ), $\beta : Q \to Q'$, $\gamma : X \to X'$ such that $\beta(q) \,.\, \gamma(x) = \beta(q \,.\, x)$ for all $x \in X$, $q \in Q$.

Lemma. In a homomorphism (β, γ) from A to A', β induces a net homomorphism from (Q, Φ) to (Q', Φ') if γ is injective.

Proof. $q \mapsto q \,.\, x \Rightarrow \beta(q) \mapsto \beta(q) \,.\, \gamma(x)$. Thus the number of inputs taking q_1 to $q_2 \leqq$ the number of inputs taking $\beta(q_1)$ to $\beta(q_2)$, since different inputs x correspond to differing $\gamma(x)$. □

2.5. Definition. If $f, g : (Q, \Phi) \to (Q', \Phi')$, define the connectivity function $\alpha_A(f, g)$ to be the number of elementary inputs $x' \in X'$ such that $x'f = g$. When $Q = Q'$ and f is itself a string of inputs we shall write $f \,.\, x' = g$.

The function α_A is called the *automata connectivity function* on the set of functions, or sometimes less generally homomorphisms, from Q to Q'. In general it may well be that $\alpha_A(f, g)$ is zero.

2.6. In Muir's net theory $\alpha_M(f, g)$ is defined to be $\min_q \Phi'(f(q), g(q))$, which, interpreted in terms of automata, means that $\alpha_M(f, g)$ is computed by observing the number of elementary inputs taking $f(q)$ to $g(q)$ for each q in Q, then selecting the minimum such number.

2.7. Definition. Given connectivities α, α' on a set X, then α is said to be *finer* than α' (α' *coarser* than α) if $\alpha(x_1, x_2) \leqq \alpha'(x_1, x_2)$ for all $x_1, x_2 \in X$.

Lemma. For automata $A = (X, Q)$, $A' = (X', Q')$ the automata connectivity function α_A is finer than α_M in the function set Q'^Q.

Proof. $\alpha_M(f, g) = \min_q$ (number of elementary inputs taking $f(q)$ to $g(q)$) $\geqq$ number of elementary inputs taking $f(q)$ to $g(q)$ for all $q = \alpha_A(f, g)$. □

When $A' = A$ we note that the roles of α_A and α_M correspond to those of the inertial and coarse function space relations respectively in the semi-group generated by X. (Warner [5]).

3. GROUP NETS

3.1. Let $A = (X, Q)$ be a permutation automaton, viz one all of whose elementary inputs are permutations. Let (Q, Φ) be its integral net structure, and let G be the group of strings of elements of X and their inverses. It is straightforward to verify that G is in fact a group under composition. Assume henceforth that G, like X, acts by right translation on Q, so that g composed with g' is written gg'.

Lemma. (G, α_A) is an integral net whose range $\operatorname{Im} \Phi \subseteq \{0, 1\} \subseteq \mathbb{Z}^{\geqq 0}$.

Proof. $\alpha_A(g, g') = 1$ if $g^{-1}g' \in X$, i.e. there exists $x \in X$ such that $gx = g'$. Such an x is of course unique. The only alternative is that $\alpha_A(g, g') = 0$. □

3.2. Lemma. α_A is invariant under left translations by elements of G.

Proof. Let $g, g', \bar{g} \in G$. $\alpha_A(\bar{g}g, \bar{g}g') = 1$ iff $(\bar{g}g)^{-1}(\bar{g}g') \in X$, i.e. $g^{-1}\bar{g}^{-1}\bar{g}g' = g^{-1}g' \in X \Leftrightarrow \alpha_A(g, g') = 1$. □

3.3. Lemma. If X is closed under conjugation by elements of G, α_A is invariant under right translations.

Proof. Let $g, g', \bar{g} \in G$. Then $\alpha_A(g\bar{g}, g'\bar{g}) = 1$ iff $\bar{g}^{-1}g^{-1}g'\bar{g} \in X$, i.e. $g^{-1}g' \in X$, and this is so iff $\alpha_A(g, g') = 1$. □

3.4. Closure under conjugation by elements of G is in fact equivalent to closure under conjugation by elements of X and their inverses. Such a set X is said to be *normal* in G.

Lemma. If the set X is normal in G then the inputs of X and their inverses are isomorphisms of the integral net (Q, Φ).

Proof. The inputs are assumed to be permutations, so we need only check preservation of connectivity.

Assume X normal in G, and consider the pair $q, q' \in Q$.

Let $q' = q \,.\, \bar{x}$, representing one "join" from q to q'. Then $\forall x \in X$, $(q, q \,.\, \bar{x}) \to$ $\to (q \,.\, x,\; q \,.\, x(x^{-1}\bar{x}x))$. Thus the join represented by $\bar{x}$ gives rise to a unique join between $q \,.\, x$ and $q \,.\, \bar{x}x$, represented by $x^{-1}\bar{x}x$. Since the same holds for $x^{-1} \in X$, it follows that x is an isomorphism, and further that x^{-1} is also an isomorphism. □

3.5. Definition. An *integral group net* is a group G, together with an integer-valued function $\alpha : G \times G \to \mathbb{Z}^{\geqq 0}$ which is left and right translation invariant.

Lemma. In a permutation automaton $A = (X, Q)$, if X is a normal subset of G, then (G, α_A) is an integral group net.

Proof. This follows immediately from Definition 3.5 and Lemmas 3.2 and 3.3.

4. HOMOGENEITY

4.1. Let $q_0 \in Q$ be a fixed base-point in the state-space. Define $\psi : G \to Q$ by $\psi(g) = g(q_0) = q_0 \cdot g$.

Lemma. ψ is a homomorphism from (G, α_A) to (Q, Φ).

Proof. $\Phi(\psi(g), \psi(g')) = \Phi(q_0 \cdot g,\; q_0 \cdot g') =$ no. of elements $x \in X$ such that $q_0 \cdot g' = q_0 \cdot gx \geqq$ no. of elements $x \in X$ such that $gx = g'$. □

4.2. Let H be the subgroup of G which fixes q_0 i.e. $g \in H$ iff $q_0 \cdot g = q_0$, and let G/H be the set of right cosets of H in G. A coset with representative g will be denoted $[g]$.

Then $g \sim g'$ iff $gg'^{-1} \in H$.

Note that in general H is not normal in G so G/H need not be a group.

Definition. Let $\hat{\alpha}_A([g], [g'])$ be the number of elements $g' \in [g']$ such that $\alpha(g, g') =$ $= 1$. Thus we require the number of $x \in X$ such that $gx \sim g'$, i.e. $gxg'^{-1} \in H$.

Lemma. $\hat{\alpha}_A : G/H \times G/H \to \mathbb{Z}^{\geqq 0}$ is a well-defined integral connectivity function on G/H.

Proof. Let $g_1 \sim g$, i.e. $g_1 = hg$. Then $gxg'^{-1} \in H$ iff $(hg)\, xg'^{-1} \in H$, and $\hat{\alpha}_A$ is well defined.

4.3. Let $\hat{\psi} : G/H \to Q$ be defined by $\hat{\psi}[g] = q_0 \cdot g$.

Lemma. $\hat{\psi}$ is a well-defined injective homomorphism from $(G/H, \hat{\alpha}_A)$ to (Q, Φ).

Proof. If $[g] = [g']$, then $g' = hg$, and $q_0 \,.\, g' = q_0 \,.\, hg = q_0 \,.\, g$. Thus $\hat{\psi}_A$ is well-defined. □

And $\hat{\psi}_A$ is $(1-1)$ since $\hat{\psi}([g]) = \hat{\psi}([g']) \Rightarrow q_0 \,.\, g = q_0 \,.\, g' \Rightarrow g \,.\, g'^{-1} \in H \Rightarrow \Rightarrow [g] = [g']$. It remains to verify that $\hat{\psi}$ is a homomorphism. Now $\Phi(\hat{\psi}[g], \hat{\psi}[g']) = = \Phi(q_0 \,.\, g, q_0 \,.\, g') =$ (the number of elements $x \in X$ such that $q_0 g' = q_0 \,.\, gx$) $=$ $=$ (number of elements $x \in X$ such that $gxg'^{-1} \in H$) $= \hat{\alpha}_A([g], [g'])$.

Definition. The permutation automaton (X, Q) is said to be *homogeneous* if for all $q, q' \in Q$, there exists $g \in G$ such that $q \,.\, g = q'$.

Theorem. In a homogeneous automaton, $\hat{\psi}$ defines an isomorphism from $(G/H, \hat{\alpha}_A)$ to (Q, Φ).

Proof. Homogeneity ensures that $\hat{\psi}$ is onto, since for $q \in Q$, let $g \in G$ such that $q_0 \,.\, g = q$. Then $\hat{\psi}([g]) = q$.

We have already observed that $\Phi(\hat{\psi}[g], \hat{\psi}[g']) = \hat{\alpha}_A([g], [g'])$, so $\hat{\psi}^{-1}$ is a homomorphism, and the theorem is proved. □

Corollary. If the input set X of a homogeneous automaton (X, Q) is a normal subset of the group G generated by it, then (Q, Φ) is isomorphic to a group-quotient automaton $(G/H, \hat{\alpha}_A)$ whose group G is an integral group net.

4.4. We have already described an alternative function space connectivity α_M. This is used by Muir [3] in conjunction with the equivalence class connectivity function $\hat{\alpha}_M([g], [g']) = \max\limits_{h \in H} \alpha_M(hg, g')$ to show that for a general net (Q, Φ) with isomorphism group G, $\hat{\psi}$ is a bijective homomorphism when Q is homogeneous. But it does not follow that $\hat{\psi}^{-1}$ is a homomorphism unless an extra condition, (very homogeneous) is imposed on (Q, Φ), namely that for all q, q', there exists $g \in G$ such $q \,.\, g = g'$ and $\Phi(q, q') = \min\limits_{\bar{q} \in Q} \Phi(\bar{q}, \bar{q} \,.\, g)$. Interpreting this in terms of elementary inputs, assume that there exist n elements $x \in X$ such that $q \,.\, x = q'$. Then we require $g \in G$ such that $q \,.\, g = q'$ and the number of elementary inputs taking $\bar{q}$ to $\bar{q} \,.\, g$ is $\geqq n$ for all $\bar{q} \in Q$. Even if g is taken to be one of the nx's there would seem to be no reason built into the automaton why it should possess this property.

We therefore argue that α_A, $\hat{\alpha}_A$ are more suitable for automata theory despite the possible loss in generality occasioned by defining α_A directly from the automaton rather than from Φ.

For a permutation automaton, α_A simply defines a relation, ($g \alpha_A g'$ iff $\alpha_A(g, g') = 1$), while Φ, $\hat{\alpha}_A$ may be thought of as 'sums of relations'. In the classical theory (cf. Warner [5]) the identification relation $\hat{\alpha}_A$ would be defined from α_A by $[g] \,\hat{\alpha}_A\, [g']$ iff there exist $g \in [g]$, $g' \in [g']$ such that $g \,\alpha_A\, g'$. In our case we have not merely sought for a $g' \in [g']$ $\hat{\alpha}_A$-related to g, but have counted the number of such g', thus gaining more information from the identification made in taking equivalence classes. We cannot threfore expect the relation group quotient theorem of (Warner

[5]) to be a special case of Theorem 4.3, – a point which is highlighted by observing that α_A is not the coarse function space relation on G. It is in fact the inertial relation for the automaton (X, G).

4.5. A semi-group quotient theorem also exists for connectivities just as for relations, but there would seem to be little virtue in undertaking the obvious formalities involved in adapting the proof. We merely observe that care should be taken in defining semi-group nets since invariance under right and left translation does not follow exactly as for groups.

5. FUZZY RELATIONS

5.1. Let (Q, Φ) be the integral net of the automaton $A = (X, Q)$, and let N be the number of elements in the set X. Then $p : Q \times Q \to I$ (the closed unit interval), where $p(q, q') = \Phi(q, q')/N$, is a fuzzy relation on Q. (Zadeh [7]). Similarly $(G/H, \hat{\alpha}_A)$ receives a fuzzy relation on division by N.

5.2. We consider the situation arising from the application of strings of inputs. Intuitively, after two successive inputs chosen at random, we are interested in the probability that the state q has become q_2. This motivates our definition of the composition of two fuzzy relations (or connectivity functions).

Definition. Let $p : Q \times Q_1 \to I$, $p' : Q_1 \times Q_2 \to I$ be fuzzy relations. The composition of p and p' is given by $p'' : Q \times Q_2 \to I$, $(q, q_2) \mapsto \sum_{q_1} p(q, q_1) \cdot p'(q_1, q_2)$, $q \in Q$, $q_1 \in Q_1$, $q_2 \in Q_2$.

Thus for the automaton A we define the two-stage fuzzy relation $p'' : Q \times Q \to I$ by composing p with itself.

Definition. The two-stage fuzzy relation $p'' : Q \times Q \to I$ is defined for the automaton A by $p''(q, q') = \sum_{\bar{q} \in Q} p(q, \bar{q}) \cdot p(\bar{q}, q')$, $q, q' \in Q$.

Clearly the composition of Φ with itself is similarly defined by $\Phi''(q, q') = \sum_{\bar{q} \in Q} \Phi(q, \bar{q}) \cdot \Phi(\bar{q}, q')$.

5.3. We may now expect a two-stage group quotient theorem by taking analogous two-stage definitions in G and G/H. The details are not presented here since they are formally the same as for the one input case. Further extensions to longer strings are obvious.

5.4. Observe that the definition of composition of fuzzy relations is not that given by Zadeh [7], who defines $p''(q, q_2)$ to be $\max_{q_1 \in Q_1} (\min (p(q, q_1), p'(q_1, q_2))$. This could be interpreted as taking a simultaneous couple of random inputs, taking the pessimistic view of the likelihood of going from q to q_2 via q_1, then maximising this by choosing the best q_1 for the purpose. Such an interpretation would seem to be inappropriate here. The same objection holds for the similar composition relation of Wee and Fu [6], namely $p''(q, q_2) = \min_{q_1 \in Q_1} (\max (p(q, q_1), p'(q_1, q_2))$.

5.5. Let $A = (X, Q)$ be a fuzzy automaton. Then the fuzzy input action is a function f from $Q \times Q \times X$ to I. A deterministic automaton is then a special case in which the range of f is restricted to the set $\{0, 1\}$. A net structure can be defined on Q by $\Phi(q_1, q_2) = \sum_{x \in X} f(q_1, q_2, x)$, and the integral connectivity for the automaton of § 2 is the appropriate special case. The remaining theory has however no obvious extension to fuzziness since an input set X can only be accused of consisting exclusively of permutations if the automaton is deterministic.

6. EXAMPLE

6.1. Let $A = (X, Q)$ have state set $Q = \{1, 2, 3\}$ and input set $X = \{x_1, x_2\}$ with action $x_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$. Thus x_1 is a cycle, while x_2 is a transposition. The group G generated by X is the dihedral group D_6. A is homogeneous.

The connectivities in Q and in G are exhibited in diagrams 1 and 2.

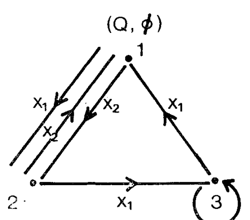

Diagram 1

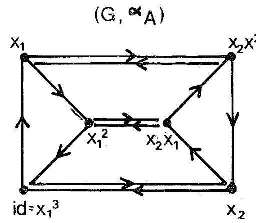

Diagram 2

Now choose $q_0 = 1$. Then $g \sim g'$ iff $1 \,.\, g = 1 \,.\, g'$. Thus $x_1 \sim x_2$, $x_1^2 \sim x_2 x_1$ and $x_1^3 \sim x_2 x_1^2$. Diagram 3 illustrates $(G/H, \hat{\alpha}_A)$ with its structure isomorphic to (Q, Φ) under $\hat{\psi}$. Here $\hat{\psi}[x_1] = 2$, $\hat{\psi}[x_1^2] = 3$, $\hat{\psi}[id] = 1$.

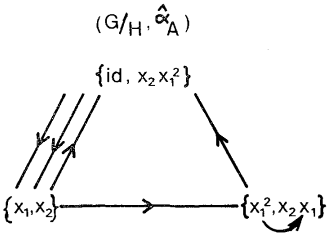

Diagram 3

The set X is not normal in G, so G is not an integral group net. For example $x_2x_1^2\alpha_A x_1$, but $x_2x_1^3\alpha_A x_1^2$.

The subgroup H of G is $\{id, x_2x_1^2\}$ which is also not normal, so G/H is not a group.

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