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APPROXIMATION OF A RANDOM SOLUTION IN EXTREMUM PROBLEMS*

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An optimization problem depending on a random parameter is considered. If the goal function is not determined exactly, another problem is investigated. The results are applied to a discrete time optimization problem.

1. INTRODUCTION

In the present paper the following problem will be considered: find a point $x \in R^n$ where the function $H(x, \xi)$ depending on a random parameter $\xi \in R^s$ attains its minimum. It is evident that a minimum-point of such function depends also on the value of ξ, and the first question that arises is: under which conditions the minimum-point $x(\xi)$ is measurable? Basing on Theorem 14 by H. E. Engl [1] it is immediately obtained that if 1) for almost every ξ the function $H(x, \xi)$ has a minimum-point $x(\xi)$, 2) $H(x, \xi)$ is continuous in x for almost every ξ and measurable for every $x \in R^n$, then $x(\xi)$ is measurable.

In what follows we shall not consider $x(\xi)$ as a function but will study the question how to find it approximately for an arbitrary value of ξ. If $H(x, \xi)$ and its derivatives for every x and for almost every ξ can be evaluated, then, in principle, for almost every value of ξ the corresponding minimum-point $x(\xi)$ can be found using some suitable minimization procedure. However, there may occur such cases when $H(x, \xi)$ cannot be evaluated exactly. For example, if $H(x, \xi) = \mathsf{E}_{\eta|\xi)} f(x, \xi, \eta)$ and the conditional distribution of η is not known, then we cannot find exactly neither the values of the conditional expectation $\mathsf{E}_{\eta|\xi} f(x, \xi, \eta)$, nor the values of its derivatives.

In this paper minimization of functions $H(x, \xi)$ and $h(x, \xi, \eta)$, where η is another random parameter, are simultaneously considered under the assumption that $\mathsf{E}_{\eta|\xi}\, h(x, \xi, \eta) = H(x, \xi)$ for every $x \in R^n$ and for almost all ξ. Hereby we shall use

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the framework and results of [3]–[4] where analogous relations between the pair of functions $H(x, \xi) = \Phi(x)$ and $h(x, \xi, \eta) = \varphi(x, \eta)$ were studied. The results will be applied to a discrete time optimization problem introduced by V. Kaňková in [2].

2. APPROXIMATION OF THE PROBLEM

Let us have a probability space (Ω, Σ, P) and consider the problem

$$(1) \qquad \min_{x \in R^n} H(x, \xi)$$

where ξ is an m-dimensional random parameter, i.e. $\xi: \Omega \to R^m$, and the function $H: R^n \times R^m \to R^1$ is continuous in x for almost every ξ, and measurable for every $x \in R^n$. Let $(R^m, \mathscr{B}, P_\xi)$ be the probability space induced in R^m by the random vector ξ. Assume that for some $B \in \mathscr{B}$, $P_\xi B > 0$ and for every $\xi \in B$ the problem (1) has a solution $x(\xi)$.

Let us consider another problem

$$(2) \qquad \min_{x \in R^n} h(x, \xi, \eta)$$

where $h: R^n \times R^m \times R^l \to R^1$, ξ is the same parameter as that in the problem (1) and $\eta: \Omega \to R^l$ is another, l-dimensional random parameter. Our aim is to determine for every $\xi \in B$ a Borel set $D(\xi) \subset R^l$ such that for every $(\xi, \eta) \in B \times D(\xi)$ the problem (2) has a solution $x(\xi, \eta)$ and to estimate the Euclidean distance $\|x(\xi) - x(\xi, \eta)\|$.

Suppose that the following conditions hold:

1. $H(x, \xi) = \mathsf{E}_{\eta|\xi}\, h(x, \xi, \eta)$ for every $\xi \in B$.
2. For every $\xi \in B$ and some $M > 0$
$$u^T H''_{xx}(x, \xi)\, u \geqq M \|u\|^2 \quad \text{for all} \quad u \in R^n .$$
3. $h''_{xx}(x, \xi, \eta)$ satisfies the Lipschitz condition
$$\|h''_{xx}(x, \xi, \eta) - h''_{xx}(y, \xi, \eta)\| \leqq C(\eta)\, \|x - y\| \quad \text{for every} \quad x, y \in R^n ,$$
$$\xi \in B , \quad \eta \in R^l .$$
4. $C(\eta)$ is a random variable with finite expectation $\mathsf{E}C(\eta)$ and finite variance $\sigma^2 C(\eta)$.
5. The conditional expectation
$$\mathsf{E}_{\eta|\xi} \|h''_{xx}(x(\xi), \xi, \eta) - \mathsf{E}_{\eta|\xi} h''_{xx}(x(\xi), \xi, \eta)\|^2 \leqq K^2$$
and conditional variances
$$\sigma^2_{\eta|\xi} h'_{x_i}(x(\xi), \xi, \eta) \leqq D_i^2 , \quad i = 1, 2, \ldots, n , \quad \text{for every} \quad \xi \in B .$$

Theorem 1. Assume that the conditions 1–5 are satisfied. If for some constants

δ_1, δ_2, $0 < \delta_1 < M$, $\delta_2 > 0$, the expression

$$p(\delta_1, \delta_2) = 1 - \frac{K^2}{\delta_1^2} - \frac{16[\mathsf{E}C(\eta) + \delta_2]^2 \sum_{i=1}^{n} \mathsf{D}_i}{(M - \delta_1)^4} - \frac{\sigma^2 C(\eta)}{\delta_2^2}$$

is positive, then for every $\xi \in B$ there exists a Borel set $D(\xi, \delta_1, \delta_2)$ such that

1) the problem (2) has a solution $x(\xi, \eta)$ for every $(\xi, \eta) \in B \times D(\xi. \delta_1, \delta_2)$,

2)
$$P_{(\xi,\eta)}[B \times D(\xi, \delta_1, \delta_2)] \geqq P_\xi B \,.\, p(\delta_1, \delta_2)$$

and

3)
$$P_{(\xi,\eta)}[B \times \{\{\eta \mid \|x(\xi, \eta) - x(\xi)\| < \varepsilon\} \cap D(\xi, \delta_1, \delta_2)\}] \geqq$$
$$\geqq P_\xi B \left[p(\delta_1, \delta_2) - \frac{\sum_{i=1}^{n} \mathsf{D}_i}{\varepsilon^2 (M - \delta_1)^2} \right] \quad \text{for arbitrary} \quad \varepsilon > 0 \,.$$

Proof. If $\xi \in B$ then, as assumed, the problem (1) has a solution $x(\xi)$. For every fixed ξ the problem (2) depends only on the random parameter η and $\mathsf{E}_{\eta|\xi}\, h(x, \xi, \eta) = H(x, \xi)$ for every $x \in R^n$ (condition 1). Relations between such kind of problems were studied in [3]. According to Theorem 1 of [3] the conditions 1–5 assure the existence of a Borel set $D(\xi, \delta_1, \delta_2) \subset R^l$ such that for every $\eta \in D(\xi, \delta_1, \delta_2)$ the problem (2) has a solution $x(\xi, \eta)$ where

$$P_{\eta|\xi}[D(\xi, \delta_1, \delta_2)] \geqq p(\delta_1, \delta_2)$$

and

$$P_{\eta|\xi}[\{\eta \mid \|x(\xi, \eta) - x(\xi)\| < \varepsilon\} \cap D(\xi, \delta_1, \delta_2)] \geqq$$
$$\geqq p(\delta_1, \delta_2) - \frac{\sum_{i=1}^{n} \mathsf{D}_i}{\varepsilon^2 (M - \delta_1)^2} \,.$$

An estimate for the probability $P_{(\xi,\eta)}[B \times D(\xi, \delta_1, \delta_2)]$ is easily obtained: Since

$$B \times D(\xi, \delta_1, \delta_2) = \{(\xi, \eta) \mid \xi \in B, \eta \in D(\xi, \delta_1, \delta_2)\} \,,$$

we have

$$P_{(\xi,\eta)}[B \times D(\xi, \delta_1, \delta_2)] \geqq P_\xi B \, P_{\eta|\xi} D(\xi, \delta_1, \delta_2) \geqq P_\xi B \,.\, p(\delta_1, \delta_2) \,.$$

Analogously we get

$$P_{(\xi,\eta)}[B \times \{\{\eta \mid \|x(\xi, \eta) - x(\xi)\| < \varepsilon\} \cap D(\xi, \delta_1, \delta_2)\}] \geqq$$
$$\geqq P_\xi B \,.\, P_{\eta|\xi}[\{\eta \mid \|x(\xi, \eta) - x(\xi)\| < \varepsilon\} \cap D(\xi, \delta_1, \delta_2)] \geqq$$
$$\geqq P_\xi B \left[p(\delta_1, \delta_2) - \frac{\sum_{i=1}^{n} \mathsf{D}_i}{\varepsilon^2 (M - \delta_1)^2} \right]. \qquad \square$$

3. APPLICATION TO A DISCRETE TIME OPTIMIZATION PROBLEM

V. Kaňková [2] has studied the following problem

$$\max \mathsf{E}[g_1(x_1, \xi_1) + \sum_{i=2}^{N} g_i(x_i, \xi_{i-1}, \xi_i)] \tag{3}$$

where ξ_i, $i = 1, 2, \ldots, N$, are m-dimensional random vectors and the functions g_i, $i = 1, 2, \ldots, N$, satisfy certain conditions. A solution of the problem (3) is a vector $(x_1^*, x_2^*(\xi_1), \ldots, x_N^*(\xi_{N-1}))$ depending on a random parameter $(\xi_1, \xi_2, \ldots, \xi_{N-1})$ and it is supposed that $x_1 \in K_1$, $x_2 \in K_2(\xi_1), \ldots, x_N \in K_N(\xi_{N-1})$ where K_1 is a compact convex nonempty set in R^n and so are $K_i(\xi_{i-1})$ for every value of ξ_{i-1}, $i = 2, 3, \ldots, N$.

Approximate solution of the problem (3) is treated in [2] in the case when the distributions of the vectors ξ_i, $i = 1, 2, \ldots, N$, are not known. The problem (3) is substituted by the problem

$$\max \left\{\frac{1}{k} \sum_{j=1}^{k} g_1(x_1, \xi_{1j}) + \frac{1}{k} \sum_{i=2}^{N} \sum_{j=1}^{k} g_i(x_i, \xi_{i-1}, \xi_{ij})\right\} \tag{4}$$

where ξ_i, ξ_{ij}, $j = 1, 2, \ldots, k$, are i.i.d. random vectors for every i. A solution of (4) is a random vector

$$(x_1^*(\xi_{11}, \ldots, \xi_{1k}), x_2^*(\xi_1, \xi_{21}, \ldots, \xi_{2k}), \ldots, x_N^*(\xi_{N-1}, \xi_{N1}, \ldots, \xi_{Nk})) \in K_1 \times \\ \times K_2(\xi_1) \times \ldots \times K_N(\xi_{N-1}) .$$

In [2] it is proved that if $k \to \infty$ the maximum-value of (4) converges almost surely to the maximum-value of (3). For convergence in probability an estimate is presented.

In the present paper we consider the relations between the problems analogous to (3) and (4) under slightly modified assumptions as those in [2] paying attention to relations between their solutions. Let us have to solve the problem

$$\min_{x_1 \in R^n, x_2(\xi_1) \in R^n, \ldots, x_N(\xi_{N-1}) \in R^n} \mathsf{E}[g_1(x_1, \xi_1) + \sum_{i=2}^{N} g_i(x_i, \xi_{i-1}, \xi_i)] . \tag{5}$$

Suppose that the problem

$$\min_{x \in R^n} \mathsf{E} g_1(x, \xi_1) \tag{6}$$

has a solution x_1^* and each of the problems

$$\min_{x \in R^n} \mathsf{E}_{\xi_i | \xi_{i-1}} g_i(x, \xi_{i-1}) \tag{6i}$$

has a solution $x^*(\xi_{i-1})$ if $\xi_{i-1} \in B_{i-1}$ where B_{i-1}, $i = 2, 3, \ldots, N$, are some Borel sets in R^m. Analogously to Lemma 1 [2] it can be shown that in this case the vector $(x_1^*, x_2^*(\xi_1), \ldots, x_N^*(\xi_{N-1}))$ is a solution of the problem (5). If the distribution of ξ_1 is not known then replace (6) by the problem

$$\min_{x \in R^n} \frac{1}{k} \sum_{j=1}^{k} g_1(x, \xi_{1j}) \tag{7}$$

where ξ_{1j}, $j = 1, 2, \ldots, k$, are independent realizations of ξ_1. Analogously, in the case of unknown distribution of ξ_i instead of (6i) solve the problem

$$\text{(7i)} \qquad \min_{x \in R^n} \frac{1}{k} \sum_{j=1}^{k} g_i(x, \xi_{i-1}, \xi_{ij})$$

when the value of ξ_{i-1} is already observed and ξ_{ij}, $j = 1, 2, \ldots, k$, are independent realizations of ξ_i, i.e. they are i.i.d. with respect to the conditional probability measure $P_{\xi_i|\xi_{i-1}}$. So a solution of (7i) $x^*(\xi_{i-1}, \xi_{i1}, \ldots, \xi_{ik})$, if it exists, depends on $k + 1$ random parameters.

To reduce the number of conditions denote

$$g_1(x, \xi_1) = g(x, \xi_0, \xi_1) \quad \text{and} \quad \mathsf{E}g_1(x, \xi_1) = \mathsf{E}_{\xi_1|\xi_0} g_1(x, \xi_0, \xi_1)\,.$$

Suppose now that for every $i = 1, 2, \ldots, N$, the following conditions hold.

1i. $u^{\mathrm{T}}\, \mathsf{E}_{\xi_i|\xi_{i-1}} g''_{ixx}(x^*(\xi_{i-1}), \xi_{i-1}, \xi_i)\, u \geqq M_i \|u\|^2$ for every $u \in R^n$.

2i. $\|g''_{ixx}(x, \xi_{i-1}, \xi_i) - g''_{ixx}(y, \xi_{i-1}, \xi_i)\| \leqq C_i(\xi_i)\, \|x - y\|$ for every $x, y \in R^n$, $\xi_{i-1} \in B_{i-1}$, $\xi_i \in R^m$.

3i. $\mathsf{E}C_i(\xi_i)$ and $\delta^2 C_i(\xi_i)$ are finite.

4i. There exist constants K_i and D_{ij}, $j = 1, 2, \ldots, n$, such that

$$\mathsf{E}_{\xi_i|\xi_{i-1}} \|g''_{ixx}(x_i^*(\xi_{i-1}), \xi_{i-1}, \xi_i) - \mathsf{E}_{\xi_i|\xi_{i-1}} g''_{ixx}(x_i^*(\xi_{i-1}), \xi_{i-1}, \xi_i)\|^2 \leqq$$
$$\leqq K_i \quad \text{and} \quad \sigma^2_{\xi_i|\xi_{i-1}} g'_{ix_j}(x^*(\xi_{i-1}), \xi_{i-1}, \xi_i) \leqq D_{ij}\,, \quad j = 1, 2, \ldots, n\,,$$

for every $\xi_i \in B_{i-1}$.

Theorem 2. Under the conditions 1i–4i for arbitrary constants δ_1, δ_2, $0 < \delta_1 < M_i$, $i = 1, 2, \ldots, N$, $\delta_2 > 0$ and for

$$k > \max_{i=1,2,\ldots,N} \left[\frac{K_i}{\delta_1^2} + \frac{16[\mathsf{E}C_i(\xi_i) + \delta_2]^2 \sum_{j=1}^{n} D_{ij}}{(M_i - \delta_1)^4} + \frac{\sigma^2 C_i(\xi_i)}{\delta_2^2} \right]$$

there exist sets $D_1(k, \delta_1, \delta_2)$, $D_i(k, \xi_{i-1}, \delta_1, \delta_2)$, $i = 2, 3, \ldots, N$, in R^{mk} such that

1) if $\xi_i \in B_i$, $i = 1, 2, \ldots, N - 1$, and $(\xi_{i1}, \xi_{i2}, \ldots, \xi_{ik}) \in D_i(k, \xi_{i-1}, \delta_1, \delta_2)$, $i = 1, 2, \ldots, N - 1$, then the problem

$$\text{(8)} \qquad \min_{x_1 \in R^n, x_2(\xi_1) \in R^n, \ldots, x_N(\xi_{N-1}) \in R^n} \left\{ \frac{1}{k} \sum_{j=1}^{k} g_1(x_1, \xi_{1j}) + \frac{1}{k} \sum_{i=2}^{N} \sum_{j=1}^{k} g_i(x_i, \xi_{i-1}, \xi_i) \right\}$$

has a solution

$$(x_{1k}^*(\xi_{11}, \ldots, \xi_{ik}), x_{2k}^*(\xi_1, \xi_{21}, \ldots, \xi_{2k}), \ldots, x_{Nk}^*(\xi_{N-1}, \xi_{N1}, \ldots, \xi_{Nk}))$$

with probability not less than

$$\prod_{i=1}^{N} P_{\xi_{i-1}} B_{i-1} \cdot p_i(k, \delta_1, \delta_2)$$

where

$$p_i(k, \delta_1, \delta_2) = 1 - \frac{1}{k}\left[\frac{K_i}{\delta_1^2} + \frac{16[EC_i(\xi_i) + \delta_2]^2 \sum_{j=1}^{n} D_{ij}}{(M_i - \delta_1)^4} + \frac{\sigma^2 C_i(\xi_i)}{\delta_2^2}\right]$$

and

$$P[\|(x_{1k}^*(\xi_{11}, \ldots, \xi_{1k}), x_{2k}^*(\xi_1, \xi_{21}, \ldots, \xi_{2k}), \ldots, x_{Nk}^*(\xi_{N-1}, \xi_{N1}, \ldots, \xi_{Nk})) -$$

$$- (x_1^*, x_2^*(\xi_1), \ldots, x_N^*(\xi_{N-1}))\| < \varepsilon] \geqq \prod_{i=1}^{N} P_{\xi_{i-1}} B_{i-1}\left[p_i(k, \delta_1, \delta_2) - \frac{N \sum_{j=1}^{n} D_{ij}}{k\varepsilon^2(M_i - \delta_1)^2}\right]$$

for arbitrary $\varepsilon > 0$, where P denotes the probability measure induced by $((\xi_{11}, \ldots, \xi_{1k})$ $\xi_1, (\xi_{21}, \ldots, \xi_{2k}), \xi_2, \ldots, (\xi_{N1}, \ldots, \xi_{Nk}))$.

To prove this theorem one has to apply Theorem 1 of [3] to the pair of problems (6), (7) and Theorem 1 above to the pairs (6i), (7i). □

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