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Kybernetika, Vol. 14 (1978), No. 3, (182)--188

Persistent URL: http://dml.cz/dmlcz/125409

KYBERNETIKA — VOLUME 14 (1978), NUMBER 3

On the Numerical Solution of Optimal Control Problems with Constraints

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The possibility of a direct treatment of control constraints using the projection technique in the connection with the so-called sequential gradient-restoration algorithm is investigated. As the result, a modified numerical algorithm is proposed for the solution of a class of optimal control problems with control constraints.

1. INTRODUCTION

The sequential gradient-restoration algorithm (SGRA) was proposed by Miele et al. [1, 2]. For the case of control constraints this approach can be described as an indirect one, because the existing constraints are treated using Lagrange multipliers. On the other hand, in a large number of cases it is possible to treat the control constraints directly applying the idea of gradient projection (clipping-off technique), e.g., see [3]. Here an attempt is made to combine the projection technique with the original SGRA. The resulting modified SGRA for optimal control problems with control constraints includes this projection and thus enables a direct treatment of these problems.

2. OPTIMAL CONTROL PROBLEM

In general, the notation of Miele et al. [1, 2] will be used, i.e., all vectors (also the gradients of various functions) are supposed to be column vectors. Further, all functions are supposed to be continuously differentiable.

The aim is to minimize the functional

$$I = \int_0^1 f(x, u, \pi, t)\, \mathrm{d}t + [g(x, \pi)]_1 \tag{1}$$

with respect to the functions $x(t)$, $u(t)$ and the parameter π which satisfy the differential constraint

$$\dot{x} - \varphi(x, u, \pi, t) = 0 , \tag{2}$$

the boundary conditions

$$(x)_0 = \text{given} . \qquad [\psi(x, \pi)]_1 = 0 , \tag{3}$$

the control constraint

$$u(t) \in U(t) , \quad t \in [0, 1] \tag{4}$$

and the parameter constraint

$$\pi \in \Pi . \tag{5}$$

Here $x(t) \in R^n$ denotes the state and $u(t) \in R^m$ the control variable at the time t and $\pi \in R^p$ the parameter. Let $U(t) \subset R^m$ denote the so-called set of admissible controls at time t and analogously let $\Pi \subset R^p$ denote the set of admissible system parameters. The various functions are defined as follows

$$\begin{aligned} f &: R^n \times R^m \times R^p \times R^1 \to R^1 , \quad g : R^n \times R^p \to R^1 , \\ \varphi &: R^n \times R^m \times R^p \times R^1 \to R^n , \quad \psi : R^n \times R^p \to R^q . \end{aligned} \tag{6}$$

Observe that this formulation includes also problems with free final time, see [1, 2].

3. GRADIENT-RESTORATION ALGORITHM

This algorithm consists of the alternate succession of gradient and restoration phases, which always determine the current changes $\Delta\, x(t)$, $\Delta\, u(t)$, Δt of the existing estimates $x(t)$, $u(t)$, π so that these values approach the desired optimum. For further details see [1, 2]. Each phase represents the solution of the linear two-point boundary-value problem

$$\begin{aligned} \dot{A} - \varphi_x^T A - \varphi_u^T B - \varphi_\pi^T C + \varrho(\dot{x} - \varphi) &= 0 , \\ (A)_0 &= 0 , \\ (\psi_x^T A + \psi_\pi^T C + \varrho\psi)_1 &= 0 , \\ \dot{\lambda} - \sigma f_x + \varphi_x \lambda &= 0 , \\ (\lambda + \sigma g_x + \psi_x \mu)_1 &= 0 , \\ B + \sigma f_u - \varphi_u \lambda &= 0 , \\ C + \int_0^1 (\sigma f_\pi - \varphi_\pi \lambda)\, \mathrm{d}t + (\sigma g_\pi + \psi_\pi \mu)_1 &= 0 , \end{aligned} \tag{7}$$

where $A(t)$, $B(t)$, C are the normalized changes (for the stepsize $\alpha = 1$) of $\Delta x(t)$, $\Delta u(t)$, $\Delta\pi$, respectively, $\lambda(t) \in R^n$ and $\mu \in R^q$ are the multipliers, and ϱ and σ characterize the gradient ($\varrho = 0$, $\sigma = 1$) phases and the restoration ($\varrho = 1$, $\sigma = 0$) phases, respectively. Let us note that the first three equations in (7) represent the linearized system description (2), (3), while the remaining ones are the necessary optimality conditions.

To evaluate various phases the following functionals are introduced ($N(a) = a^T a$)

$$P = \int_0^1 N(\dot{x} - \varphi)\,\mathrm{d}t + N(\psi)_1\,, \tag{8}$$

$$J = \int_0^1 [f + \lambda^T(\dot{x} - \varphi)]\,\mathrm{d}t + (g + \mu^T\psi)_1\,. \tag{9}$$

Functional P measures the cumulative error in constraints and J is the so-called augmented functional.

4. CONSTRAINTS TREATMENT

The given constraints (4) and (5) will be included directly using the projection technique. It means that the problem (1)–(5) is being solved, in principle, as being unconstrained with respect to $u(t)$ and π. The new estimates of $u(t)$, π are always tested whether or not (4) and (5) are met. If not, those parts of $u(t)$ and π, which are overlapping $U(t)$ and Π, are simply neglected. The efficiency of such an approach strongly depends on the concrete form of $U(t)$ and Π. However, there exists a number of practically important cases, where these sets are given as parallelepipeds, balls, spheres, etc., when the required projection is easily performed.

More exactly, for a given $u(t)$ its projection on $U(t)$ is defined as

$$\begin{aligned} u^*(t) &= \operatorname{proj}\{u(t) \mid U(t)\} = \\ &= \{v \in U(t) \mid \|u(t) - v\| \leqq \|u(t) - \bar{v}\|,\ \bar{v} \in U(t)\}\,, \end{aligned} \tag{10}$$

where $\|\cdot\|$ is the Euclidean norm. Otherwise speaking, $u^*(t)$ is the "nearest" point of the set $U(t)$ to $u(t)$. To simplify some further considerations we assume that such a unique $u^*(t)$ always exists. Similar considerations hold also for Π.

5. MODIFIED ALGORITHM

Here the subscripts G and R will be used to distinguish gradient and restoration phases in the following algorithm.

STEP 1. Select the initial solution estimates $x(t)$, $u(t)$, π and evaluate P according to (8). If $P \leqq \varepsilon_P$ (ε_P is the permitted constraints error), then continue with Step 2; else go to Step 4.

STEP 2. Set $\varrho = 0$, $\sigma = 1$ for the gradient phase and solve (7). Using the resulting $A_G(t)$, $B_G(t)$, C_G, $\lambda_G(t)$, μ_G form the one-parameter family of new solution estimates

$$x(t) + \alpha_G A_G(t) , \quad u(t) + \alpha_G B_G(t) , \quad \pi + \alpha_G C_G$$

and the functional $J(\alpha_G)$ and $P(\alpha_G)$. The gradient stepsize α_G is to be chosen such that $J(\alpha_G) < J(0)$ to guarantee the convergence, and $P(\alpha_G) \leqq \varepsilon_G$ to limit the excessive constraints violation during the gradient phase. Satisfactory α_G^* is obtained by employing a bisection process starting from a suitably selected reference stepsize $\bar{\alpha}_G$ (determined by quadratic or cubic interpolation of $J(\alpha_G)$).

STEP 3. Compute the feasible control and parameter changes during the gradient phase

$$\Delta u^*(t) = \operatorname{proj} \{u(t) + \alpha_G^* B_G(t) \mid U(t)\} - u(t) ,$$
$$\Delta \pi^* \quad = \operatorname{proj} \{\pi + \alpha_G^* C_G \mid \Pi\} - \pi$$

and the corresponding $\Delta x^*(t)$ from the equation

$$\Delta \dot{x}^* = \varphi_x^T \Delta x^* + \varphi_u^T \Delta u^* + \varphi_\pi^T \Delta \pi^* , \quad (\Delta x^*)_0 = 0 .$$

Then set

$$x(t) \triangleq x(t) + \Delta x^*(t) , \quad u(t) \triangleq u(t) + \Delta u^*(t) , \quad \pi \triangleq \pi + \Delta \pi^*$$

and go to Step 6.

STEP 4. Set $\varrho = 1$, $\sigma = 0$ for the restoration phase and solve (7). Using the resulting $A_R(t)$, $B_R(t)$, C_R, $\lambda_R(t)$, μ_R form the one-parameter family of the corrected solution estimates

$$x(t) + \alpha_R A_R(t) , \quad u(t) + \alpha_R B_R(t) , \quad \pi + \alpha_R C_R$$

and the functional $P(\alpha_R)$. The restoration stepsize α_R is to be chosen such $P(\alpha_R) < < P(0)$, to reduce the constraints violation. Satisfactory α_R^* is obtained by emploing a bisection process starting from the reference stepsize $\bar{\alpha}_R = 1$.

STEP 5. Compute the feasible control and parameter changes during the restoration phase

$$\Delta u^*(t) = \operatorname{proj} \{u(t) + \alpha_R^* B_R(t) \mid U(t)\} - u(t) ,$$
$$\Delta \pi^* \quad = \operatorname{proj} \{\pi + \alpha_R^* C_R^* \mid \Pi\} - \pi$$

and the corresponding $\Delta x^*(t)$ from the equation

$$\Delta \dot{x}^* = \varphi_x^T \Delta x^* + \varphi_u^T \Delta u^* + \varphi_\pi^T \Delta \pi^* , \quad (\Delta x^*)_0 = 0 .$$

Then set

$$x(t) \triangleq x(t) + \Delta x^*(t)\,, \quad u(t) \triangleq u(t) + \Delta u^*(t)\,, \quad \pi \triangleq \pi + \Delta\pi^*$$

and go to Step 6.

STEP 6. Evaluate P according to (8). If $P \leqq \varepsilon_P$, then continue with Step 7; else go to Step 4.

STEP 7. Compute the functional I according to (1) and compare this value with the value $\bar{I}$ stored after completion of the preceding gradient-restoration cycle. If $I < \bar{I}$, then continue with Step 8; else set $\alpha_G^* \triangleq \alpha_G^*/2$ and go to Step 3.

STEP 8. Compute

$$\Delta u(t) = u(t) = \bar{u}(t)\,, \quad \Delta\pi = \pi - \bar{\pi}\,,$$

where $\bar{u}(t)$, $\bar{\pi}$ are stored values after completion of the preceding gradient-restoration cycle. Compute the functional

$$S = \int_0^1 N(\Delta u)\,\mathrm{d}t + N(\Delta\pi)\,.$$

If $S \leqq \varepsilon_S$ (ε_S is the stopping condition), then stop the computations; else go to Step 2 and start the new gradient-restoration cycle.

6. EXAMPLE

Minimize the functional

$$I = \int_0^1 (u_1^2 + u_2^2)\,\mathrm{d}t$$

subject to the differential constraints

$$\dot{x}_1 = x_2^2 + u_1 u_2\,,$$
$$\dot{x}_2 = x_1 + u_2\,,$$

the boundary conditions

$$x_1(0) = x_2(0) = 1\,, \quad x_1(1) = x_2(1) = 0$$

and the control constraints

$$-2 \leqq u_i(t) \leqq 1{\cdot}5\,, \quad i = 1, 2\,, \quad t \in [0, 1]\,.$$

The nominal estimates

$$x_1(t) = x_2(t) = 1 - t\,, \quad u_1(t) = u_2(t) = 0\,, \quad t \in [0, 1]\,,$$

were applied with $\varepsilon_P = 10^{-6}$, $\varepsilon_S = 10^{-4}$ and $\varepsilon_G = 1$.

Table 1. Convergence history of the algorithm for the illustrative example.

N_*	N_R	I	P	S
0	2	3·5286456	$0{\cdot}9 \times 10^{-6}$	$0{\cdot}2 \times 10^{1}$
1	5	2·8043016	$0{\cdot}4 \times 10^{-6}$	$0{\cdot}8 \times 10^{0}$
2	7	2·7640325	$0{\cdot}6 \times 10^{-8}$	$0{\cdot}3 \times 10^{-3}$
3	9	2·7639796	$0{\cdot}8 \times 10^{-6}$	$0{\cdot}5 \times 10^{-4}$

The described algorithm was implemented on the IBM 370/135 computer in double-precision arithmetic. The integration was performed using the modified Euler method and the interval of integration was divided into 50 steps. For the solution of (7) the method of particular solutions [1, 2] was used. The definite integrals were

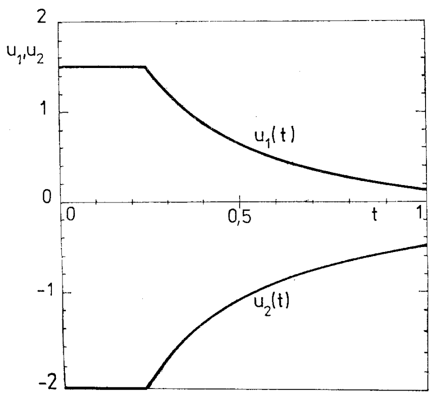

Fig. 1. Control histories for the illustrative example.

computed using Simpson's rule. Algorithm converged in overall 3 gradient-restoration cycles. More details can be found in Table 1, where N_* denotes the current cycle and N_R is number of restoration phases per cycle. Other symbols were introduced earlier. Quadratic interpolation was always applied to determine $\tilde{\alpha}_G$. One bisection occured during the first restoration phase in the last two cycles. The obtained time histories for both control variables are depicted in Fig. 1.

7. CONCLUDING REMARKS

The described algorithm provides an alternative treatment of control constraints using the original sequential gradient-restoration algorithm [1]. It enables the exact satisfaction of the given constraints, which is not necessarily true when applying the indirect approach [2]. The more detailed description of the suggested algorithm will be published elsewhere [4].

(Received November 3, 1977.)

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