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IMPROVEMENT OF EXTRAPOLATION IN MULTIVARIATE STATIONARY PROCESSES

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If we are not satisfied with the accuracy of the single-step extrapolation $\hat{X}_t$ in a discrete stationary multivariate $\{X_t\}$ we can try to improve it by means of an additional multivariate proces $\{Y_t\}$. Necessary and sufficient conditions for the case that the knowledge of values $Y_{t+s}, Y_{t+s-1}, \ldots$ do not improve this extrapolation are derived in the paper provided the compound process $\{(X_t', Y_t')'\}$ is a process AR(n) or MA(m). These conditions are formulated in terms of (matrix) parameters of the corresponding models. Further the problem of uncorrelated processes $\{X_t\}$ and $\{Y_t\}$ in connection with the improvement of extrapolation is solved.

1. INTRODUCTION AND SUMMARY

Let $\{W_t\} = \{(X_t', Y_t')'\}$ be a multivariate discrete stationary process with zero mean value and (vector) components $\{X_t\}$ and $\{Y_t\}$. Let $\hat{X}_t$ be the extrapolation of X_t based on $X_{t-1}, X_{t-2}, \ldots$ (i.e. the particular scalar components of $\hat{X}_t$ are the best linear approximations of the corresponding components of X_t in the Hilbert space generated by all scalar components of all vectors $X_{t-1}, X_{t-2}, \ldots$) and analogously let $\hat{X}_t(a, b)$ be the extrapolation of X_t based on $X_{t-a}, X_{t-a-1}, \ldots, Y_{t-b}, Y_{t-b-1}, \ldots$. The latter extrapolation plays an important role in such cases when particular components of a multivariate process are observed through various time periods. The following case is also very frequent: provided we are not satisfied with the accuracy of the extrapolation $\hat{X}_t$ we can try to improve it by means of an additional process $\{Y_t\}$. The accuracy of the extrapolations is measured by the matrices

$$\Delta_X = \mathsf{E}[X_t - \hat{X}_t]\,[X_t - \hat{X}_t]'\,, \quad \Delta_X(1, b) = \mathsf{E}[X_t - \hat{X}_t(1, b)]\,[X_t - \hat{X}_t(1, b)]'\,.$$

Obviously, it holds that

$$\Delta_X \geqq \ldots \geqq \Delta_X(1, 1) \geqq \Delta_X(1, 0) \geqq \Delta_X(1, -1) \geqq \ldots\,, \tag{1.1}$$

where e.g. $\Delta_X(1, 1) \geqq \Delta_X(1, 0)$ denotes that the matrix $\Delta_X(1, 1) - \Delta_X(1, 0)$ is positive semidefinite. If $\Delta_X = \Delta_X(1, b)$ we can say that the process $\{Y_t\}$ does not improve the extrapolation for one step in the process $\{X_t\}$.

Provided $\{W_t\}$ is a process AR(n) or MA(m) the necessary and sufficient conditions for particular equalities in (1.1) are derived in this paper. These conditions are expressed in the form of certain relations among the (matrix) parameters of the corresponding models. Further it is shown that some equalities in (1.1) imply that the processes $\{X_t\}$ and $\{Y_t\}$ are uncorrelated. The formulas in the proofs of some assertions in this paper can be applied for the actual construction of the mentioned extrapolations $\hat{X}_t(1, b)$.

Anděl in [1], [2] and [3] has solved this problem for some equalities in (1.1) for models AR(1), MA(1) and partially for ARMA(1, 1). His technique is generalized and applied in this paper. The result of Anderson [4] for a model AR(1) is also a special case of the results derived in this paper (cf. Theorem 3.1). Finally, it is necessary to mention the concept of the causal relationship (or causality) among time series investigated e.g. by Granger [6] or by Pierce and Haugh [7]. A time series $\{Y_t\}$ causes (causes instantaneously) another time series $\{X_t\}$ if a current value of $\{X_t\}$ can be extrapolated better by using past values (past and current values) of $\{Y_t\}$ than by not doing so. Therefore the investigation of causality is a special case of the aim of this paper.

2. PRELIMINARIES

A brief survey of the used mathematical tools is given in this section. The proofs of the assertions in the paper take advantage mainly of the following two principles. The first principle consists in the well-known theorem on the "successive projection":

Theorem 2.1. Let u, v_t and w_s (for all integers t and s) be elements of a Hilbert space H. Denote $H\{x, y, \ldots\}$ the Hilbert space generated by elements $x, y, \ldots \in H$. If $\hat{u}$ is the projection of u on $H\{\{v_t\}, \{w_s\}\}$, $\tilde{u}$ the projection of u on $H\{\{v_t\}\}$, $\tilde{w}_s$ the projection of w_s on $H\{\{v_t\}\}$ and $\bar{u}$ the projection of u on $H\{\{w_s - \tilde{w}_s\}\}$, then the relation

$$\hat{u} = \tilde{u} + \bar{u} \tag{2.1}$$

holds.

Remark 2.1. In the following text Theorem 2.1 will be used exclusively for the case when the series $\{w_s\}$ is finite (this series will be always formed by the scalar components of a random vector Y_t). It holds that $\|u - \hat{u}\| = \|u - \tilde{u}\|$ if and only if $\bar{u} = 0$ (since $\|u - \tilde{u}\|^2 = \|u - \hat{u}\|^2 + \|\hat{u} - \tilde{u}\|^2 = \|u - \hat{u}\|^2 + \|\bar{u}\|^2$).

Remark 2.2. Let X and Y be random vectors of arbitrary finite dimensions with zero mean values and finite second moments. Let the variance matrix of Y be regular.

Then the extrapolation of X based on Y is given by $\hat{X} = \operatorname{cov}(X, Y)(\operatorname{var} Y)^{-1} Y$. Further $\hat{X} = 0$ if and only if $\operatorname{cov}(X, Y) = 0$ (the latter assertion holds without the assumption of regularity of var Y).

The second principle consists in a method how to extrapolate in a (multivariate) process $\{X_t\}$ when a model for a compound process $\{W_t\} = \{(X_t', Y_t')'\}$ is only known. Anděl [3] described a very convenient method for the case that $\{W_t\}$ is a process ARMA(m, n). In this method an explicit ARMA model is constructed such that the corresponding process has the same spectral properties (and therefore the same extrapolation properties) as the process $\{X_t\}$. In this paper we use only the following assertion derived by means of this method (cf. [3]):

Let $\{W_t\} = \{(X_t', Y_t')'\}$ be a r-dimensional ARMA(m, n) process defined by

$$\sum_{k=0}^{n} A_k W_{t-k} = \sum_{j=0}^{m} B_j Z_{t-j}, \tag{2.2}$$

where the process $\{X_t\}$ is p-dimensional and $\{Y_t\}$ is q-dimensional $(p + q = r)$, $\{Z_t\}$ is a r-dimensional white noise (i.e. $\mathsf{E}Z_t = 0$, var $Z_t = I$ (an unit matrix), $\operatorname{cov}(Z_s, Z_t) = 0$ for $s \neq t$). Denote

$$\sum_{k=0}^{n} A_k z^k = \begin{pmatrix} K(z), & L(z) \\ M(z), & N(z) \end{pmatrix}, \qquad \sum_{j=0}^{m} B_j z^j = \begin{pmatrix} P(z), & Q(z) \\ R(z), & S(z) \end{pmatrix}, \tag{2.3}$$

$$N_0(z) = \operatorname{adj} N(z), \quad \nu(z) = \det N(z), \tag{2.4}$$

$$\nu(z) P(z) - L(z) N_0(z) R(z) = \sum_{j=0}^{nq+m} G_j z^j, \tag{2.5}$$

$$\nu(z) Q(z) - L(z) N_0(z) S(z) = \sum_{j=0}^{nq+m} H_j z^j, \tag{2.6}$$

where the blocks $K(z)$ and $P(z)$ are $p \times p$ matrices, $N(z)^{-1} = (1/\nu(z)) N_0(z)$ and $p \times p$ matrices G_j and $p \times q$ matrices H_j do not depend on z.

Theorem 2.2. Assume that

$$\det\left(\sum_{k=0}^{n} A_k z^k\right) \neq 0, \quad \det\left(\sum_{j=0}^{m} B_j z^j\right) \neq 0 \quad \text{for} \quad |z| \leqq 1 \tag{2.7}$$

and $\det N(z) \neq 0$ for $|z| \leqq 1$ is fulfilled in the model (2.2). Then the equality $\Delta_X = \Delta_X(1, 1)$ holds if and only if $p \times p$ matrices $D_0 = I, D_1, \ldots, D_{nq+m}$ exist such that

$$(G_j, H_j) = D_j(G_0, H_0), \quad j = 0, 1, \ldots, nq + m. \tag{2.8}$$

In this paper we use these symbols and conventions: $\{Z_t\}$ is a r-dimensional white noise (cf. (2.2)), all matrices without upper indices are $r \times r$ matrices and we devide vectors and matrices into the blocks according to the following patterns:

$$W_t = \begin{pmatrix} X_t \\ Y_t \end{pmatrix}, \quad Z_t = \begin{pmatrix} Z_t^1 \\ Z_t^2 \end{pmatrix}, \quad A_k = \begin{pmatrix} A_k^{11}, & A_k^{12} \\ A_k^{21}, & A_k^{22} \end{pmatrix},$$

etc., where the vectors X_t and Z_t^1 are p-dimensional, Y_t and Z_t^2 are q-dimensional and the block A_k^{11} is a $p \times p$ matrix $(p + q = r)$.

And finally, the following assertion (cf. [3]) will be also used in this paper:

Theorem 2.3. Let assumptions (2.7) and $\det B_0^{22} \neq 0$ hold in the model (2.2) where $A_0 = I$. Then the equality $\Delta_X(1, 1) = \Delta_X(1, 0)$ holds if and only if

$$B_0^{11} B_0^{21\prime} + B_0^{12} B_0^{22\prime} = 0 \,. \tag{2.9}$$

The proofs of all following assertions will be brief. The details are described in [5].

3. MODEL AR(n)

Consider a r-dimensional AR(n) process $\{W_t\}$ defined by

$$\sum_{k=0}^{n} A_k W_{t-k} = Z_t \,. \tag{3.1}$$

Assume that

$$\det \left(\sum_{k=0}^{n} A_k z^k \right) \neq 0 \,, \quad \det \left(\sum_{k=0}^{n} A_k^{22} z^k \right) \neq 0 \quad \text{for} \quad |z| \leqq 1 \,, \tag{3.2}$$

$$\det A_0^{11} \neq 0 \,. \tag{3.3}$$

Remark 3.1. The assumptions (3.2) imply regularity of the matrices A_0 and A_0^{22}. The first assumption in (3.2) is usual for the process AR(n).

Theorem 3.1. Let $s \geqq 0$ be a given integer and let the assumptions (3.2) and (3.3) be fulfilled in the model (3.1). Then the equality $\Delta_X = \Delta_X(1, -s)$ holds if and only if the following conditions are fulfilled simultaneously:

$$U_k^{12} = 0 \,, \quad k = 1, \ldots, n \,, \tag{3.4}$$

$$M^{11} M^{21\prime} + M^{12} M^{22\prime} = 0 \,, \tag{3.5}$$

$$U_k^{21} = 0 \,, \quad k = 1, \ldots, s \,, \tag{3.6}$$

where $M = A_0^{-1}$, $U_k = -A_0^{-1} A_k$, $k = 1, \ldots, n$.

Remark 3.2. The conditions (3.4)–(3.6) refer to the basic form of the model (3.1):

$$W_t = \sum_{k=1}^{n} U_k W_{t-k} + M Z_t \,, \tag{3.7}$$

where $M = A_0^{-1}$, $U_k = -A_0^{-1} A_k$. For $s = 0$ the condition (3.6) is omitted. To simplify the considerations put $U_k = 0$ and $A_k = 0$ for $k > n$ and $U_0 = 0$.

Proof. We shall use the method of induction. For $s = 0$ it is necessary to show that $\Delta_X = \Delta_X(1, 1) = \Delta_X(1, 0)$ if and only if the conditions (3.4) and (3.5) hold.

According to Theorem 2.2 the equality $\Delta_X = \Delta_X(1, 1)$ holds if and only if $p \times p$ matrices $D_0 = I, D_1, \ldots, D_{nq}$ exist such that

$$\nu(z) I = D(z)\, G_0 ,$$

$$-L(z)\, N_0(z) = D(z)\, H_0 ,$$

where $D(z) = \sum_{j=0}^{nq} D_j z^j$ and the denotation (2.3)–(2.6) is used (clearly, $P(z) = I$, $Q(z) = 0$, $R(z) = 0$, $S(z) = I$ in the model (3.1)). Hence for $z = 0$ the matrices G_0 and H_0 can be calculated so that after some treatments we get

$$\sum_{k=0}^{n} A_k^{12} z^k = A_0^{12}(A_0^{22})^{-1} \sum_{k=0}^{n} A_k^{22} z^k . \tag{3.8}$$

Since $U_k^{12} = -[A_0^{11} - A_0^{12}(A_0^{22})^{-1} A_0^{21}]^{-1} [A_k^{12} - A_0^{12}(A_0^{22})^{-1} A_k^{22}]$ (it follows from the theorem on the inverse of a matrix devided into blocks which is applied for $U_k = -A_0^{-1} A_k$), the relation (3.8) is equivalent to (3.4).

Further according to Theorem 2.3 the equality $\Delta_X(1, 1) = \Delta_X(1, 0)$ holds if and only if (3.5) is fulfilled.

To finish the proof assume validity of the relations (3.4)–(3.6) for some arbitrary fixed $s \geqq 0$ and they are supposed to be equivalent to the equality $\Delta_X = \Delta_X(1, -s)$. Under these assumptions it will be shown that $\Delta_X(1, -s) = \Delta_X(1, -(s+1))$ if and only if $U_{s+1}^{21} = 0$. According to Theorem 2.1 and Remark 2.2 we can write

$$\hat{X}_t(1, -(s+1)) = \hat{X}_t(1, -s) + y , \tag{3.9}$$

where

$$y = \operatorname{cov}[X_t, Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, 1)] \,. \tag{3.10}$$
$$. \{\operatorname{var}[Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, 1)]\}^{-1} [Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, 1)]$$

and $\hat{Y}_t(a, b)$ denotes the extrapolation of Y_t based on $X_{t-a}, X_{t-a-1}, \ldots, Y_{t-b}, Y_{t-b-1}, \ldots$. Analogously, using successively Theorem 2.1 we get

$$Y_{t+s+1} - Y_{t+s+1}(s+2, 1) = \tag{3.11}$$
$$= Y_{t+s+1} - Y_{t+s+1}(s+2, s+2) - y_1 - \ldots - y_{s+1} ,$$

where for $k = 1, \ldots, s+1$

$$y_k = \operatorname{cov}[Y_{t+s+1}, Y_{t+s+1-k} - \hat{Y}_{t+s+1-k}(s+2-k, 1)] \,. \tag{3.12}$$
$$. \{\operatorname{var}[Y_{t+s+1-k} - \hat{Y}_{t+s+1-k}(s+2-k, 1)]\}^{-1} \,.$$
$$. [Y_{t+s+1-k} - \hat{Y}_{t+s+1-k}(s+2-k, 1)] \,.$$

The assumption $\Delta_X = \Delta_X(1, -s)$ implies according to Remarks 2.1 and 2.2 that $\operatorname{cov}[X_t, Y_{t+k} - \hat{Y}_{t+k}(k+1, 1)] = 0$ for $k = 1, \ldots, s$. Hence substituting (3.11) and

(3.12) to (3.10) we get after some calculation that $\Delta_X(1, -s) = \Delta_X(1, -(s+1))$ if and only if

$$(3.13) \qquad \operatorname{cov}[X_t, Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, 1)] =$$

$$= \operatorname{cov}[X_t - \hat{X}_t(1, 1), Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, s+2)] -$$

$$- \operatorname{cov}[X_t - \hat{X}_t(1, 1), Y_t - \hat{Y}_t(1, 1)] \{\operatorname{var}[Y_t - \hat{Y}_t(1, 1)]\}^{-1} .$$

$$. \operatorname{cov}[Y_{t+s+1}, Y_t - \hat{Y}_t(1, 1)]' = 0 .$$

It can be easily derived for the model (3.1) that

$$X_t - \hat{X}_t(1, 1) = M^{11}Z_t^1 + M^{12}Z_t^2 , \quad Y_t - \hat{Y}_t(1, 1) = M^{21}Z_t^1 + M^{22}Z_t^2 ,$$

so that according to (3.13) and the assumption (3.5) the equality $\Delta_X(1, -s) = \Delta_X(1, -(s+1))$ holds if and only if

$$(3.14) \qquad \operatorname{cov}[M^{11}Z_t^1 + M^{12}Z_t^2, Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, s+2)] = 0 .$$

Further it follows from (3.7) that

$$(3.15) \qquad W_t = MZ_t + \sum_{v=1}^{\infty} \sum_{u=1}^{v} \sum_{i_1+\ldots+i_u=v} U_{i_1} \ldots U_{i_u} MZ_{t-v}$$

and hence

$$W_{t+s+1} - \hat{W}_{t+s+1}(s+2, s+2) = MZ_{t+s+1} +$$

$$+ \sum_{v=1}^{s+1} \sum_{u=1}^{v} \sum_{i_1+\ldots+i_u=v} U_{i_1} \ldots U_{i_u} MZ_{t+s+1-v} .$$

Therefore for the process $\{Y_t\}$ we can write

$$(3.16) \qquad Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, s+2) = f(Z_{t+s+1}, \ldots, Z_{t+1}) +$$

$$+ (\sum_{u=1}^{s+1} \sum_{i_1+\ldots+i_u=s+1} U_{i_1} \ldots U_{i_u})^{21} (M^{11}Z_t^1 + M^{12}Z_t^2) +$$

$$+ (\sum_{u=1}^{s+1} \sum_{i_1+\ldots+i_u=s+1} U_{i_1} \ldots U_{i_u})^{22} (M^{21}Z_t^1 + M^{22}Z_t^2) ,$$

where f is a (vector) linear function. Substituting (3.16) to (3.14) and using (3.5) and (3.6) we have that $\Delta_X(1, -s) = \Delta_X(1, -(s+1))$ if and only if

$$(3.17) \qquad (M^{11}M^{11\prime} + M^{12}M^{12\prime}) \{(\sum_{u=1}^{s+1} \sum_{i_1+\ldots+i_u=s+1} U_{i_1} \ldots U_{i_u})^{21}\}' =$$

$$= (M^{11}M^{11\prime} + M^{12}M^{12\prime}) U_{s+1}^{21\prime} = 0 .$$

Since the matrix $M^{11}M^{11\prime} + M^{12}M^{12\prime}$ is regular it follows from (3.17) finally that $\Delta_X(1, -s) = \Delta_X(1, -(s+1))$ holds if and only if $U_{s+1}^{21} = 0$ and the theorem is proved. □

The following assertion follows easily from Theorem 3.1:

Corollary 3.2. In the model from Theorem 3.1 the following implication holds:

$$[\Delta_X = \Delta_X(1, -n)] \Rightarrow [\Delta_X = \Delta_X(1, b) \text{ for all integers } b]\,. \tag{3.18}$$

Remark 3.3. Corollary 3.2 has an important practical meaning: provided the process $\{Y_t\}$ observed till the moment $t + n$ does not improve the extrapolation $\hat{X}_t$ it is useless for this purpose to consider the process $\{Y_t\}$ at all.

Now the problem of the uncorrelated processes $\{X_t\}$ and $\{Y_t\}$ can be solved as follows:

Theorem 3.3. Let the assumptions (3.2) and (3.3) hold in the model (3.1). Then the equality $\Delta_X = \Delta_X(1, -n)$ holds if and only if the processes $\{X_t\}$ and $\{Y_t\}$ are uncorrelated, i.e. $\operatorname{cov}(X_s, Y_t) = 0$ for all integers s, t.

Proof. If the processes $\{X_t\}$ and $\{Y_t\}$ are uncorrelated then the equality $\Delta_X = \Delta_X(1, -n)$ is obvious.

Therefore assume that $\Delta_X = \Delta_X(1, -n)$. According to Theorem 3.1 this equality implies relations (3.4), (3.5) and moreover

$$U_k^{21} = 0\,, \quad k = 1, \ldots, n\,. \tag{3.19}$$

Denote $R(s) = \mathsf{E} W_{t+s} W_t'$ the covariance function of the process $\{W_t\}$. Then using (3.15) we get for all integers s:

$$(R(s))^{12} = 0\,, \quad (R(s))^{21} = 0\,, \tag{3.20}$$

because the product of block-diagonal matrices is again a block-diagonal matrix. In other words, the processes $\{X_t\}$ and $\{Y_t\}$ are uncorrelated. □

4. MODEL MA(m)

In this section we shall investigate a r-dimensional process MA(m) defined by

$$W_t = \sum_{j=0}^{m} B_j Z_{t-j}\,. \tag{4.1}$$

Assume that

$$\det\Big(\sum_{j=0}^{m} B_j z^j\Big) \neq 0 \quad \text{for} \quad |z| \leq 1\,, \tag{4.2}$$

$$\det B_0^{11} \neq 0\,, \quad \det B_0^{22} \neq 0\,. \tag{4.3}$$

Remark 4.1. According to the assumption (4.2) the model (4.1) is invertible. This assumption also guarantees regularity of the matrix B_0.

The following assertion is analogous to Theorem 3.1:

Theorem 4.1. Let $s \geqq 0$ be a given integer and let the assumptions (4.2) and (4.3) be fulfilled in the model (4.1). Then the equality $\Delta_X = \Delta_X(1, -s)$ holds if and only if the following conditions are fulfilled simultaneously:

$$B_j^{12} - B_j^{11}(B_0^{11})^{-1} B_0^{12} = 0 \,, \quad j = 1, \ldots, m \,, \tag{4.4}$$

$$B_0^{11} B_j^{21\prime} + B_0^{12} B_j^{22\prime} = 0 \,, \qquad j = 0, 1, \ldots, s \tag{4.5}$$

(we put $B_j = 0$ for $j > m$).

Proof. The method of the proof will be similar to that of Theorem 3.1. The proof will be again carried out by induction. For $s = 0$ it will be shown that $\Delta_X = \Delta_X(1, 1) = \Delta_X(1, 0)$ if and only if (4.4) and

$$B_0^{11} B_0^{21\prime} + B_0^{12} B_0^{22\prime} = 0 \tag{4.6}$$

hold simultaneously.

Similarly as in the proof of Theorem 3.1 the equality $\Delta_X = \Delta_X(1, 1)$ holds if and only if $p \times p$ matrices $D_0 = I, D_1, \ldots, D_m$ exist such that

$$\sum_{j=0}^{m} B_j^{11} z^j = \left(\sum_{j=0}^{m} D_j z^j\right) G_0 \,, \tag{4.7}$$

$$\sum_{j=0}^{m} B_j^{12} z^j = \left(\sum_{j=0}^{m} D_j z^j\right) H_0 \,, \tag{4.8}$$

where G_0 and H_0 are defined in (2.5) and (2.6) (we took advantage of the fact that $K(z) = I$, $L(z) = 0$, $M(z) = 0$, $N(z) = I$, $v(z) = 1$ for the model (4.1)). Obviously, the conditions (4.7) and (4.8) are equivalent to

$$B_j^{11} = D_j B_0^{11} \,, \quad B_j^{12} = D_j B_0^{12} \,, \quad j = 1, \ldots, m \,, \tag{4.9}$$

which are further equivalent to (4.4) when regularity of the matrix B_0^{11} is used.

The equivalence of the equality $\Delta_X(1, 1) = \Delta_X(1, 0)$ with (4.6) follows from Theorem 2.3 immediately.

Assume now that conditions (4.4) and (4.5) are fulfilled for some $s \geqq 0$ and they are supposed to be equivalent to the equality $\Delta_X = \Delta_X(1, -s)$. Then we shall prove that $\Delta_X(1, -s) = \Delta_X(1, -(s + 1))$ if and only if

$$B_0^{11} B_{s+1}^{21\prime} + B_0^{12} B_{s+1}^{22\prime} = 0 \,. \tag{4.10}$$

According to the proof of Theorem 3.1 the equality $\Delta_X(1, -s) = \Delta_X(1, -(s + 1))$ is fulfilled if and only if (3.13) holds (this part of the proof of Theorem 3.1 is applicable without any changes also to the process MA(m)). Since the model (4.1) is invertible it can be easily derived that

$$X_t - \hat{X}_t(1, 1) = B_0^{11} Z_t^1 + B_0^{12} Z_t^2 \,, \tag{4.11}$$

$$Y_t - \hat{Y}_t(1, 1) = B_0^{21} Z_t^1 + B_0^{22} Z_t^2 \,, \tag{4.12}$$

$$(4.13) \qquad Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, s+2) = B_0^{21} Z_{t+s+1}^1 + B_0^{22} Z_{t+s+1}^2 + \dots$$
$$\dots + B_{s+1}^{21} Z_t^1 + B_{s+1}^{22} Z_t^2 .$$

Substituting (4.11)–(4.13) to (3.13) and using (4.6) we get

$$\operatorname{cov}[X_t, Y_{t+s+1} - \hat{Y}_{t+s+1}(s+2, 1)] =$$
$$= \operatorname{cov}[B_0^{11} Z_t^1 + B_0^{12} Z_t^2, B_0^{21} Z_{t+s+1}^1 + B_0^{22} Z_{t+s+1}^2 + \dots + B_{s+1}^{21} Z_t^1 + B_{s+1}^{22} Z_t^2] -$$
$$- (B_0^{11} B_0^{21\prime} + B_0^{12} B_0^{22\prime}) \{\operatorname{var}[Y_t - \hat{Y}_t(1, 1)]\}^{-1} .$$
$$. \{\operatorname{cov}[Y_{t+s+1}, Y_t - \hat{Y}_t(1, 1)]\}' = B_0^{11} B_{s+1}^{21\prime} + B_0^{12} B_{s+1}^{22\prime} .$$

Hence $\Delta_X(1, -s) = \Delta_X(1, -(s+1))$ if and only if (4.10) holds so that the proof is finished. □

Corollary 4.2. For the model from Theorem 4.1 the following implication holds:

$$(4.14) \qquad [\Delta_X = \Delta_X(1, -m)] \Rightarrow [\Delta_X = \Delta_X(1, b) \text{ for all integers } b] .$$

Theorem 4.3. Let the assumptions (4.2) and (4.3) hold in the model (4.1). Then the equality $\Delta_X = \Delta_X(1, -n)$ holds if and only if the processes $\{X_t\}$ and $\{Y_t\}$ are uncorrelated.

Proof. It is again sufficient to assume that $\Delta_X = \Delta_X(1, -m)$ is fulfilled because the opposite implication is obvious. Therefore according to Theorem 4.1 the conditions

$$(4.15) \qquad B_j^{12} - B_j^{11} (B_0^{11})^{-1} B_0^{12} = 0 , \quad j = 0, 1, \dots ,$$

$$(4.16) \qquad B_0^{11} B_j^{21\prime} + B_0^{12} B_j^{22\prime} = 0 , \qquad j = 0, 1, \dots$$

hold. Obviously the covariance function $R(s)$ has the following form in the model (4.1):

$$(4.17) \qquad R(s) = B_s B_0' + B_{s+1} B_1' + \dots , \quad s = 0, 1, \dots .$$

The proof will be completed if we show that

$$(4.18) \qquad B_{s+i}^{11} B_i^{21\prime} + B_{s+i}^{12} B_i^{22\prime} = 0 ,$$

$$(4.19) \qquad B_{s+i}^{21} B_i^{11\prime} + B_{s+i}^{22} B_i^{12\prime} = 0$$

for all nonnegative integers i and s. To verify (4.18) write

$$B_{s+i}^{11} B_i^{21\prime} + B_{s+i}^{12} B_i^{22\prime} = B_{s+i}^{11} B_i^{21\prime} + B_{s+i}^{11} (B_0^{11})^{-1} B_0^{12} B_i^{22\prime} =$$
$$= B_{s+i}^{11} (B_0^{11})^{-1} (B_0^{11} B_i^{21\prime} + B_0^{12} B_i^{22\prime}) = 0$$

where we used (4.15) for $j = s + i$ and (4.16) for $j = i$. The verification of (4.19) is analogous. □

5. CONCLUSION

The problem of improvement of extrapolation in the models AR(n) and MA(m) was solved in this paper. As far as the model ARMA(m, n) is concerned the author of this paper has dealt also with this case. The assertions analogous to Theorems 3.1, 4.1, 3.3 and 4.3 have been proved for this model but they are much more complicated and demand further auxiliary assertions of the matrix calculus. These assertions and their proofs are given in [5]. Moreover, there is not such a straightforward solution of the problem of the uncorrelated processes in the model ARMA(m, n) as in the model AR(n) and MA(m).

Another problem consists in improvement of the extrapolation $\hat{X}_t(a)$ for $a > 1$ (i.e. the extrapolation of X_t based on $X_{t-a}, X_{t-a-1}, \ldots$). The author has solved the problem of equality $\Delta_X(a) = \Delta_X(a, a)$ in the model ARMA(m, n) but it is only a small part of the whole problem.

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REFERENCES

[1] J. Anděl: Measures of dependence in discrete stationary processes. Math. Operationsforsch. Statist., Ser. Statistics *10* (1979), 107–126.

[2] J. Anděl: On extrapolation in two-dimensional stationary processes. Math. Operationsforsch. Statist., Ser. Statistics *11* (1980), 315–326.

[3] J. Anděl: Some Measures of Dependence in Discrete Stationary Processes (in Czech). Doctoral dissertation, Dept. of Statistics, Charles University, Prague, 1980.

[4] T. W. Anderson: Repeated measurements on autoregressive processes. JASA *73* (1978), 371–378.

[5] T. Cipra: Correlation and Improvement of Prediction in Multivariate Stationary Processes (in Czech). Ph. D. dissertation, Dept. of Statistics, Charles University, Prague, 1980.

[6] C. W. J. Granger: Investigating causal relations by econometric models and cross spectral methods. Econometrica *37* (1969), 424–438.

[7] D. A. Pierce, L. D. Haugh: Causality in temporal systems. Journal of Econometrics *5* (1977), 265–293.

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