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RECURSIVE ESTIMATION AS AN OPTIMALLY CONTROLLED PROCESS

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A recursive estimation by means of two simple stochastic approximation algorithms is treated as a controlled process with the gain as the control variable. The optimal control is derived for the case when excitation and noise are mutually independent and uncorrelated random processes.

1. INTRODUCTION

In recent years the behaviour of stochastic approximation algorithms was studied in many of papers – see e.g. [1, 2, 3, 4]. Almost all of them are concerned with the asymptotical performance and the deepest results formulate the asymptotically optimal algorithms for the given external conditions of estimation. The results of this kind are very interesting from the mathematical point of view according to the behaviour in a finite number of initial steps is not important. However, the practical point of view is quite different: we are always in a finite distance from the beginning of estimation and therefore we are first of all interested in the initial behaviour. In practice we need not have the exact estimates in infinity, but we need to reach estimates with the prescribed accuracy in a finite number of steps as soon as possible. Unfortunately, the stochastic approximation algorithms, including the asymptotically optimal ones, exhibit generally a very poor initial convergence. Many of the proposed asymptotically optimal algorithms can even diverge at the very beginning of estimation.

The present paper deals with two well-known simple identification algorithms: standard (simple) stochastic approximation (SSA) and normalized stochastic approximation (NSA). The estimation by means of these two algorithms is treated as a controlled process with the gain as a control variable. For simple external conditions – when the excitation and noise are independent and uncorrelated random processes – the optimal control is derived. The derived algorithms have an optimal behaviour from the very beginning of estimation and their asymptotic optimality is a pure consequence of the overall optimality.

The structure of the paper is as follows. First of all the identification model and the estimation algorithms are described (Section 2). Then the estimation process is defined as a controlled process and the notion of an optimal trajectory of identification process is introduced (Section 3). In Section 4 the special case of a constant control is discussed. The main results – the laws of optimal control of estimation – are contained in Section 5. In Section 6 some generalizations of the main results are briefly mentioned.

The paper represents an extension of the results presented in [5].

2. THE IDENTIFICATION MODEL

The following notation will be used:

$\mathbf{z}_t$ n-dimensional input vector

y_t scalar output

$\boldsymbol{b}$ n-dimensional vector of unknown parameters

$\hat{\boldsymbol{b}}_t$ estimate of the vector $\boldsymbol{b}$

$\boldsymbol{d}_t$ error of the estimate, $\boldsymbol{d}_t = \boldsymbol{b} - \hat{\boldsymbol{b}}_t$

η_t random component of the input-output mapping: $\mathbf{z}_t \rightarrow y_t$

$\varkappa_t$ gain (scalar) of the estimation algorithm

The subscript t denotes discrete time $t = 0, 1, 2, \ldots$

The two considered algorithms for parameter estimation of the stochastic linear system

$$y_t = \mathbf{z}_t^{\mathrm{T}} \boldsymbol{b} + \eta_t \tag{2.1}$$

have the form

$$\hat{\boldsymbol{b}}_t = \hat{\boldsymbol{b}}_{t-1} + \varkappa_t (y_t - \mathbf{z}_t^{\mathrm{T}} \hat{\boldsymbol{b}}_{t-1})\, \mathbf{z}_t \qquad \text{(SSA)} \tag{2.2}$$

(simple standard stochastic approximation) and

$$\hat{\boldsymbol{b}}_t = \hat{\boldsymbol{b}}_{t-1} + \varkappa_t \frac{y_t - \mathbf{z}_t^{\mathrm{T}} \hat{\boldsymbol{b}}_{t-1}}{\mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t}\, \mathbf{z}_t \qquad \text{(NSA)} \tag{2.3}$$

(normalized stochastic approximation).

The sequences $\{\mathbf{z}_t\}$, $\{\eta_t\}$ are assumed to be random, independent and with the following statistics:

$$\mathsf{E}\{\mathbf{z}_t\} = \mathbf{0} \tag{2.4}$$

$$\mathsf{E}\{\mathbf{z}_t \mathbf{z}_s^{\mathrm{T}}\} = \begin{cases} \sigma_z^2 \boldsymbol{I},\ \sigma_z^2 > 0 & \text{for } t = s \\ \mathbf{0} & \phantom{\text{for }} t \neq s \end{cases} \tag{2.5}$$

$$\mathsf{E}\{\eta_t\} = 0 \tag{2.6}$$

$$\mathsf{E}\{\eta_t \eta_s\} = \begin{cases} \sigma_\eta^2 & \text{for } t = s \\ 0 & \phantom{\text{for }} t \neq s \end{cases} \tag{2.7}$$

From the independence of random sequences $\{\mathbf{z}_t\}$, $\{\eta_t\}$ and the assumptions (2.4), (2.6) we conclude

$$\mathsf{E}\{\mathbf{z}_t^{\mathrm{T}}\eta_s\} = 0 \tag{2.8}$$

It is further assumed that the inputs $\mathbf{z}_t$ have gaussian distribution, but no assumption is made about the kind of the noise distribution.

The following lemma will be useful in the sequel.

Lemma 2.1. Let the coordinates of an n-dimensional vector $\mathbf{z}$ be independent random variables, gaussian distributed, with zero means and with common variance σ_z^2. Then it holds:

$$\mathsf{E}\{(\mathbf{z}\mathbf{z}^{\mathrm{T}})^2\} = (n+2)\,\sigma_z^4 \boldsymbol{I} \tag{2.9}$$

$$\mathsf{E}\left\{\frac{1}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}\right\} = \frac{1}{(n-2)\,\sigma_z^2} \quad \text{for} \quad n > 2 \quad \text{(othervise undefined)} \tag{2.10}$$

$$\mathsf{E}\left\{\frac{\mathbf{z}}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}\right\} = \mathbf{0} \tag{2.11}$$

$$\mathsf{E}\left\{\frac{\mathbf{z}\mathbf{z}^{\mathrm{T}}}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}\right\} = \frac{1}{n}\boldsymbol{I} \tag{2.12}$$

Proof. The proof of (2.9) is straightforward and can be omitted. The proofs of (2.10)–(2.12) are similar and therefore only the first of these equations will be proved. To prove (2.10) we transform the rectangular coordinates $z_1 z_2, \ldots, z_n$ of the vector $\mathbf{z}$ (the subscripts exeptionally do not indicate the time, as in all other parts of the paper, but the coordinates) to the polar coordinates $\varrho, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$

$$z_i = \varrho \cos(\alpha_i) \,.\, \prod_{k=1}^{i-i} \sin(\alpha_k); \quad i = 1, 2, \ldots, n-1 \tag{2.13}$$

$$z_n = \varrho \,.\, \prod_{k=1}^{n-1} \sin(\alpha_k)$$

where $\varrho \geqq 0$, $0 \leqq \alpha_i < \pi$ for $i = 1, 2, \ldots, n-2$, $0 \leqq \alpha_{n-1} < 2\pi$. From (2.13) it follows (can be proved by induction):

$$\mathbf{z}^{\mathrm{T}}\mathbf{z} = \varrho^2 \tag{2.14}$$

$$\mathrm{d}z_1\, \mathrm{d}z_2 \ldots \mathrm{d}z_n = \varrho^{n-1}\, \mathrm{d}\varrho \prod_{k=1}^{n-1} (\sin^{n-k-1}(\alpha_k)\, \mathrm{d}\alpha_k)$$

Our aim is to determine the mean value

$$\mathsf{E}\left\{\frac{1}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \frac{1}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}\, \frac{1}{(2\pi)^{n/2}\sigma_z^n} \exp\left(\frac{-1}{2\sigma_z^2}\mathbf{z}^{\mathrm{T}}\mathbf{z}\right) \mathrm{d}z_1\, \mathrm{d}z_2 \ldots \mathrm{d}z_n \,. \tag{2.15}$$

Introducing (2.14) into (2.15) we get

(2.16)

$$\mathsf{E}\left\{\frac{1}{\mathbf{z}^{\mathrm{T}}\mathbf{z}}\right\} = \frac{1}{(2\pi)^{n/2}\sigma_z^n} \int_0^{\infty} \varrho^{n-3} \exp\left(\frac{-\varrho^2}{2\sigma_z^2}\right) \mathrm{d}\varrho \prod_{k=1}^{n-2} \left(\int_0^{\pi} \sin^{n-k-1}(\alpha_k)\, \mathrm{d}\alpha_k\right) \int_0^{2\pi} \mathrm{d}\alpha_{n-1}$$

Using substitution $\varrho^2/2\sigma_z^2 = x$ we have for the first integral in (2.16)

$$(2.17)\quad \int_0^\infty \varrho^{n-3} \exp\left(\frac{-\varrho^2}{2\sigma_z^2}\right) d\varrho = 2^{(n/2)-2}\,\sigma_z^{n-2} \int_0^\infty x^{((n/2)-1)-1} \exp(-x)\, dx =$$

$$= 2^{(n/2)-2}\,\sigma_z^{n-2}\,\Gamma\left(\frac{n}{2} - 1\right)$$

Note that the gamma function in (2.17) is defined only for $n > 2$. The others integrals in (2.16) can also be expressed using the gamma function

$$(2.18)\qquad \int_0^\pi \sin^{n-k-1}(\alpha_k)\, d\alpha_k = \sqrt{(2\pi)}\, \frac{\Gamma\left(\dfrac{n-k}{n}\right)}{\Gamma\left(\dfrac{n-k+1}{2}\right)}$$

Substituting (2.17), (2.18) into (2.16) and using elementary properties of the gamma function equation (2.10) can be obtained after some transformations.

3. CONTROL OF THE ESTIMATION PROCESS

We choose the gain $\varkappa_t$ as the control variable of the estimation process. Similarly the mean value of the squared distance between the vector $\boldsymbol{b}$ (object) and its estimate $\hat{\boldsymbol{b}}_t$ (model) is chosen as the state variable of the estimation process, i.e.

$$(3.1)\qquad S_t = \mathsf{E}\{\boldsymbol{d}_t^{\mathrm{T}}\boldsymbol{d}_t\} = \mathsf{E}\{(\boldsymbol{b} - \hat{\boldsymbol{b}}_t)^{\mathrm{T}}(\boldsymbol{b} - \hat{\boldsymbol{b}}_t)\}$$

The mean value in (3.1) is taken over all random variables $\mathbf{z}_1, \mathbf{z}_2, \ldots, \mathbf{z}_t$ and $\eta_1, \eta_2, \ldots$ $\ldots, \eta_t$. The sequence

$$(3.2)\qquad \{S_t;\ t = 0, 1, 2, \ldots\}$$

is called the trajectory of the estimation process (or briefly the process) and the sequence

$$(3.3)\qquad \{\varkappa_t;\ t = 1, 2, \ldots\}$$

the control of the estimation process. Providing the initial state and the control (3.3) is given, the trajectory of the estimation process is uniquely determined by the state equation

$$(3.4)\qquad S_t = f(S_{t-1}, \varkappa_t)$$

To compare two trajectories $\{S_t\}$ and $\{S'_t\}$ with the common initial state S_0, we introduce the following notions. We say that the process $\{S_t\}$ has a better *initial convergence* than the process $\{S'_t\}$ if there exists $t_0 \geqq 1$ such that

$$(3.5)\qquad S_t < S'_t$$

holds for $0 < t \leqq t_0$. If the inequality (3.5) holds for all $t \geqq t_0$, we say that the process $\{S_t\}$ has a better *final convergence* than $\{S_t\}$. If the inequality $S_t = S'_t$ holds for all $t = 0, 1, 2. \ldots$ and the strict inequality is valid for at least one t, then we say that the process $\{S_t\}$ has a better *global convergence* than the process $\{S'_t\}$.

Our main goal is to find such a control that gives the best global convergence, i.e. infimal trajectory of the estimation process. Such a control will be called the *globally optimal control*. The control $\{\varkappa_t^*\}$ defined for all $t = 1, 2, \ldots$ by the relation

$$f(S_{t-1}, \varkappa_t^*) = \min_{\varkappa_t} f(S_{t-1}, \varkappa_t) \tag{3.6}$$

is called the *locally optimal control* (control optimal at each time instant). Note that the locally optimal control need not be globally optimal and that globally optimal control need not exist at all. The following lemma states the condition that guarantees existence of the globally optimal control.

Lemma 3.1. Let the implication

$$S < S' \Rightarrow f(S, \varkappa) < f(S', \varkappa) \tag{3.7}$$

be true for all admissible values of the control variable $\varkappa$. Then the locally optimal control is also globally optimal.

Proof. Let us compare the process $\{S_t^*\}$ corresponding to locally optimal control $\{\varkappa_t^*\}$ with the process $\{S_t\}$ corresponding to any other control $\{\varkappa_t\} \neq \{\varkappa_t^*\}$. Suppose $S_{t-1}^* \leqq S_{t-1}$. Then from (3.7) and (3.6) it follows

$$S_t^* = f(S_{t-1}^* \varkappa_t^*) \leqq f(S_{t-1}, \varkappa_t^*) \leqq f(S_{t-1}, \varkappa_t) = S_t \,,$$

and consequently $S_t^* \leqq S_t$ (the equality $S_t^* = S_t$ holds if and only if contemporarily $S_{t-1}^* = S_{t-1}$ and $\varkappa_t^* = \varkappa_t$). Let us denote $t_0 = \min\{t \mid \varkappa_t \neq \varkappa_t^*\}$. Since $S_0^* = S_0$, we get

$$S_1^* = S_1 \,, \quad S_2^* = S_2, \ldots, S_{t_0-1}^* = S_{t_0-1} \,, \quad S_{t_0}^* < S_{t_0} \,, \quad S_{t_0+1}^* < S_{t_0+1}, \ldots$$

i.e. the locally optimal control is globally optimal □

Now, we turn our attention to the estimation processes generated by the estimators SSA (2.2) and NSA (2.3).

Theorem 3.1. Assume the external conditions of estimation to be the same as in Section 2. Then the state equation (3.4) for the estimation processes generated by the estimators SSA (n arbitrary) and NSA ($n > 2$) has the following forms

$$S_t = f_{\mathrm{SSA}}(S_{t-1}, \varkappa_t) = [1 - 2\sigma_z^2 \varkappa_t + (n + 2)\, \sigma_z^4 \varkappa_t^2]\, S_{t-1} + n\sigma_z^2 \sigma_\eta^2 \varkappa_t^2 \tag{3.8}$$

$$S_t = f_{\mathrm{NSA}}(S_{t-1}, \varkappa_t) = \left[1 - \frac{2}{n} \varkappa_t + \frac{1}{n} \varkappa_t^2\right] S_{t-1} + \frac{1}{n-2} \frac{\sigma_\eta^2}{\sigma_z^2} \varkappa_t^2 \tag{3.9}$$

Proof. We rewrite recursions (2.2) and (2.3) for estimates $\hat{\boldsymbol{b}}_t$ in terms of the estimation errors $\mathbf{d}_t = \mathbf{b} - \hat{\mathbf{b}}_t$. Using (2.1) we get

$$\mathbf{d}_t = (\mathbf{I} - \varkappa_t \mathbf{z}_t \mathbf{z}_t^{\mathrm{T}})\, \mathbf{d}_{t-1} - \varkappa_t \eta_t \mathbf{z}_t \quad \text{for SSA} \tag{3.10}$$

$$\mathbf{d}_t = \left(\mathbf{I} - \varkappa_t \frac{\mathbf{z}_t \mathbf{z}_t^{\mathrm{T}}}{\mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t}\right) \mathbf{d}_{t-1} - \varkappa_t \frac{\eta_t}{\mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t}\, \mathbf{z}_t \quad \text{for NSA} \tag{3.11}$$

and hence after some calculations

$$\begin{aligned} \mathbf{d}_t^{\mathrm{T}} \mathbf{d}_t = {} & \mathbf{d}_{t-1}^{\mathrm{T}} \mathbf{d}_{t-1} - 2\varkappa_t \mathbf{d}_{t-1}^{\mathrm{T}} \mathbf{z}_t \mathbf{z}_t^{\mathrm{T}} \mathbf{d}_{t-1} + \varkappa^2 \mathbf{d}_{t-1}^{\mathrm{T}} (\mathbf{z}_t \mathbf{z}_t^{\mathrm{T}})^2\, \mathbf{d}_{t-1} - \\ & - 2\varkappa_t \eta_t \mathbf{z}_t^{\mathrm{T}} \mathbf{d}_{t-1} + 2\varkappa_t^2 \eta_t \mathbf{d}_{t-1}^{\mathrm{T}} \mathbf{z}_t \mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t + \varkappa_t^2 \eta_t^2 \mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t \quad \text{for SSA} \end{aligned} \tag{3.12}$$

$$\begin{aligned} \mathbf{d}_t^{\mathrm{T}} \mathbf{d}_t = {} & \mathbf{d}_{t-1}^{\mathrm{T}} \mathbf{d}_{t-1} - \varkappa_t (2 - \varkappa_t)\, \mathbf{d}_{t-1}^{\mathrm{T}} \frac{\mathbf{z}_t \mathbf{z}_t^{\mathrm{T}}}{\mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t}\, \mathbf{d}_{t-1} + \\ & + 2\varkappa_t (\varkappa_t - 1)\, \eta_t \frac{\mathbf{d}_{t-1}^{\mathrm{T}} \mathbf{z}_t}{\mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t} + \varkappa_t^2 \eta_t^2 \frac{1}{\mathbf{z}_t^{\mathrm{T}} \mathbf{z}_t} \quad \text{for NSA} \end{aligned} \tag{3.13}$$

The mean value $S_t = \mathsf{E}\{\mathbf{d}_t^{\mathrm{T}} \mathbf{d}_t\}$ can be calculated using the well-known rule of probability theory

$$\mathop{\mathsf{E}}_{x,y}\{g(x, y)\} = \mathop{\mathsf{E}}_{x}\{\mathop{\mathsf{E}}_{y}\{g(x, y) \mid x\}\}\,.$$

Following this rule we apply the operator of conditional mean $\mathsf{E}\{\cdot \mid \mathbf{d}_{t-1}\}$ (the expectation is considered for $\mathbf{z}_t$, η_t) on the equations (3.12), (3.13) at first .By Lemma 2.1, assumptions (2.6), (2.7) and the independence of random sequences $\{\mathbf{z}_t\}$, $\{\eta_t\}$ we obtain

$$\mathsf{E}\{\mathbf{d}_t^{\mathrm{T}} \mathbf{d}_t \mid \mathbf{d}_{t-1}\} = [1 - 2\sigma_z^2 \varkappa_t + (n + 2)\, \sigma_z^4 \varkappa_t^2]\, \mathbf{d}_{t-1}^{\mathrm{T}} \mathbf{d}_{t-1} + n \sigma_z^2 \sigma_\eta^2 \varkappa_t^2 \quad \text{for SSA} \tag{3.14}$$

$$\mathsf{E}\{\mathbf{d}_t^{\mathrm{T}} \mathbf{d}_t \mid \mathbf{d}_{t-1}\} = \left[1 - \frac{2}{n} \varkappa_t + \frac{1}{n} \varkappa_t^2\right] \mathbf{d}_{t-1}^{\mathrm{T}} \mathbf{d}_{t-1} + \frac{1}{n - 2} \frac{\sigma_\eta^2}{\sigma_z^2} \varkappa_t \quad \text{for NSA} \tag{3.15}$$

Finally, applying the operator $\mathsf{E}\{\cdot\}$ (the expectation is considered for $\mathbf{d}_{t-1}$) on (3.14), (3.15) we get the state equations (3.8), (3.9). □

Theorem 3.2. Let the external conditions of estimation are the same as in Section 2. Then the locally optimal control of estimation by SSA or NSA ($n > 2$) is also globally optimal.

Proof. According to Lemma 3.2, we have to prove the implication (3.7) for $f = f_{\mathrm{SSA}}$ and $f = f_{\mathrm{NSA}}$. From (3.8), (3.9) it follows that this implication is equivalent to the following inequality

$$1 - 2\sigma_z^2 \varkappa + (n + 2)\, \sigma_z^4 \varkappa^2 > 0 \quad \text{for SSA} \tag{3.16}$$

$$1 - \frac{2}{n} \varkappa + \frac{1}{n} \varkappa^2 > 0 \quad \text{for NSA} \tag{3.17}$$

It is easy to see that the inequality (3.16) holds for arbitrary n and the inequality (3.17) holds for $n > 1$ and hence also for $n > 2$ as demanded. □

The just proven theorem enables us to omit the adverbs "locally" and "globally" and to speak simply about optimal control.

4. THE CONSTANT CONTROL

In this section we shall study the estimation process in the trivial case when the control variable is chosen constant, i.e. $\varkappa_t \equiv \varkappa$.

Theorem 4.1. Assume the same external conditions of estimation as in Section 2. The estimation processes generated by the estimator SSA (dimension n arbitrary) and NSA (dimension $n > 2$) with the constant gain $\varkappa_t \equiv \varkappa$ are described by the formula

(4.1) $$S_t = (S_0 - R)\, Q^t + R\,; \quad t = 1, 2, \ldots$$

where quantities Q and R are constants specified for SSA and NSA respectively as follows

(4.2) $$Q = 1 - 2\sigma_z^2 \varkappa + (n + 2)\, \sigma_z^4 \varkappa^2$$

(4.3) $$R = \frac{n \sigma_\eta^2 \varkappa}{2 + (n + 2)\, \sigma_z^2 \varkappa}$$

for SSA

(4.4) $$Q = 1 - \frac{2}{n} \varkappa + \frac{1}{n} \varkappa^2$$

(4.5) $$R = \frac{n}{n - 2} \frac{\sigma_\eta^2}{\sigma_z^2} \frac{\varkappa}{2 - \varkappa}$$

for NSA

Proof. If $\varkappa_t \equiv \varkappa$ then the state equations (3.8) for SSA and (3.9) for NSA have the form of a simple difference equation

(4.6) $$S_t = Q S_{t-1} + P$$

with constant coefficients Q, P. The solution of (4.6) is

(4.7) $$S_t = \left(S_0 - \frac{P}{1 - Q}\right) Q^t + \frac{P}{1 - Q}$$

If we substitute for P, Q into (4.7) in accordance with (3.8) or (3.9) we obtain the formulas (4.1), (4.2), (4.3) or (4.1), (4.4), (4.5) respectively. □

From equation (4.1) it follows: if $|Q| < 1$ then the sequence $\{S_t\}$ converges and has a limit R (the so called residual or stationary error of estimation).

Simple analysis of formulas (4.2), (4.3) gives the following conclusions for the

SSA estimator. The estimation process converges for

$$\varkappa \in \left(0; \frac{2}{(n + 2)\,\sigma_z^2}\right) \tag{4.8}$$

The function $Q = Q_{\mathrm{SSA}}(\varkappa)$ given by (4.2) is decreasing (the rate of convergence increasing) for $\varkappa \in (0; 1/((n + 2)\,\sigma_z^2))$ and increasing (the rate of convergence decreasing) for $\varkappa \in (1/((n + 2)\,\sigma_z^2); 2/((n + 2)\,\sigma_z^2))$. Providing $\sigma_\eta^2 > 0$, the function $R = R_{\mathrm{SSA}}(\varkappa)$ given by (4.3) increases (the stationary accuracy of estimation decreases) in the whole interval of convergence (4.8). Every reasonable choice of a constant gain $\varkappa$ is therefore from the interval

$$\varkappa \in \left(0; \frac{1}{(n + 2)\,\sigma_z^2}\right) \tag{4.9}$$

and it always represents a compromise between the contradictional requirements for rate and for accuracy of estimation.

Quite similar results can be derived for the estimator NSA by the analysis of formulas (4.4), (4.5). The estimation process converges for

$$\varkappa \in (0; 2) \tag{4.10}$$

but the reasonable choice of a constant gain (balancing the contradictory requirements for rate and accuracy) is from the interval

$$\varkappa \in (0; 1)\,. \tag{4.11}$$

5. THE OPTIMAL CONTROL

From the theory of stochastic approximation it is well known that the exact estimates will be reached if we apply time-variant control satisfying the conditions

$$\varkappa_t > 0\,, \quad \sum_{t=1}^{\infty} \varkappa_t = \infty\,, \quad \sum_{t=1}^{\infty} \varkappa_t^2 < \infty \tag{5.1}$$

The most frequently used control is the following standard control

$$\varkappa_t = \frac{1}{\alpha t}, \quad \alpha > 0 \tag{5.2}$$

which evidently satisfies conditions (5.1). It is easy to verify that the recursive form of (5.2) is

$$\varkappa_t = \frac{\varkappa_{t-1}}{1 + \alpha \varkappa_{t-1}}; \quad t = 2, 3, \ldots \tag{5.3}$$

In this section the recursive law of the optimal control is derived and it is shown that the standard control just mentioned does not produce an optimal estimation.

Theorem 5.1. Suppose the same external conditions of estimation as in Section 2. Then the optimal control of the estimation process generated by the estimator SSA is given by

$$\varkappa_1^* = \frac{S_0}{(n+2)\,\sigma_z^2\,S_0 + n\sigma_\eta^2} \tag{5.4}$$

$$\varkappa_t^* = \frac{1 - \sigma_z^2\,\varkappa_{t-1}^*}{1 - (n+2)\,\sigma_z^4\varkappa_{t-1}^{*2}}\,\varkappa_{t-1}^*; \quad t = 2, 3, \ldots \tag{5.5}$$

For the estimator NSA ($n > 2$) we have the following formulas

$$\varkappa_1^* = \frac{(n-2)\,\sigma_z^2 S_0}{(n-2)\,\sigma_z^2\,S_0 + n\sigma_\eta^2} \tag{5.6}$$

$$\varkappa_t^* = \frac{n - \varkappa_{t-1}^*}{n - \varkappa_{t-1}^{*2}}\,\varkappa_{t-1}^*; \quad t = 2, 3, \ldots. \tag{5.7}$$

Proof. To prove the theorem we shall find the optimal control at an arbitrary time instant t (in an arbitrary state S_{t-1}), i.e. we shall solve the optimization problem (3.6) where $f = f_{\mathrm{SSA}}$ or $f = f_{\mathrm{NSA}}$ – see (3.8) or (3.9). By the differential calculus we get the solution

$$\varkappa_t^* = \frac{S_{t-1}}{(n+2)\,\sigma_z^2 S_{t-1} + n\sigma_\eta^2} \quad \text{for SSA} \tag{5.8}$$

$$\varkappa_t^* = \frac{(n-2)\,\sigma_z^2 S_{t-1}}{(n-2)\,\sigma_z^2 S_{t-1} + n\sigma_\eta^2} \quad \text{for NSA} \tag{5.9}$$

Hence, for $t = 1$, we get (5.4) for SSA or (5.6) for NSA. Combining relation (5.8) resp. (5.9) with (3.8) resp. (3.9) we obtain the recursion (5.5) resp. (5.7) after some calculations. □

The mappings $\varkappa_{t-1}^* \to \varkappa_t^*$ defined by the relations (5.5) or (5.7) respectively have the following interesting properties:

1) The end-points of the resonable intervals (4.9) or (4.11) are the fixed points of the mappings: the points 0 and $1/((n+2)\,\sigma_z^2)$ for the mapping (5.5) and the points 0 and 1 for the mapping (5.7).
2) If $\sigma_\eta^2 = 0$, then for SSA from (5.4) it follows $\varkappa_1^* = 1/((n+2)\,\sigma_z^2)$ and hence from the first property $\varkappa_t^* = 1/((n+2)\,\sigma_z^2)$ for all t. Similarly for the estimator NSA from (5.6) it follows $\varkappa_1^* = 1$, and hence, since $\varkappa_1^* = 1$ is a fixed point of (5.7), $\varkappa_t^* = 1$ for all t. Thus, for deterministic systems the optimal control is the constant control.
3) Let $\sigma_\eta^2 \neq 0$ and $\varkappa_1^* \in (0; 1/((n+2)\,\sigma_z^2))$ for SSA or $\varkappa_1^* \in (0; 1)$ for NSA. Then the following inequalities

$$0 < \varkappa_t^* < \varkappa_{t-1}^*$$

are valid and the sequences $\{\varkappa_t^*\}$, $\{S_t^*\}$ converge to zero (the formal proof of this property is based on the fixed point theorem).

In practice, we do not know the exact values of quantities S_0, σ_η^2 contained in formulas (5.4), (5.6) and that is why these formulas are not suitable for practical determination of the initial value of an optimal control. Luckilly, the run of estimation is not too much sensitive to the error of determination of the first control action $\varkappa_1^*$. Moreover, in the majority of real estimation problems we can assume $(n + 2) \cdot \sigma_z^2 S_0 \gg n\sigma_\eta^2$ for SSA, or $(n - 2)\, \sigma_z^2 S_0 \gg n\sigma_\eta^2$ for NSA, and therefore it is reasonable to choose the first control action nearly equal to $1/((n + 2)\, \sigma_z^2)$ for SSA or 1 for NSA, but always a little lesser that these values. We can also employ the following theorem.

Theorem 5.2. Let us denote $R_\varkappa = \lim_{t \to \infty} S_t$ the residual error of estimation provided the constant control $\varkappa_t \equiv \varkappa$ is applied. The first optimal control action for the initial state $S_0 = R_\varkappa$ of the estimation process is given by the formula

$$\varkappa_1^* = \frac{\varkappa}{2} \tag{5.10}$$

The formulas (5.10) is valid both for the estimator SSA and NSA.

Proof. Substitute into (5.4) for S_0 the R according to (4.3) (for the estimator SSA) or into (5.6) according to (4.5) (for the estimator NSA). In both cases we obtain (5.10). □

In practice we can use Theorem 5.2 in such a way. We apply constant control $\varkappa_t = \varkappa$ first and after some time when it is possible to assume that stationary state is reached, we switch on optimal control given by (5.5) or (5.7) and with the initial value of control variable stated by (5.10).

Now we compare the estimation processes generated by the optimally controlled estimators SSA and NSA.

Theorem 5.3. Assume the same experimental conditions of estimation as in Section 2 and consider the estimation process generated by the estimator SSA (2.2) with optimal control (5.4), (5.5) and the estimation process generated by the estimator NSA (2.3) with optimal control (5.6), (5.7). Then for $n > 2$, $S_0 > 0$, $\sigma_z^2 > 0$ the following assertions are true:

1) If $\sigma_\eta^2 = 0$ (deterministic case) then the estimation process generated by NSA has always (independently of S_0, n, σ_z^2) a better global convergence that the estimation process generated by SSA.
2) If $\sigma_\eta^2 > 0$ (stochastic case) then the algorithm SSA provides always (regardless of the values of S_0, n, σ_z^2, σ_η^2) a better final convergence than NSA. If the inequality

$$S_0 < \frac{n}{n - 2} \frac{\sigma_\eta^2}{\sigma_z^2} \tag{5.11}$$

is valid then the algorithm SSA has also a better initial convergence than the NSA and, consequently, a better global convergence too. If the converse of (5.11) is true then the algorithm NSA exhibits a better initial convergence than the SSA.

Proof. The runs of the two optimal controlled estimation processes are described by the state equations

$$S_t = f^*_{SSA}(S_{t-1}) = f_{SSA}(S_{t-1}, \varkappa_t^{*SSA}(S_{t-1}))$$
$$S_t = f^*_{NSA}(S_{t-1}) = f_{NSA}(S_{t-1}, \varkappa_t^{*NSA}(S_{t-1}))$$

where the functions f_{SSA}, f_{NSA} are defined by formulas (3.8), (3.9) and the quantities $\varkappa^{*SSA}(S_{t-1})$, $\varkappa^{*NSA}(S_{t-1})$ by the equations (5.8) and (5.9) respectively. Inserting (5.8) into (3.8) resp. (5.9) into (3.9) we finally get (the time indexes are omitted):

$$f^*_{SSA}(S) = S\left(1 - \sigma_z^2 \frac{S}{(n+2)\,\sigma_z^2 S + n\sigma_\eta^2}\right) \tag{5.12}$$

$$f^*_{NSA}(S) = S\left(1 - \frac{1}{n}\frac{(n-2)\,\sigma_z^2 S}{(n-2)\,\sigma_z^2 S + n\sigma_\eta^2}\right) \tag{5.13}$$

The assertions of the theorem follow from the relations

$$f^*_{NSA}(S)\begin{Bmatrix} < \\ = \\ > \end{Bmatrix} f^*_{SSA}(S) \Leftrightarrow S \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} \frac{n}{n-2}\frac{\sigma_\eta^2}{\sigma_z^2}$$

that can be easily verified on the basis of (5.12), (5.13). □

Remark, that 1) the real conditions of estimation are mostly such that the inequality (5.11) is false and the NSA estimator has better initial convergence than SSA, 2) the difference between the rates of final conveıgence of both estimators is practically negligible, 3) the convergence interval and the recursive law of optimal control for the NSA estimation does not depend on the input variance σ_z^2 (in contrast to the SSA estimation). For all these reasons we generally prefer the NSA estimator to the SSA one.

Finally, we make some comments regarding the standard control of estimation (5.2) or (5.3). The comparison of the recursions (5.5), (5.7) with (5.3) shows that the standard control can never be globally optimal (by no choice of parameter α). Nevertheless, we try to find such standard control $\varkappa_t$ that implies asymptotically optimal behaviour. For this purpose assume $\varkappa^*_{t-1} = \varkappa_{t-1}$ and divide (5.5) or (5.7) by (5.3). Neglecting the terms of second order we get respectively

$$\frac{\varkappa_t^*}{\varkappa_t} = 1 - (\alpha - \sigma_z^2)\,\varkappa_{t-1} \quad \text{for} \quad \text{SSA} \tag{5.14}$$

$$\frac{\varkappa_t^*}{\varkappa_t} = 1 + \left(\alpha - \frac{1}{n}\right)\varkappa_{t-1} \quad \text{for} \quad \text{NSA} \tag{5.15}$$

The standard control $\{\varkappa_t\}$ will be asymptotically the same as the optimal control $\{\varkappa_t^*\}$ if

$$\lim_{t\to\infty} \frac{\varkappa_t^*}{\varkappa_t} = 1 \tag{5.16}$$

From (5.14), (5.15) it follows that (5.16) will be satisfied if $\alpha = \sigma_z^2$ for SSA and $\alpha = 1/n$ for NSA. Consequently the standard controls

$$\varkappa_t = \frac{1}{\sigma_z^2 t} \quad \text{for} \quad \text{SSA} \tag{5.17}$$

$$\varkappa_t = \frac{n}{t} \quad \text{for} \quad \text{NSA} \tag{5.18}$$

are asymptotically optimal for estimation conditions stated in Section 2. The formula (5.17) can also be derived from the more general results contained in [3]. Nevertheless we cannot recommend the controls (5.17), (5.18) for practical use because of their divergence at the very beginning of the estimation. To improve the initial behaviour it is reasonable to restrict the possible values of control actions to intervals (4.9), (4.11) as follows

$$\varkappa_t = \begin{cases} \dfrac{1}{(n+2)\,\sigma_z^2} & \text{for} \quad t < n+2 \\ \dfrac{1}{\sigma_z^2 t} & \quad t \geqq n+2 \end{cases} \qquad \text{(SSA)} \tag{5.19}$$

$$\varkappa_t = \begin{cases} 1 & \text{for} \quad t < n \\ \dfrac{n}{t} & \quad t \geqq n \end{cases} \qquad \text{(NSA)} \tag{5.20}$$

6. SOME GENERALISATIONS

The results presented in the previous section can be extended in several directions. We briefly mention two of such generalisations.

1) The assumption of gaussian distribution of the inputs is not essential. The optimal control of the SSA estimator (formulas (5.4), (5.5)) can be generalised as follows

$$\varkappa_1^* = \frac{S_0}{(n+\beta)\,\sigma_z^2 S_0 + n\sigma_n^2} \tag{6.1}$$

$$\varkappa_t^* = \frac{1 - \sigma_z^2 \varkappa_{t-1}^*}{1 + (n+\beta)\,\sigma_z^2 S_0 \varkappa_{t-1}^*}\, \varkappa_{t-1}^* \,; \quad t = 2, 3, \ldots \tag{6.2}$$

where the parameter β depends on the type of input's distribution. The value $\beta = 2$

corresponds to the normal distribution, the value $\beta = \frac{4}{5}$ to the uniform distribution and the value $\beta = 0$ to the binary distribution (the random variable takes on only two values σ_z, $-\sigma_z$ with the same probability $\frac{1}{2}$). The assertion was proved for normal distribution (see Theorem 5.1, Theorem 3.1, Lemma 2.1 and their proofs), the proofs for uniform and binary distribution are quite similar.

The optimal control of the NSA estimator for binary distributed inputs is given by the formulas

$$\varkappa_1^* = \frac{\sigma_z^2 S_0}{\sigma_z^2 S_0 + \sigma_\eta^2} \tag{6.3}$$

$$\varkappa_t^* = \frac{n - \varkappa_{t-1}^*}{n - \varkappa_{t-1}^{*2}} \varkappa_{t-1}^* \,; \quad t = 2, 3, \dots \tag{6.4}$$

Note that the recursive law (6.4) of optimal control is the same as that for normally distributed inputs – see (5.7). The optimal control of the NSA estimator seems to be much less sensitive to the type of input's distribution than the optimal control of the SSA estimator.

2) The second extension generalizes the model of the system to be identified. In this generalization the estimated system can be not only stochastic but also time-variant. Equation (2.1), describing the system, is replaced by the following more general equation

$$y_t = \mathbf{z}_t^{\mathrm{T}} \boldsymbol{b}_t + \eta_t \tag{6.5}$$

where $\boldsymbol{b}_t$ is the time-variant vector of the system parameters. The evolution of this vector is described by

$$\boldsymbol{b}_t = \boldsymbol{b}_{t-1} + \boldsymbol{g}_t$$

where $\{\boldsymbol{g}_t\}$ is a random sequence with the statistics

$$\mathsf{E}\{\boldsymbol{g}_t\} = \mathbf{0}$$

$$\mathsf{E}\{\boldsymbol{g}_t \boldsymbol{g}_s^{\mathrm{T}}\} = \begin{cases} \mathbf{G} & \text{for } t = s \\ \mathbf{0} & \phantom{\text{for }} t \neq s \end{cases}$$

The sequence $\{\boldsymbol{g}_t\}$ is assumed to be independent of the input and noise random sequences $\{\mathbf{z}_t\}$, $\{\eta_t\}$. All assumptions on $\{\mathbf{z}_t\}$, $\{\eta_t\}$ made in Section 2 remain unchanged.

We shall consider only the estimator NSA because it is more convenient for estimation of nonstationary systems (the results for SSA are similar). For generalized model we can formulate the following theorems:

Theorem 6.1. Suppose the generalised external conditions of estimation as stated above. Then for $n > 2$ the estimation process generated by the estimator NSA with constant control $\varkappa_t \equiv \varkappa$ is described by the formulas:

$$S_t = (S_0 - R)\, Q^t + R; \quad t = 1, 2, \dots$$

$$Q = 1 - \frac{2}{n} \varkappa + \frac{1}{n} \varkappa^2$$

$$R = R_1 + R_2$$

$$R_1 = \frac{n}{n-2} \frac{\sigma_\eta^2}{\sigma_z^2} \frac{\varkappa}{2 - \varkappa}$$

$$R_2 = \left(\frac{n}{\varkappa(2 - \varkappa)} - 1 \right) \operatorname{tr} \mathbf{G}$$

Observe that the stationary error R consists of two components R_1, R_2, the first of which depends on the noise-to-signal ratio $(\sigma_\eta^2/\sigma_z^2)$ and the second on the degree of the time variability ($\operatorname{tr} \mathbf{G}$). A simpe analysis shows that with the control parameter $\varkappa$ growing from 0 to 1 the first component is growing too, but the second is decreasing. Such a value $\varkappa^* \in (0; 1\rangle$ exists for which the overall stationary error $R = R_1 + R_2$ has the minimal value $R^* = R(\varkappa^*) = \min R(\varkappa)$.

Theorem 6.2. Suppose $n > 2$ and the generalised external conditions of the estimation. Then the optimal control of the estimator NSA is given by the formulas

$$\varkappa_1^* = \frac{(n-2)\, \sigma_z^2 (S_0 + \operatorname{tr} \mathbf{G})}{(n-2)\, \sigma_z^2 (S_0 + \operatorname{tr} \mathbf{G}) + n \sigma_\eta^2}$$

$$\varkappa_t^* = \frac{(n - \varkappa_{t-1}^*)\, \varkappa_{t-1}^* + (1 - \varkappa_{t-1}^*)\, D}{n - \varkappa_{t-1}^{*2} + (1 - \varkappa_{t-1}^*)\, D}; \quad t = 2, 3, \ldots$$

where

$$D = \frac{(n-2)\, \sigma_z^2 \operatorname{tr} \mathbf{G}}{\sigma_\eta^2}$$

If we apply the optimal control according to Theorem 6.2, then we reach the minimal value of the stationary error R^* along the optimal (infimal) trajectory of the estimation process, i.e. with the maximal possible rate. If nonstationarity occurs ($\operatorname{tr} \mathbf{G} \neq 0$) then the sequences $\{\varkappa_t^*\}$, $\{S_t^*\}$ do not tend to zero, as in the stationary case, but to the non-zero values $\varkappa^*$, R^* mentioned above.

7. CONCLUSION

In this paper new stochastic approximation algorithms for system identification have been presented. In contrast to the well-known standard stochastic approximation, they possess the property of overall (global) optimality. However, it should be stressed that the assumed external conditions of identification are the most simple ones. In practice we always meet more difficult external conditions. The presented algorithms can be applied also under these conditions but without pretension to optimality. Nevertheless, the experience shows that regardless of the external conditions the presented algorithms exhibit better convergence than the standard ones.

For practical use the generalisation of NSA estimator, proposed in [6], that radically accelerates the initial convergence, can be recommended. The recursive law of the gain evolution derived in this paper can be used for the generalised algorithm too; naturally again without pretension to optimality.

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