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Kybernetika, Vol. 30 (1994), No. 6, 689--699

Persistent URL: http://dml.cz/dmlcz/125487

KYBERNETIKA — VOLUME 30 (1994), NUMBER 6, PAGES 689-699

DEAD-BEAT RESPONSE OF SISO SYSTEMS TO PARABOLIC INPUTS WITH OPTIMUM STEP AND RAMP RESPONSES

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The design problem of optimum control systems with dead-beat response to parabolic inputs is considered. The optimization is performed by the minimization of the sum of the squared-error coefficients of the system due to step and ramp inputs, where different weighting factors are used for each error sequence. Response characteristics are illustrated through diagrams of typical prototype responses and normalized overshoot and cost function curves.

1. INTRODUCTION

The design of control systems that exhibit dead-beat response to parabolic inputs with minimum squared-error restrictions on step and ramp responses was introduced by J. L. Pokoski and D. A. Pierre [1]. The performance measure they considered was

$$I = \sum_{k=0}^{\infty} e_s^2(kT) + he_r^2(kT),$$

where $e_s(kT)$ and $e_r(kT)$ are the unit step and ramp errors at the kth sampling instant and h is a weighting factor. For the system to exhibit dead-beat response to a parabolic input, they also considered the two well known conditions for the overall transfer function $M(z)$ of the system, that is

$$M(z) = b_1 z^{-1} + b_2 z^{-2} + \cdots + b_n z^{-n}$$

and

$$1 - M(z) = \left(1 - z^{-1}\right)^3 \left(1 + a_1 z^{-1} + \cdots + a_{n-3} z^{-n+3}\right).$$

Stability considerations impose that all unstable (or critically stable) poles of the plant must be included in $1 - M(z)$ as zeros, and all zeros of the plant that lie on or outside the unit circle must be included in $M(z)$ as zeros [2]. The equations resulting in the a_k which minimize I may be found by setting

$$\frac{\partial I}{\partial a_k} = 0 \quad \text{for } k = 1, 2, \ldots, n-3.$$

This results in $n-3$ equations which may be solved for the $n-3$ values of a_k, but, according to the authors [1], since it is difficult to solve the resulting equations for the a_k as functions of both h and n, they are solved for the limiting cases, $h=\infty$ (only ramp error considered) and $h=0$ (only step error considered). The responses for the intermediate values of h are generally expected to lie between those of the preceding cases.

In this work is proposed a new approach to the design of optimum dead-beat response systems to parabolic inputs based on the same performance criterion. Recently introduced necessary and sufficient error conditions [3] are elaborated, and the optimization is performed by the minimization of the sum of the squared step and ramp response error sequence coefficients of the system, in which each error sequence is regarded with a different weighting factor, thus obtaining a more general solution. Response characteristics are illustrated through diagrams with typical prototype responses and normalized overshoot and cost function curves. The response of the system to a complex input is also examined.

2. ERROR CONDITIONS FOR DEAD-BEAT RESPONSE TO PARABOLIC INPUTS

Consider the digital control system shown in Figure 1. The z-transform of the error sequence is

$$E(z)=\frac{R(z)}{1+D(z)\,G_h(z)} \tag{1}$$

from which the transfer function of the digital controller can be obtained as

$$D(z)=\frac{1}{G_h(z)}\left(\frac{R(z)}{E(z)}-1\right). \tag{2}$$

Under the above mentioned stability conditions, from equation (2) is readily implied that for the design of the digital controller, the z-transform of the error sequence $E(z)$ in response to a given input signal $R(z)$ must be determined. A system exhibiting dead-beat response to parabolic inputs, will exhibit dead-beat response to step and ramp inputs, as well [3].

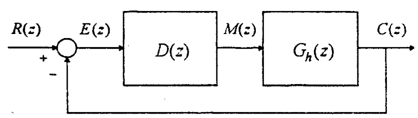

Fig. 1. Digital control system.

Suppose there are n steps before the settling of the output signal when the system is forced by a step, a ramp or a parabolic input. By denoting as a_k, b_k and c_k the terms of the error sequences in response to a step, a ramp and a parabolic input, respectively, we have

$$E_0(z) = \sum_{k=0}^{n} a_k z^{-k}, \quad E_1(z) = \sum_{k=0}^{n} b_k z^{-k}, \quad E_2(z) = \sum_{k=0}^{n} c_k z^{-k}. \tag{3}$$

These error sequences are not independent to each other, but are related through

$$b_k = T \sum_{i=0}^{k-1} a_i \quad \text{for all } k \tag{4}$$

and

$$c_k = T^2 \sum_{i=0}^{k-1} \left(a_i + 2 \sum_{l=0}^{i-1} a_l \right) \quad \text{for all } k, \tag{5}$$

where T is the sampling period. Equations (4) and (5) for $k = n + 1$ lead to

$$\sum_{i=0}^{n} a_i = 0 \tag{6}$$

and

$$\sum_{i=0}^{n} \sum_{l=0}^{i-1} a_l = 0. \tag{7}$$

The last two equations have been proven to be necessary and sufficient conditions for the system to exhibit dead-beat response to parabolic inputs [3]. These equations are elaborated in the sequel for the solution of the optimization problem. It is readily apparent that for the design of the digital controller, the determination of only one of the three error sequences is sufficient, since they are associated by explicit relations.

3. OPTIMIZATION

The error coefficients can be determined by optimizing the system's response in accordance to an objective function which may be arbitrarily selected. As such a performance criterion, a function of the squared values of the error sequences due to step and ramp inputs is chosen. In particular, by defining that

$$J_0 = \sum_{k=0}^{n} a_k^2 \tag{8}$$

and

$$J_1 = \sum_{k=0}^{n} b_k^2 \tag{9}$$

then as an objective function can be chosen the following

$$J = sJ_0 + rJ_1 = \sum_{k=0}^{n} \left(s a_k^2 + r b_k^2 \right), \tag{10}$$

where s and r are weighting factors.

Since the error coefficients b_k can be expressed as a function of the error coefficients a_k, the optimization of the system's performance in accordance to the specified objective function will be achieved through the minimization of J with respect to the coefficients a_k, with the subsidiary constraints expressed by equations (6) and (7). Thus, it is a problem of static optimization under linear constraints, which can be solved using the Lagrange method of undetermined multipliers. The modified function

$$I = J + \lambda_1 \sum_{i=0}^{n} a_i + \lambda_2 \sum_{i=0}^{n} \sum_{l=0}^{i-1} a_l \tag{11}$$

is considered with no constraints, and is optimized with respect to the coefficients a_k (for $k = 1, 2, \ldots, n$, since $a_0 = 1$) for which I becomes optimum are given from

$$\begin{cases} \dfrac{\partial I}{\partial a_j} = 0 \quad \text{for } j = 1, 2, \ldots, n \\ \dfrac{\partial I}{\partial \lambda_i} = 0 \quad \text{for } i = 1, 2. \end{cases} \tag{12}$$

The partial derivatives of the last set of equations can be written as

$$\begin{cases} \dfrac{\partial I}{\partial a_j} = \dfrac{\partial J}{\partial a_j} + \lambda_1 \dfrac{\partial}{\partial a_j} \left(\sum_{i=0}^{n} a_i \right) + \lambda_2 \dfrac{\partial}{\partial a_j} \left(\sum_{i=0}^{n} \sum_{l=0}^{i-1} a_l \right) \quad \text{for } j = 1, 2, \ldots, n \\ \dfrac{\partial I}{\partial \lambda_1} = \sum_{i=0}^{n} a_i \\ \dfrac{\partial I}{\partial \lambda_2} = \sum_{i=0}^{n} \sum_{l=0}^{i-1} a_l \end{cases} \tag{13}$$

and the set of equations (12) is equivalent to

$$\begin{cases} \dfrac{\partial J}{\partial a_j} + \lambda_1 \dfrac{\partial}{\partial a_j} \left(\sum_{i=0}^{n} a_i \right) + \lambda_2 \dfrac{\partial}{\partial a_j} \left(\sum_{i=0}^{n} \sum_{l=0}^{i-1} a_l \right) = 0 \quad \text{for } j = 1, 2, \ldots, n \\ \sum_{i=0}^{n} a_i = 0 \\ \sum_{i=0}^{n} \sum_{l=0}^{i-1} a_l = 0 \end{cases} \tag{14}$$

now from equation (10)

$$\frac{\partial J}{\partial a_j} = s \frac{\partial J_0}{\partial a_j} + r \frac{\partial J_1}{\partial a_j} \quad \text{for } j = 1, 2, \ldots, n, \tag{15}$$

where by using equation (8)

$$\frac{\partial J_0}{\partial a_j} = 2a_j \quad \text{for } j = 1, 2, \ldots, n \tag{16}$$

and equation (9)

$$\frac{\partial J_1}{\partial a_j} = 2T^2 \sum_{k=j+1}^{n} \sum_{m=0}^{k-1} a_m \quad \text{for } j = 1, 2, \ldots, n, \tag{17}$$

In this way, the first of (14) transforms to the equation

$$2sa_j + 2rT^2 \sum_{k=j+1}^{n} \sum_{m=0}^{k-1} a_m + \lambda_1 + \lambda_2(n-j) = 0 \quad \text{for } j = 1, 2, \ldots, n. \tag{18}$$

Expanding the left-hand side series of (18) and taking into account that $a_0 = 1$, we have in matrix form

$$\begin{bmatrix} s+(n-1)\,rT^2 & (n-2)\,rT^2 & \cdots & rT^2 & 0 \\ (n-2)\,rT^2 & s+(n-2)\,rT^2 & \cdots & rT^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rT^2 & rT^2 & \cdots & s+rT^2 & 0 \\ 0 & 0 & \cdots & 0 & s \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = - \left(rT^2 + \frac{\lambda_2}{2} \right) \begin{bmatrix} n-1 \\ n-2 \\ \vdots \\ 1 \\ 0 \end{bmatrix} - \frac{\lambda_1}{2} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}. \tag{19}$$

Solving the set of linear equations (19) for the unknown coefficients a_k we have

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{n-1} \\ a_n \end{bmatrix} = - \begin{bmatrix} s+(n-1)\,rT^2 & (n-2)\,rT^2 & \cdots & rT^2 & 0 \\ (n-2)\,rT^2 & s+(n-2)\,rT^2 & \cdots & rT^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ rT^2 & rT^2 & \cdots & s+rT^2 & 0 \\ 0 & 0 & \cdots & 0 & s \end{bmatrix}^{-1} \cdot \left(\left(rT^2 + \frac{\lambda_2}{2} \right) \begin{bmatrix} n-1 \\ n-2 \\ \vdots \\ 1 \\ 0 \end{bmatrix} + \frac{\lambda_1}{2} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} \right). \tag{20}$$

From equation (20) the coefficients a_k are obtained as a function of the Lagrange multipliers λ_1 and λ_2. Elimination of these parameters is achieved using equations (6) and (7). As a result, coefficients a_k are expressed as a function of T, s, r and n, only. Furthermore, substitution of these coefficients into equation (3) leads to the reduction of the z-transform of the error sequence $E_0(z)$, and finally by use of equation (2), the transfer function of the digital controller $D(z)$ is obtained.

In the specific case that the plant pulse transfer function $G_h(z)$ contains more than one discrete delays, say $L+1$, the designed closed-loop system must involve at least the same magnitude of transportation delay [2], and the optimization of all coefficients a_k, for $k = 1, 2, ..., n$ is not possible, since the first L values of the error sequence stay intact, that is $a_k = 1$, for $k = 1, 2, ..., L$. Nevertheless, the remaining $n - L$ error coefficients can be optimized by the procedure developed above.

4. SPECIAL CASES

4.1. No cost on ramp response ($r = 0$)

In this case, the transient response of the system is optimized by the minimization of the sum of the squared step-error coefficients only, with no cost on the ramp response. In this way the first of (14) transforms to the equation

$$2a_j + \lambda_1 + \lambda_2(n-j) = 0 \quad \text{for } j = 1, 2, \ldots, n \tag{21}$$

which can be written as

$$a_j = -\frac{1}{2}\left[\lambda_1 + (n-j)\,\lambda_2\right] \quad \text{for } j = 1, 2, \ldots, n. \tag{22}$$

Substituting the last equation into equations (6) and (7), and simplifying the sums that appear, the following set of equations is derived

$$\begin{cases} 2n\lambda_1 + n(n-1)\,\lambda_2 = 4 \\ 3(n-1)\,\lambda_1 + (n-1)(2n-1)\,\lambda_2 = 12 \end{cases} \tag{23}$$

and the values of the Lagrange multipliers are reduced from its solution as

$$\lambda_1 = -\frac{4}{n}, \quad \lambda_2 = \frac{12}{n(n-1)}. \tag{24}$$

Substitution of these parameters into equation (22) leads to

$$a_j = -2\frac{(2n+1-3j)}{n(n-1)} \quad \text{for } j = 1, 2, \ldots, n \tag{25}$$

with $a_j = 0$ for $j > n$, and the transfer function of the digital controller is obtained as

$$D(z) = \frac{1}{G_h(z)} \cdot \frac{\frac{n+4}{n+2}z^{-1} - \frac{n+1}{n-1}z^{-2} - \frac{2}{n-1}z^{-(n+1)} - \frac{2}{n+2}z^{-(n+2)}}{\frac{n}{n-1} - 2z^{-1} + \frac{n+1}{n-1}z^{-2} - \frac{2}{n-1}z^{-(n+1)} + \frac{2}{n+2}z^{-(n+2)}}. \tag{26}$$

From equation (25) it follows that after a maximum overshoot of $\frac{4}{n} \cdot 100\%$ at the first sampling instant, there is a linear decrease of the system output, until a maximum undershoot of $\frac{2}{n} \cdot 100\%$ is reached at the nth sampling instant.

4.2. No cost on step response ($s = 0$)

In this case, the transient response of the system is optimized by the minimization of the sum of the squared ramp-error coefficients only, with no cost on the step response. In this way the first of (14) transforms to the equation

$$2T^2 \sum_{k=j+1}^{n} \sum_{m=0}^{k-1} a_m + \lambda_1 + \lambda_2(n-j) = 0 \quad \text{for } j = 1, 2, \ldots, n. \tag{27}$$

The last equation for $j = n$ leads to

$$\lambda_1 = 0 \tag{28}$$

while for $j = 1$ and use of condition (7) leads to

$$\lambda_2 = \frac{2T^2}{n-1}. \tag{29}$$

Substitution of the Lagrange multipliers from equations (28) and (29) into equation (27), and solution of the resulting equation yields

$$a_k = \begin{cases} -\frac{n}{n-1} & \text{for } k = 1 \\ 0 & \text{for } 1 < k < n \\ \frac{1}{n-1} & \text{for } k = n \end{cases} \tag{30}$$

with $a_k = 0$ for $k > n$, and the transfer function of the digital controller is obtained as

$$D(z) = \frac{1}{G_h(z)} \cdot \frac{\left(-1+z^{-1}\right) z^{-n} - nz^{-2} + (2n-1)z^{-1}}{\left(-1+z^{-1}\right)\left(-z^{-n} + nz^{-1} - n + 1\right)}. \tag{31}$$

From equation (30) it follows that besides the first sampling instant when an overshoot of $\frac{n}{n-1} \cdot 100$ % is observed, and the nth sampling instant when an undershoot of $\frac{1}{n-1} \cdot 100$ % results, the system output presents zero error at step inputs.

5. OPTIMUM RESPONSE CHARACTERISTICS

In the following are examined the general properties of the systems designed with the proposed procedure. A new normalized variable m is introduced, which is defined by $m = \frac{s}{rT^2}$. In this way, certain features of the system, such as the overshoot, depend only on the variable m and the number of sampling periods n for settling. In Figure 2 is presented the step response of a system designed with the proposed procedure, for $n = 5$ and for various values of the parameter m. As it can be observed, the maximum step response overshoot occurs at the first sampling instant after the application of the input, and becomes smaller as the parameter m increases. This happens because greater values of m enhance the step response of the system. After the first sampling period, the output of the system gradually approaches the input, but it presents an undershoot, which is maximum at the nth sampling instant. In

Figure 3 are presented the normalized curves for the step response, and in Figure 4 the normalized cost function curves, all as a function of the variable m, for $n = 2, 3, \ldots, 10$.

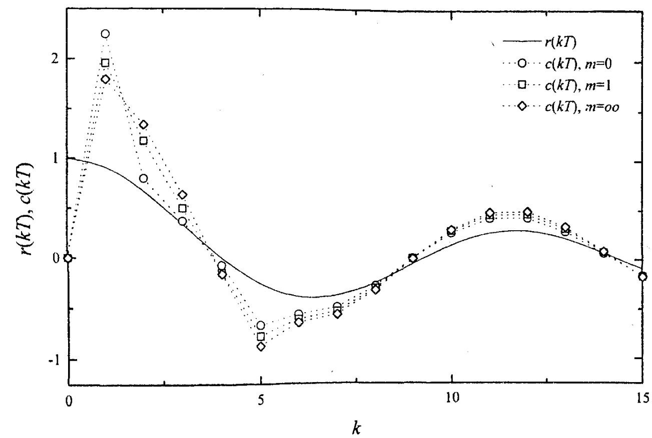

Fig. 2. Step response for $n = 5$.

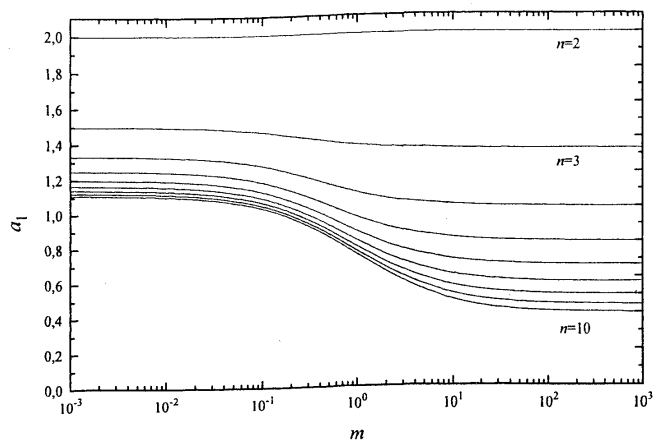

Fig. 3. Normalized maximum overshoot curves.

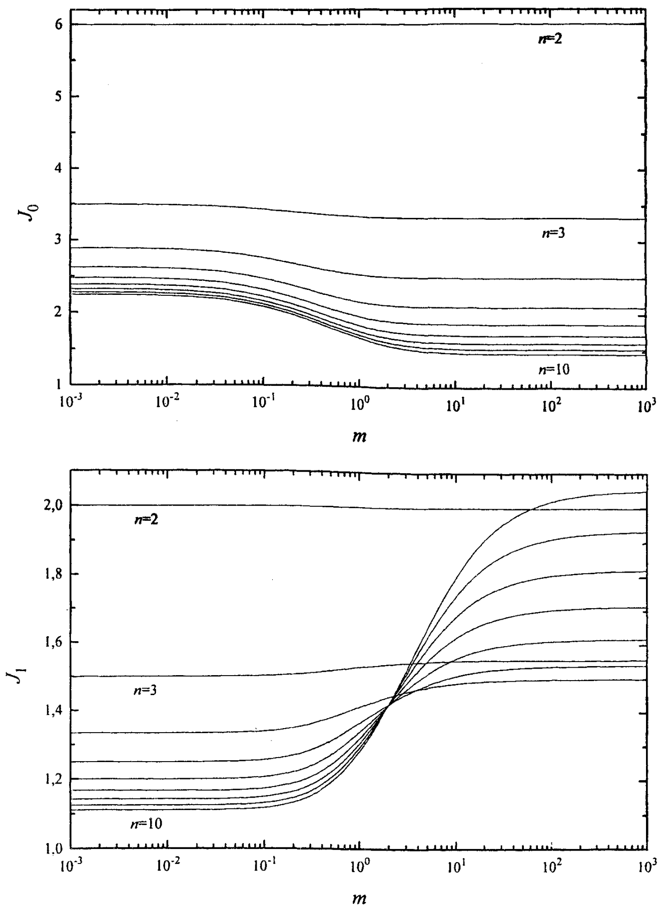

Fig. 4. Normalized cost-function diagrams.

In Figure 5 are presented the maximum overshoot and undershoot curves as a function of the number n for the two special cases considered above. In Figure 6 are presented the responses of three particular systems to a rather complex input signal,

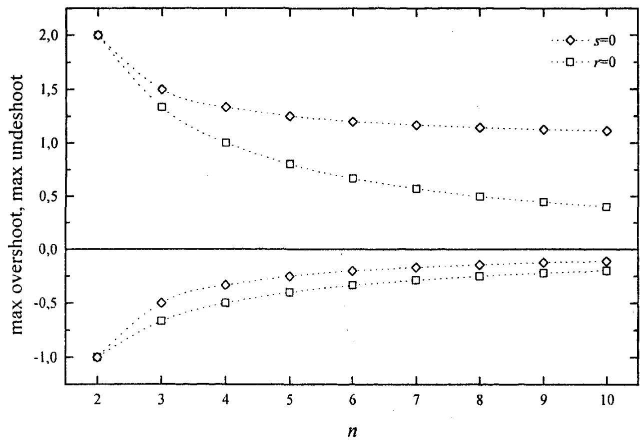

Fig. 5. Maximum overshoot and undershoot curves for the special cases.

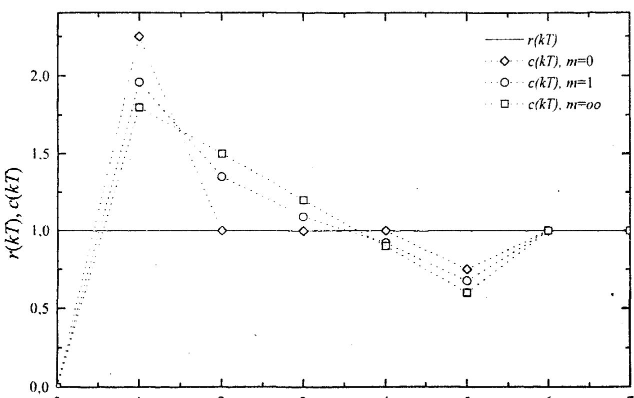

Fig. 6. Response to a complex input signal.

namely to the Bessel function of zero order $J_0(kT)$. The sampling period T is taken equal to 0.6, and the number of sampling periods to settling is taken $n = 5$. The three systems regarded are the two ones corresponding to the special cases mentioned and a system designed with $m = 1$. As it is expected, the system with $m = 0$ presents the maximum overshoot, but attains the best tracking performance. The system with $m = \infty$ obtains the minimum overshoot, but has not as good tracking performance as the previous one. The system designed for $m = 1$ has a moderate response to the complex input, since it presents lower overshoot than the minimum prototype one, while at the same time attains a good tracking performance.

6. CONCLUSIONS

In this work was presented a generalized approach to the design of optimum dead-beat response systems to parabolic inputs on the basis of an extendedperformance criterion. The optimization was performed by the minimization of the sum of the squared error coefficients of the system due to step and ramp inputs, with a different weighting factor regarded for each error sequence. This was accomplished by the utilization of recently introduced necessary and sufficient conditions for the error sequences of a dead-beat response system. General features of the optimum systems resulting from the proposed design procedure were readily apparent from the normalized diagrams presented.

(Received November 12, 1993.)

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