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A Stochastic Approach to Robot Plan Formation

IVAN M. HAVEL, IVAN KRAMOSIL

A new approach to robot plan formation is suggested and discussed, based on probabilistic ideas. The concept of incidental phenomena from [18] is extended into a randomized form and then it is used as a background for exploring approximate plans (plans with a certain degree of execution reliability and of fitness to a given task). This makes it possible to develop plans with loops, which would otherwise require infinitistic tools.

1. INTRODUCTION

Complexity, in all its various appearances, is one of the nightmares of workers in artificial intelligence and cognitive robotics. It is a common belief that heuristic approaches, combined with strong constraints on the relevant problem domain, are the only available efficient tools for developing systems of intelligent performance. The aim of the present paper is to develop and discuss a novel approach for the area of robot plan formation, based on probabilistic ideas, which is a potential alternative to the heuristic approach. In addition, this technique enables us to introduce certain constructs (loops in plans) that are hard to handle by conventional tools. It must be confessed that the paper contains more definitions than theorems and is more of a contemplative, than of a persuasive character.

Interestingly enough, the previous work in robot plan formation only seldom utilizes probabilistic and decision-theoretic means. Of course, many authors are aware of the possibility but they do not explore it in detail (cf., e.g., [14], [4], [7], [16]). Robot planning is viewed as a decision-making activity in papers [9], [3], [1], [5]. A rather natural idea of branching plans with probabilistically evaluated branches was suggested in [4] and [15], and led to the study of truncated plans in [11] and [12]. In [10] an emphasis was placed on the random character of environment. The advantages of unprecise plans are pointed out in [6]. (Most of the papers men-

144 tioned above are either suggestive or experimentally oriented, e.g., [1]). The past approaches as well as some future possibilities of stochastic robot decision making are outlined in [8].

The approach suggested in the present paper is characterized by its attention to one aspect of robot planning: on the basis of randomized representation of the changes of environment induced by the robot's actions the planning subsystem of the robot generates approximate plans with a controlled level of reliability as well as of suitability (fitness for the original problem). Our formal framework is logically oriented. It is built on the concept of an image space with branching plans [17] extended by the idea of incidental phenomena [18], the relevant formalism is surveyed in Section 2. In Section 3 the idea of incidental phenomena is extended into a randomized form and the first of our variants of stochastic plans, the so called (ϵ, δ) -stochastic plan, is defined. Motivation and formalism for incorporating (potentially infinite) loops into plans and representing them by a special symbol (Kleene star) is the topic of Section 4, where the main concepts of a stochastic star plan and of an ϵ -stochastic star plan are introduced. These concepts enable to work with plans similar as in [17] but infinite due to unbounded loops and thus intractable in the framework of classical logic. In Section 5 two performance measures (the reliability and the suitability) are investigated asymptotically, in their dependence on the number of times a loop is executed. In the last Section 6 some ideas for general algorithmic means for producing cycles are presented.

The new concepts in this paper are mostly introduced in a flexible way covering a variety of possible alternatives. Great care is given to elucidating all notions by informal discussion and by including illustrative examples.

2. IMAGE SPACE WITH INCIDENTAL PHENOMENA: THE DETERMINISTIC CASE

In this section we briefly review the basic concepts of the logical theory of robot problem solving in its original deterministic setting. Our presentation is self-contained but a reader familiar with the approach of [17] and [18] might find it much easier to see the proper intuitive background behind our formalism. We start with some preliminaries.

Since we shall very frequently work with finite sequences of elements of a given set it will be advantageous to adopt certain notational conventions common in formal language theory. Let Ξ be a set of abstract symbols. A finite sequence (x_1, x_2, \dots, x_n) of elements of Ξ will be written as a *string*, $x_1x_2\dots x_n$; the set of all such strings, including the *empty string* Λ (to allow for $n = 0$), is denoted by Ξ^* . Note that $\Xi \subseteq \Xi^*$. By *concatenating* two strings $u \in \Xi^*$ and $v \in \Xi^*$ we obtain another string

$w = uv \in \Xi^*$; u is then called a *prefix* of w . For a set $L \subseteq \Xi^*$ we define

$$\text{Pref } L = \{u \in \Xi^* \mid uv \in L \text{ for some } v \in \Xi^*\}$$

(the set of all prefixes of strings in L) and

$$F_u(L) = \{x \in \Xi \mid ux \in \text{Pref } L\}$$

(the set of symbols which follow u in L). The string $uu\dots u$ (repeated n -times, $n \geq 0$) is abbreviated by u^n ; in particular, $u^0 = \Lambda$. We shall find very useful the Kleene star operation:

$$u^* = \{u^n \mid 0 \leq n < \infty\}.$$

There are two general and theoretically elaborated frameworks for robot problem solving: the image space and the situation calculus. Here we shall deal mostly with the image space (the situation calculus will be briefly mentioned in Section 4). An *image space* I is a collection of formal theories (axiomatic systems) of the first-order predicate calculus and a collection of operators transforming one theory into another. The theories are called *images* and share a common language \mathcal{L}_I (in the sequel we denote by \mathcal{L}_I the set of all well-formed formulas of the language, in particular, *true*, *false* $\in \mathcal{L}_I$) and a common subset of axioms, T_I , called the *core theory* of I . The core theory represents permanent truths about the world. Each image can be viewed as an extension $T_I[A]$ of T_I by a "specific" axiom $A \in \mathcal{L}_I$ and represents the known facts about a particular instantaneous state of the world.

The collection of *operators* is denoted by Σ (unlike in [17] and [18] we do not work here with operator schemata). Each operator $\varphi \in \Sigma$ is specified by a pair of formulas $\langle C_\varphi, R_\varphi \rangle \in \mathcal{L}_I^2$ (the *condition* and the *result* of φ , respectively). An operator φ is *applicable* in $T_I[A]$ iff $T_I[A] \vdash C_\varphi$ (i.e., C_φ is provable in $T_I[A]$), the outcome of such application is the image $T_I[R_\varphi]$. Operators represent concrete (physical) actions of the robot in the environment.

A *problem* in I is a pair of variable-free formulas $\langle X, Y \rangle \in \mathcal{L}_I^2$ (the *initial* and the *goal* formula, respectively). A *solution* to such problem has, in the simplest case, the form of a single operator sequence $\varphi_1\varphi_2\dots\varphi_n \in \Sigma^*$ (elements of Σ^* are called *operator sequences*) such that

$$T_I[X] \vdash C_{\varphi_1},$$

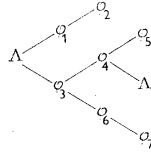
$$T_I[R_{\varphi_i}] \vdash C_{\varphi_{i+1}} \quad (1 \leq i < n)$$

and

$$T_I[R_{\varphi_n}] \vdash Y;$$

($n = 0$ only if $T_I[X] \vdash Y$). Such an operator sequence is called a *straight-line plan* for $\langle X, Y \rangle$. A more general solution may obtain the form of a *branching plan*, which is a finite set K of operator sequences, $K \subseteq \Sigma^*$. It is convenient to imagine K as a tree obtained by joining common prefixes of sequences in K . For instance, the

146 set $K = \{\varphi_1\varphi_2, \varphi_3\varphi_4, \varphi_3\varphi_4\varphi_5, \varphi_3\varphi_6\varphi_7\}$ can be represented by the tree



Correspondingly, any element $a \in K$ will be also called a *branch* in K (a path from the root to a terminal node). Any branch in a plan intuitively represents an *execution sequence* of the plan and a branching point occurs where the decision about the next operator to be executed is not made in the planning stage (it is postponed to the execution stage). For instance, in (1), we might have

$$T_1[R_{\varphi_3}] \vdash C_{\varphi_4} \vee C_{\varphi_6}, \quad T_1[R_{\varphi_4}] \vdash C_{\varphi_5} \vee Y.$$

Whenever in the sequel we use the phrase “execution of an operator” we do not mean its application in the image space (yielding a new image) but rather the execution (in real world) of a physical action represented by the operator.

The idea of incidental phenomena introduced in [18] enriches, in a certain way, the mechanism of plan formation. A pair of formulas $\langle A, B \rangle \in \mathcal{L}_1^2$ is an *incidental phenomenon* of φ if the validity of A (in addition to C_φ) before the application of φ assures the validity of B after the application. The difference between an incidental phenomenon $\langle A, B \rangle$ and the condition-result pair $\langle C_\varphi, R_\varphi \rangle$ is more practical than theoretical. While $\langle C_\varphi, R_\varphi \rangle$ represents a change of the world substantial for φ (in fact it defines φ), an incidental phenomenon $\langle A, B \rangle$ represents a side-effect of φ or, if $A = B$, an unchanged fact: part of the “frame” of φ which may be paid attention to in one application of φ , but neglected in another application. During the plan formation the incidental phenomena play a role of “catalysts” and do not appear in the obtained plan.

To associate incidental phenomena with the image space I in a formal way we first associate with every operator $\varphi \in \Sigma$ a set $\text{Inc}_\varphi \subseteq \mathcal{L}_1^2$ of all incidental phenomena of φ . (In practical cases the set Inc_φ may be infinite but easy to generate or recognize.) A natural requirement is that $\langle \text{true}, \text{true} \rangle \in \text{Inc}_\varphi$ and that $\langle A, B \rangle \in \text{Inc}_\varphi$ together with $\langle A', B' \rangle \in \text{Inc}_\varphi$ implies $\langle A \& A', B \& B' \rangle \in \text{Inc}_\varphi$. Denote by Inc_1 the system of all sets Inc_φ , $\varphi \in \Sigma$. The basic elements of planning in the image space with incidental phenomena are now triples $\langle \varphi, A, B \rangle$ where $\varphi \in \Sigma$ and $\langle A, B \rangle \in \text{Inc}_\varphi$. We call such triples *transitions* and denote the set of them by Σ_t , $\Sigma_t \subseteq \Sigma \times \mathcal{L}_1^2$. Strings in Σ_t^* are called *transition sequences*. For a notational convenience a transition $\tau \in \Sigma_t$ will be also written in the form $\tau = \langle \text{op}(\tau), A_\tau, B_\tau \rangle$; here ‘op’ is treated as a mapping $\text{op} : \Sigma_t \rightarrow \Sigma$ which is, in a natural way, extended to a length preserving

string homomorphism $\text{op} : \Sigma_t^* \rightarrow \Sigma^*$. Furthermore, for $\Gamma \subseteq \Sigma_t^*$ we write $\text{op}(\Gamma) = \{\text{op}(\alpha) \mid \alpha \in \Gamma\}$. (We denote transition sequences by Greek letters α, β, \dots to distinguish them from operator sequences denoted by letters of the Latin alphabet, a, b, \dots).

Now we can give a precise definition of a branching plan in image space with incidental phenomena. In fact, we define simultaneously two notions: a plan and an analyzed plan. The latter notion appears to be very important for our further study.

Definition 1. Let I be an image space with incidental phenomena Inc_I and $\langle X, Y \rangle$ a problem in I . A finite nonempty subset $K \subseteq \Sigma^*$ is a (deterministic) *plan for* $\langle X, Y \rangle$ and a subset $\Gamma \subseteq \Sigma_t^*$ is an *analyzed plan for* $\langle X, Y \rangle$ corresponding to K , iff $K = \text{op}(\Gamma)$ and Γ satisfies:

$$(i) \quad T_i[X] \vdash \bigvee_{\tau \in F_A(\Gamma)} (C_{\text{op}(\tau)} \& A_\tau) \vee Y(\Lambda),$$

and, for any $\alpha = \alpha' \tau' \in \text{Pref } \Gamma$,

$$(ii) \quad T_i[R_{\text{op}(\tau')} \& B_{\tau'}] \vdash \bigvee_{\tau \in F_A(\Gamma)} (C_{\text{op}(\tau)} \& A_\tau) \vee Y(\alpha),$$

where, for $\beta \in \Sigma_t^*$, $Y(\beta)$ is Y if $\beta \in \Gamma$ and *false* otherwise.

The plan K , composed of operator sequences, provides all the necessary information to the execution subsystem of the robot. The analyzed plan Γ , composed of transition sequences, offers some additional information about those assumptions about environment which were chosen (in the form of incidental phenomena) by the planning subsystem. Since we shall be interested in estimating how much an a priori formed plan is prone to error we shall mostly deal with analyzed plans. The adjective 'deterministic' will be later used for plans in the above sense to distinguish them from various types of stochastic plans.

Note that the mapping op as a function $\text{op} : \text{Pref } \Gamma \rightarrow \text{Pref } K$ is not, in general, a bijection. The case when $\text{op}(\alpha) = \text{op}(\beta)$ for some $\alpha, \beta \in \text{Pref}(\Gamma)$, $\alpha \neq \beta$, is called *concealed branching* in [18] and requires a careful treatment.

In the following, whenever convenient, we shall use a special function $f_K : K \rightarrow \Sigma_t^*$, where K is a subset of Σ^* such that for each $a \in K$

$$\text{op}(f_K(a)) = a.$$

This function will be called an *analyzing function* (for K). Since K will be always understood we shall write simply f instead of f_K .

Remark. There is an alternative inductive way of defining (analyzed) plans. One may require that Γ satisfies (i) of Definition 1 and for each $\tau \in F_A(\Gamma)$, the set $\{\alpha \in \Sigma_t^* \mid \tau\alpha \in \Gamma\}$ is an analyzed plan for the problem $\langle R_{\text{op}(\tau)} \& B_\tau, Y \rangle$. This inductive definition is possible due to the finiteness of Γ .

The model based on the notions of a branching plan and incidental phenomena seems to be quite adequate for some problems of artificial intelligence and, using this model, a number of interesting theoretical results can be achieved. Nevertheless, there are at least two open topics, namely dealing with "near misses" and unbounded cycles in plans, whose treatment — and even formulation — are beyond the capabilities of that model. In this and the next sections we shall try to modify our system in order to overcome this limitation; we shall see, moreover, that both topics are in a certain way related.

The set Inc_φ of incidental phenomena associated with an operator (schema) φ can be understood as a representation of possible *deterministic* changes (or non-changes) of the environment, accompanying the operator execution. Thus, if $\langle A, B \rangle \in \text{Inc}_\varphi$ one may say that "if φ is executable and if A holds before its execution then B holds after". However, the changes taking place in the surrounding us world could be treated as deterministic only to a certain degree, depending on the level of the robot's (or designer's) apriori knowledge of its dynamic properties. Thus we should actually work with changes of the form "if φ is executable and if A holds before its execution then it is likely that B holds after". It is just this way of reasoning which enables us to avoid considering various unexpected, very improbable (yet logically not excludable) consequences of action — for the cost, of course, of a positive probability of a failure.

Hence, to introduce randomized incidental phenomena, we define, instead of the set Inc_φ for each operator φ , a function P_φ ascribing to an (arbitrary) pair of formulas a certain real number $P_\varphi(A, B)$ between 0 and 1. Moreover, to account for a possible stochastic dependence of successive operator executions, we extend the definition to sequences of operators. This yields for any operator sequence $a = \varphi_1 \varphi_2 \dots \varphi_n$ and any two formulas A, B a real number, written $P_a(A, B)$ or $P_{\varphi_1 \varphi_2 \dots \varphi_n}(A, B)$, which can be intuitively interpreted as the conditional probability that B holds in a certain state of the world, under the condition that this state was achieved by executing the operator a sequence a (in the proper order) from a state in which A held. We admit also the special case of $a = \Lambda$ (i.e. $n = 0$) in order to express, by means of $P_\Lambda(A, B)$, the conditional probability that B holds in a state under the condition that A holds in the same state.

To formalize this probabilistically we would have first to interpret formulas as random events. A possible approach would be to make formula represent the set of all models in which it is true. This possibility is developed in detail in [13]. To avoid a lengthy presentation we shall base our further discussion on the following axiomatic definition of randomized incidental phenomena.

Definition 2. Let I be an image space with the set of operators Σ . A system \mathbf{P}_I of randomized incidental phenomena (associated with I) is a mapping ascribing

to each $a \in \Sigma^*$ a function $P_a : \mathcal{L}_1^2 \rightarrow \langle 0, 1 \rangle$, called the *stochastic dependence function* such that for any $A, B \in \mathcal{L}_1$, $a \in \Sigma^*$ and $\varphi \in \Sigma$:

- (i) if $T_1[R_\varphi] \vdash B$ then $P_{a\varphi}(A, B) = 1$;
- (ii) if $T_1[A] \vdash B$ then $P_\Lambda(A, B) = 1$;
- (iii) if $T_1[R_\varphi] \vdash \neg(B \& C)$ then $P_{a\varphi}(A, B) + P_{a\varphi}(A, C) \leq 1$;
- (iv) if $T_1[A] \vdash \neg(B \& C)$ then $P_\Lambda(A, B) + P_\Lambda(A, C) \leq 1$.

All the four axioms (i)–(iv) are in agreement with the intuitive interpretation of the stochastic dependence functions mentioned earlier. By axioms (i) and (ii) all provable facts certainly hold; as a special case of (i)

$$(2) \quad P_{a\varphi}(A, R_\varphi) = 1$$

(this reflects the assumption that operators themselves are foolproof, at least from the point of view of the planning system – hardware failures are taken care of by lower level subsystems); of course, by (i) and (ii),

$$(3) \quad P_a(A, \text{true}) = 1$$

for any $a \in \Sigma^*$. By (iii) and (iv) the probabilities of incompatible facts cannot add to a number greater than one. A special case is

$$(4) \quad P_a(A, B) + P_a(A, \neg B) \leq 1$$

for any $a \in \Sigma^*$.

Purposely we have chosen the axioms weaker than one might expect on the basis of the suggested interpretation. In particular, in (4) we do not claim the equality. The reason for this is to allow that functions P_a do not represent the probabilities themselves but only their *lower estimates*. While it is assumed that their values were obtained on the basis of logical interferences within a given image (hence the equalities in (i) and (ii)), it would be too presumptuous to expect the knowledge (whether on the side of the robot or of the designer) of all the hidden laws of changes in the world. Thus the validity of C_φ & A in one state may always imply the validity of B in the new state obtained by φ , yet $P_\varphi(A, B)$ may not equal to one. Accordingly, when talking in the sequel about “probability” we always mean only its lower estimate *accessible to the robot*.

Furthermore, we do not stipulate the converse of (ii), that is that $P_\Lambda(A, B) = 1$ implies $T_1 \vdash A \rightarrow B$; in fact, the possibility of both $P_\Lambda(A, B) = 1$ and $T_1 \nvdash A \rightarrow B$ might enable to consider “multiple-goal plans” within our framework.

We do not suppose, neither that the equality

$$(5) \quad P_a(A \& A', B \& B') = P_a(A, B) \cdot P_a(A', B')$$

holds in general: the difference between the two sides of (5) enables to express statistical dependencies between the incidental phenomena $\langle A, B \rangle$ and $\langle A' B' \rangle$, if they are interpreted as random events.

Let us note that the following implications may be also considered as possible candidates for axioms:

$$(6) \quad T_1[\mathbf{R}_\varphi] \vdash B \rightarrow C \Rightarrow P_{a\varphi}(A, B) \leq P_{a\varphi}(A, C),$$

$$(7) \quad T_1[A] \vdash B \rightarrow C \Rightarrow P_\lambda(A, B) \leq P_\lambda(A, C).$$

It can be easily shown that (3) and (6) imply (i), and (3) and (7) imply (ii).

We can associate, with any system \mathbf{P}_1 , a collection \mathbf{Inc}_1 of sets Inc_φ , $\varphi \in \Sigma$, representing the incidental phenomena in the “deterministic” sense of Section 2:

$$(8) \quad \text{Inc}_\varphi = \{ \langle A, B \rangle \in \mathcal{L}_1^2 \mid P_{a\varphi}(A, B) = 1 \text{ for all } a \in \Sigma^* \}.$$

However, for formal reasons we shall behave as if $\text{Inc}_\varphi = \mathcal{L}_1^2$, that means we shall use further the term *transition* for any triple $\langle \varphi, A, B \rangle$, $\varphi \in \Sigma$, $A, B \in \mathcal{L}_1$ and denote $\Sigma_1 = \Sigma \times \mathcal{L}_1^2$.

Our next objective will be to introduce a measure of “reliability” of transition sequences on the basis of stochastic dependence functions. We shall define three variants of such measure.

Definition 3. Let \mathbf{P}_1 be a system of randomized incidental phenomena and let $\alpha = \langle \varphi_1, A_1, B_1 \rangle \langle \varphi_2, A_2, B_2 \rangle \dots \langle \varphi_n, A_n, B_n \rangle$ be a transition sequence, $n \geq 1$. The *simple reliability* RS, *path-dependent reliability* RP, and *Markov reliability* RM of α with respect to \mathbf{P}_1 are defined, respectively, as follows.

$$\text{RS}(\alpha, \mathbf{P}_1) = \prod_{i=1}^n P_{\varphi_i}(A_i, B_i),$$

$$\text{RP}(\alpha, \mathbf{P}_1) = \prod_{i=1}^n P_{\varphi_1 \dots \varphi_i}(A_i, B_i),$$

$$\text{RM}(\alpha, \mathbf{P}_1) = P_{\varphi_1}(A_1, B_1) \cdot \prod_{i=1}^{n-1} P_{\varphi_i \varphi_{i+1}}(A_i, B_{i+1}).$$

Moreover, we define $\text{RS}(\Lambda, \mathbf{P}_1) = \text{RP}(\Lambda, \mathbf{P}_1) = \text{RM}(\Lambda, \mathbf{P}_1) = 1$. Any of the three functions RS, RP and RM is called a *reliability* and denoted by R.

(Assuming \mathbf{P}_1 fixed in the sequel, we write $R(\alpha)$ instead of $R(\alpha, \mathbf{P}_1)$).

The reliabilities RM and RP enable to express the possible statistical dependence among the applications of various operators and the corresponding incidental phenomena. On the other hand, the simple reliability RS is much close to a practical application because of the fact that the values $P_\varphi(A, B)$, $\varphi \in \Sigma$, which are the only necessary to define RS are more easy to obtain, or at least estimate, on the base of a statistical treatment than the more general values of $P_a(A, B)$, $a \in \Sigma^*$.

The reliability function and the stochastic dependence function are, according to Definition 3, closely interrelated. The former should be understood rather as a measure of “structural correctness” of transition sequences and depends on the incidental phenomena occurring in it; the latter expresses more the dynamical properties of the world with respect to a change indicated by an operator sequence.

The reliability, together with the function P_Λ (the stochastic “distance” of formulas), are two measures which we shall use for evaluating approximate plans. As our first step towards introducing approximate plans we define an auxiliary structure, called ‘quasi-plan’, by abstaining from the requirement that the goal is achieved in a “uniform” way as in the case of deterministic plans of Definition 1, while retaining the inherent branching structure of the plan. (From now to the rest of the paper we keep fixed the underlying image space I , thus giving a permanent meaning to the symbols $\mathcal{L}_I, T_I, \Sigma, \Sigma_I, \mathbf{P}_I, R_I$.)

Definition 4. Let $X \in \mathcal{L}_I$ be a formula, let $\Gamma \subseteq \Sigma_I^*$ be a nonempty (possibly infinite) set of transition sequences, and let $Y: \Sigma_I^* \rightarrow \mathcal{L}_I$ be a function with the property that $Y(\alpha) = \text{false}$ for all $\alpha \in \Sigma_I^* - \Gamma$. We shall call Γ a *quasi-plan for X* , and Y a *goal assignment for Γ* , iff the following two conditions hold:

- (i) $T_I[X] \vdash \bigvee_{\tau \in F_\Lambda(\Gamma)} (C_{\text{op}(\tau)} \ \& \ A_\tau) \vee Y(\Lambda)$;
- (ii) for each $\alpha = \alpha' \tau' \in \text{Pref } \Gamma$,
 $T_I[R_{\text{op}(\tau')} \ \& \ B_{\tau'}] \vdash \bigvee_{\tau \in F_\Sigma(\Gamma)} (C_{\text{op}(\tau)} \ \& \ A_\tau) \vee Y(\alpha)$.

Note that, if Γ is finite, by taking $Y(\beta) = Y$ for each $\beta \in \Gamma$ we obtain, as a particular case, the analyzed deterministic plan for $\langle X, Y \rangle$.

Definition 5. Let $\langle X, Y \rangle$ be a problem in I , let $\varepsilon, \delta \geq 0$ be two real numbers. A finite nonempty set $K \subseteq \Sigma^*$ is an (ε, δ) -stochastic plan for $\langle X, Y \rangle$ and $\Gamma \subseteq \Sigma_I^*$ is an analyzed (ε, δ) -stochastic plan for $\langle X, Y \rangle$ (corresponding to K) iff $K = \text{op}(\Gamma)$ and Γ is a quasi-plan for X with a goal assignment Y such that for any $\alpha \in \Gamma$

- (i) $R(\alpha) \geq 1 - \varepsilon$
- (ii) $P_\Lambda(Y(\alpha), Y) \geq 1 - \delta$.

The intuitive idea behind the concept of an (ε, δ) -stochastic plan can be explained as follows. We start with the analyzed deterministic plan for $\langle X, Y \rangle$ and generalize it by assuming that every branch terminates not necessarily reaching the goal Y , but some another formula $Y(\alpha)$ which is connected to Y only stochastically; the condition (ii) expresses, in a stochastic way, the fact that $Y(\alpha)$ is “close” to Y , or more precisely, that in a world satisfying $Y(\alpha)$ there is a high probability (at least $1 - \delta$) that Y holds, too. The condition (i) expresses the requirement that the reliability of any branch is at least $1 - \varepsilon$.

An example of a possible concrete goal assignment is $Y(\Lambda) = X$ if $\Lambda \in \Gamma$ and $Y(\alpha\tau) = R_{\text{op}(\tau)} \& B_\tau$, if $\alpha\tau \in \Gamma$. The intuition for this choice may be that Γ is a fragment of a (larger) deterministic analyzed plan Γ' ; (i.e. $\Gamma \subseteq \text{Pref } \Gamma'$); the only result known to have been achieved by a truncated branch ending by τ is just $R_{\text{op}(\tau)} \& B_\tau$. Hence, the considered goal assignment makes easier to test such fragments whether they are (ε, δ) -stochastic plans or not.

The following two theorems establish the mutual relationship of (ε, δ) -stochastic plans (with $\varepsilon = \delta = 0$ and considering the simple reliability RS) and deterministic plans from Section 2. Let I be an image space with either Inc_1 (a system of deterministic incidental phenomena) or \mathbf{P}_1 (a system of randomized incidental phenomena). Let $\langle X, Y \rangle$ be a problem in I .

Theorem 1. Assume that, for every $\varphi \in \Sigma$, $\langle A, B \rangle \in \text{Inc}_\varphi$ implies $P_\varphi(A, B) = 1$ and that R is the simple reliability RS. Then every (analyzed) deterministic plan for $\langle X, Y \rangle$ is also (analyzed) $(0, 0)$ -stochastic plan for $\langle X, Y \rangle$.

Proof. Under the conditions of the theorem let K be a deterministic plan, and Γ a corresponding analyzed plan. Let $\alpha \in \Gamma$, then

$$R(\alpha) = \text{RS}(\alpha) = \prod_{i=1}^n P_{\varphi_i}(A_i, B_i),$$

where $\text{op}(\alpha) = \varphi_1 \varphi_2 \dots \varphi_n$, $\langle A_i, B_i \rangle \in \text{Inc}_{\varphi_i}$, $i = 1, 2, \dots, n$. Hence $R(\alpha) = 1$. As any analyzed deterministic plan Γ is also a quasi-plan with respect to the uniform goal assignment $Y(\alpha) = Y$ for all $\alpha \in \Gamma$, $T_1 \vdash Y \rightarrow Y$, hence, $P_\Lambda(Y, Y) = 1$ and thus $K = \text{op}(\Gamma)$ is a $(0, 0)$ -stochastic plan and Γ a corresponding analyzed $(0, 0)$ -stochastic plan. Q.E.D.

Theorem 2. Assume that for every $\varphi \in \Sigma$

$$\text{Inc}_\varphi = \{ \langle A, B \rangle \in \mathcal{L}_1^2 \mid P_\varphi(A, B) = 1 \},$$

moreover, assume that $P_\Lambda(A, B) = 1$ implies $T_1 \vdash A \rightarrow B$ for any $A, B \in \mathcal{L}_1^2$ and that R is RS. Then every (analyzed) $(0, 0)$ -stochastic plan for $\langle X, Y \rangle$ is also (analyzed) deterministic plan for $\langle X, Y \rangle$.

Proof. Under the conditions of the theorem let K be a $(0, 0)$ -stochastic plan and Γ a corresponding analyzed $(0, 0)$ -stochastic plan). Then Γ is a quasi-plan with a goal assignment Y such that $P_\Lambda(Y(\alpha), Y) = 1$ for any $\alpha \in \Gamma$. Thus $T_1 \vdash Y(\alpha) \rightarrow Y$ which enables to replace $Y(\alpha)$ by Y in (ii) of Definition 4. We have

$$R(\alpha) = \text{RS}(\alpha) = \prod_{i=1}^n P_{\varphi_i}(A_i, B_i),$$

where $op(x) = \varphi_1, \dots, \varphi_n$. By the condition $R(x) = 1$ we have $P_{\varphi_i}(A_i, B_i) = 1$, $i = 1, 2, \dots, n$, hence $\langle A_i, B_i \rangle \in Inc_{\varphi_i}$. The definition of a quasi-plan for this case reduces to the definition of an analyzed deterministic plan, which yields the result for Γ and thus also for $K = op(\Gamma)$. Q.E.D.

Let us close this section by mentioning the fact that the model developed here for representing the random character of problem domain is close to the model introduced in [10]. The difference consists in the fact that in [10] the real-time process of building a formal representation is emphasized and that this representation is of a special type.

4. UNBOUNDED PLANS: PLANS WITH LOOPS

Let us start our discussion of the second important modification of branching plans with an illustrative example. It is a real-world situation requiring a plan with iterations resembling WHILE loops in computer programs.

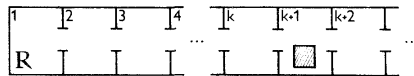


Fig. 1.

Example 1. Consider a long chain of rooms, numbered 1, 2, 3, ... each pair of adjacent rooms connected by a door (cf. Fig. 1). There is a box in one of the rooms, and a robot, initially in room 1, contemplates the problem of how to paint the box.

The robot's abilities consist of two operators

- φ : go to the next room,
- ψ : paint the box.

It is a fact of principal importance for our further reasoning that a solution of this problem cannot be obtained on the basis of classical predicate logic by standard methods of automatic plan formation, *unless* the total number of rooms is specified at the beginning. In the latter case, i.e. if the total number of rooms is known to be n ($n \geq 1$), we can easily imagine a branching plan of the form

$$(9) \quad \{\varphi^k \psi \mid 0 \leq k < n\}$$

obtained, let us say, on the basis of the situation calculus or the image space (cf. [17]). The execution of plan (9) would follow a particular branch $\varphi^k \psi$ iff the box is in the $(k + 1)$ st room (the robot has to enter k empty rooms before finding the box). Thus the plan guarantees achieving the goal for any position of the box.

Intuitively, however, there is no essential difference between the case of an a priori given number of rooms and the case when this number is unknown. Indeed, an infinite branching “plan” of the form

$$(10) \quad \{\varphi^k\psi \mid 0 \leq k < \infty\}$$

seems to be completely plausible for *both* cases. Moreover, it can be conveniently expressed by a single “expression”

$$(11) \quad \varphi^*\psi$$

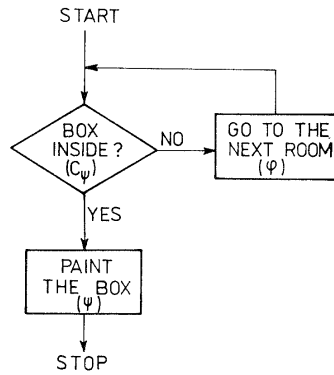


Fig. 2.

with the star indicating arbitrary number of repetitions. Expression (11) corresponds to a flowchart containing a single while loop (cf. Fig. 2). In general, a star expression may contain several stars (just as a flowchart may have several loops). For instance, the above problem extended to any (unknown but finite) number of boxes in any room (with the total number of rooms also unknown but finite) would be solved by the “plan” $(\psi^*\varphi)^*$.

Our aim in the rest of this paper is to formalize such “star plans” and use stochastic way of reasoning as a tool for their approximation and construction.

First we introduce some notation and terminology concerning repetitions in sequences. So far the notation φ^k was just a convenient abbreviation for the sequence $\varphi\varphi \dots \varphi$ (repeated k -times). We now formalize this notation by treating numerical exponents (and parentheses) as abstract symbols. (The following definitions have meaning for an arbitrary original set Ξ of abstract symbols; however, we formulate them for the set Σ of operators which is the only case we need here.)

Definition 6. The set $\text{Rp}(\Sigma)$ of *repetition expressions over* Σ is the smallest set of formal expressions including Σ^* and such that with each $\mathbf{a}, \mathbf{b} \in \text{Rp}(\Sigma)$ and $n \in \{1, 2, 3, \dots\}$ also $\mathbf{ab} \in \text{Rp}(\Sigma)$ and $(\mathbf{a})^n \in \text{Rp}(\Sigma)$.

In this context we treat the numerals $1, 2, 3, \dots, n, \dots$ as abstract symbols (codes) representing natural numbers. We interpret the repetition expressions in the natural way as sequences over Σ : for $\mathbf{a} = a \in \Sigma^*$ define $|\mathbf{a}| = a$ and for $\mathbf{a}, \mathbf{b} \in \text{Rp}(\Sigma)$, $n \geq 1$ define $|\mathbf{ab}| = |\mathbf{a}||\mathbf{b}|$ and $|(\mathbf{a})^n| = \mathbf{aa} \dots \mathbf{a}$ (n -times). Note that $|\mathbf{a}| = |\mathbf{b}|$ does not, in general, imply $\mathbf{a} = \mathbf{b}$. For instance, $|(\varphi\psi)^3 \varphi| = |\varphi(\psi\varphi)^3| = \varphi\psi\varphi\psi\varphi\psi\varphi$. The numerals occurring in a repetition expression \mathbf{a} are called the *repetition indices in* \mathbf{a} .

We introduce star expressions as an extension of repetition expressions:

Definition 7. The set $\text{St}(\Sigma)$ of *star expressions over* Σ is the smallest set of formal expressions including Σ^* and such that with each $\bar{\mathbf{a}}, \bar{\mathbf{b}} \in \text{St}(\Sigma)$ and $n \in \{1, 2, \dots\}$ also $\bar{\mathbf{a}}\bar{\mathbf{b}} \in \text{St}(\Sigma)$, $(\bar{\mathbf{a}})^n \in \text{St}(\Sigma)$ and $(\bar{\mathbf{a}})^* \in \text{St}(\Sigma)$.

Clearly $\Sigma^* \subseteq \text{Rp}(\Sigma) \subseteq \text{St}(\Sigma)$. Unlike the repetition expressions the star expressions represent *sets* of sequences in accordance with the common interpretation of the Kleene star in formal language theory. Formally we define

$$\begin{aligned} \mathcal{S}(\bar{\mathbf{a}}) &= \{a\}, \quad \text{if } \bar{\mathbf{a}} = a \in \Sigma^* ; \\ \mathcal{S}(\bar{\mathbf{a}}\bar{\mathbf{b}}) &= \{ab \mid a \in \mathcal{S}(\bar{\mathbf{a}}), b \in \mathcal{S}(\bar{\mathbf{b}})\} ; \\ \mathcal{S}((\bar{\mathbf{a}})^n) &= \{a_1 a_2 \dots a_n \mid a_i \in \mathcal{S}(\bar{\mathbf{a}}) \text{ for } 1 \leq i \leq n\} ; \\ \mathcal{S}((\bar{\mathbf{a}})^*) &= \{\Lambda\} \cup \{a_1 a_2 \dots a_n \mid 1 \leq n < \infty \text{ and } a_i \in \mathcal{S}(\bar{\mathbf{a}}) \text{ for } 1 \leq i \leq n\} . \end{aligned}$$

Sequences in $\mathcal{S}(\bar{\mathbf{a}})$ are called the *repetition instances of* $\bar{\mathbf{a}}$. We shall mostly work with a less general, but easier to manipulate, special class $\mathcal{S}_u(\bar{\mathbf{a}})$ of *uniform repetition instances*: $a \in \mathcal{S}_u(\bar{\mathbf{a}})$ iff $a = |\mathbf{a}|$ where \mathbf{a} is a repetition expression obtained from $\bar{\mathbf{a}}$ by formal replacement of all stars occurring in $\bar{\mathbf{a}}$ by some repetition indices. We shall use a special notation for uniform repetition instances: assume $\bar{\mathbf{a}}$ has $n \geq 1$ occurrences of stars and let $\bar{k} = (k_1, \dots, k_n) \in \mathbb{N}^n$ be the n -tuple (\mathbb{N} is the set of positive integers) of the repetition indices in the repetition expression \mathbf{a} , replacing all the stars of $\bar{\mathbf{a}}$ (e.g. from the left to the right). Then we denote by $\bar{\mathbf{a}}[\bar{k}]$ or $\bar{\mathbf{a}}[k_1, \dots, k_n]$ the corresponding uniform repetition instance of $\bar{\mathbf{a}}$:

$$\bar{\mathbf{a}}[\bar{k}] = |\mathbf{a}| .$$

Thus, e.g.,

$$((\varphi)^* \psi)^* [2, 3] = \varphi\varphi\psi\varphi\varphi\psi\varphi\varphi\psi$$

(note that nonuniform repetition instances, e.g. $\varphi\psi\varphi\varphi\psi$, cannot be written in this way). In the following, whenever we write $\bar{\mathbf{a}}[\bar{k}]$ we tacitly assume that $\bar{k} \in \mathbb{N}^n$, where $n \geq 1$ is the number of occurrences of stars in $\bar{\mathbf{a}}$.

We shall now introduce the main concept of this section.

Definition 8. Let $\langle X, Y \rangle$ be a problem in the image space I . A finite nonempty set $\tilde{K} \subseteq \text{St}(\Sigma)$ is a *stochastic star plan* for $\langle X, Y \rangle$, and a (possibly infinite) set $\Gamma \subseteq \Sigma^*$ is an *analyzed stochastic star plan* for $\langle X, Y \rangle$ (corresponding to \tilde{K}), iff

$$\mathcal{S}(\tilde{K}) = \text{op}(\Gamma)$$

and Γ is a quasi-plan for X with a goal assignment Y such that for every $\varepsilon > 0$, $\delta > 0$ and every $\bar{\alpha} \in \tilde{K}$ there exists $\alpha \in \Gamma$ such that $\text{op}(\alpha)$ is a uniform repetition of $\bar{\alpha}$ and

- (i) $R(\alpha) \geq 1 - \varepsilon$,
- (ii) $P_{\Lambda}(Y(\alpha), Y) \geq 1 - \delta$.

In a strict analogy with our previous variants of the concept of a plan (Definition 1 and Definitions 4 and 5) we might be tempted to call a “plan” actually the set $\mathcal{S}(\tilde{K})$, rather than \tilde{K} . We make this conceptual shift in order to preserve the common intuition of a plan as a finite structure which can serve as an input to a physical executing system. Here a plan \tilde{K} is just a finitary presentation of the possibly infinite set $\mathcal{S}(\tilde{K})$ which can be itself viewed as the set of all *possible realizations* of \tilde{K} . (As a matter of fact, already in the case of a deterministic plan we worked with the set of all possible realizations of a “real plan” represented, e.g., by a flow diagram.)

In the special case when \tilde{K} happens to be a subset of Σ^* the concept of an (analyzed) stochastic star plan coincides with that of an (analyzed) $(0, 0)$ -stochastic plan because of the finiteness of \tilde{K} (cf., also, Theorems 1 and 2).

The intuitive meaning of the last part of Definition 8 (beginning “for every $\varepsilon > 0$, $\delta > 0 \dots$ ”) can be explained as follows.

In a typical case one may expect that with increasing number of repetitions in $\text{op}(\alpha)$, and thus with increasing length of α , the possibility that the goal has been achieved increases. Thus for arbitrarily small δ the sequence α in (ii) may tend to be large. If, at the same time, the reliability is growing for large α , the condition (i) may be satisfied as well.

However, in reality every additional repetition is likely to increase the probability of a failure and consequently the function $R(\alpha)$ might, in fact, decrease. This is indeed the case when R is the simple reliability RS (cf. Definition 4); we shall return to this case below.

For this reason it may be interesting to give an example of a reliability function increasing with the number of repetitions. Consider two operators φ, ψ and a star expression $(\varphi)^* \psi$ with an analyzing function f :

$$f((\varphi)^* \psi) = \langle \varphi, A_1, B_1 \rangle \langle \varphi, A_2, B_2 \rangle \dots \langle \varphi, A_n, B_n \rangle \langle \psi, A_0, B_0 \rangle.$$

Let

$$P_{\varphi}(A_1, B_i) = 1$$

for $i = 1, 2, \dots, n$ and let

$$P_{\varphi^i\psi}(A_i, B_0) = 1 - \frac{1}{i+1}.$$

When R is RP we have

$$R(f(\varphi^n\psi)) = \left[\prod_{i=1}^n P_{\varphi^i}(A_i, B_i) \right] \cdot P_{\varphi^n\psi}(A_1, B_0) = 1 - \frac{1}{n+1}.$$

This function is increasing and $\lim_{n \rightarrow \infty} R(f(\varphi^n\psi)) = 1$. (Intuition: think that φ is “strike the nail by a hammer” and ψ is “hang a picture on the nail”).

Now consider the case of $R = RS$. The following theorem shows that under this (and some other) general assumptions any set \tilde{K} , in order to be a plan in the sense of Definition 8, has to possess a rather strong property, which, in fact, eliminates the advantages of taking the reliability into account.

Let $\bar{a} \in \text{St}(\Sigma)$ and let $a, b \in \mathcal{S}_u(\bar{a})$. We write $a \leq b$ if either $a = b$ or $a = \bar{a}[k_1, \dots, k_n]$ and $b = \bar{a}[k'_1, \dots, k'_n]$ where $k_i \leq k'_i$ for all $i = 1, \dots, n$. Let us call an analyzing function f for \bar{a} normal iff for each $a, b \in \mathcal{S}_u(\bar{a})$, if $c \leq b$, then $f(a)$ is a subsequence of $f(b)$ (i.e., all transitions in $f(a)$ occur also in $f(b)$).

Theorem 3. Let $R = RS$ and let \tilde{K} be a stochastic star plan for a problem $\langle X, Y \rangle$ with an analyzing function f which is normal for all $\bar{a} \in \tilde{K}$. Then for every $\bar{a} \in \tilde{K}$ and every $\delta > 0$ there exists $a \in \mathcal{S}_u(\bar{a})$ such that

$$R(f(a)) = 1$$

and

$$P_A(Y(f(a)), Y) \geq 1 - \delta,$$

Proof. Let $\bar{a} \in \tilde{K}$. If \bar{a} is star-free then the result follows directly from Definition 8. Assume, therefore, that there is at least one occurrence of the star in \bar{a} . Denote by $a_1, a_2, \dots \in \mathcal{S}_u(\bar{a})$ a sequence of uniform repetition instances of \bar{a} such that $R(f(a_i)) \geq 1 - 1/i$ and $P_A(Y(f(a_i)), Y) \geq 1 - 1/i$; such a sequence exists according to the definition of stochastic star plan. $R = RS$ gives that, for any $\alpha = (\langle \varphi_i, A_i, B_i \rangle)_{i=1}^n \in \Sigma_t^*$

$$R(\alpha) = \prod_{i=1}^n P_{\varphi_i}(A_i, B_i).$$

The assumption on f gives that if $a_1 = a_2$, then any factor in the product for $R(f(a_1))$ occurs also in the product for $R(f(a_2))$, so $R(f(a_1)) \geq R(f(a_2))$. The number of star occurrences in \bar{a} is finite, hence, starting with certain element a_{n_0} in the sequence above, any a_i is comparable with some a_j , $j < i$. Hence, for any $\varepsilon_0 > 0$ there is only a finite number of values $R(f(a_i))$ in $(1 - \varepsilon_0, 1)$. Consequently there is an index

158 i_0 such that $i \geq i_0$ implies $R(f(a_i)) = 1$. Taking $j \geq i_0$ such that $P_\lambda(Y(f(a_j)), Y) > \delta$ (that is always possible) and setting $a = a_j$ we obtain the conclusion of the theorem. Q.E.D.

This result reveals a rather restrictive nature of the notion of a stochastic plan, especially in combination with the simple reliability function. The whole idea of introducing randomness into the plans is reduced just to the case when the "proximity" to the goal is evaluated. When the goal is approached only by a limiting process, the definition requires existence of arbitrarily long transition sequences of reliability one.

In the following variant of Definition 8 the reliability of sequences is kept only above certain predetermined value.

Definition 9. The ε -stochastic star plan and the corresponding analyzed ε -stochastic star plan are defined in the same way as stochastic star plans in Definition 8 except that the phrase "for every $\varepsilon > 0, \delta > 0$ " is replaced by "for a fixed $\varepsilon \geq 0$ and every $\delta > 0$ ".

The following implications between the predicates 'stochastic' and ' ε -stochastic' can be easily verified for star plans (and corresponding analyzed star plans) for a given problem:

- (i) '0-stochastic' implies 'stochastic';
- (ii) 'stochastic' implies ' ε -stochastic' for any $\varepsilon > 0$;
- (iii) ' ε -stochastic' implies ' ε' -stochastic' for any $\varepsilon' \geq \varepsilon$.

Neither of these implications can be reversed, in general (note that the goal assignment involved in the definitions may depend on ε). A particular case when 'stochastic' implies '0-stochastic' is represented in Theorem 3.

Let us illustrate the concepts of a stochastic and ε -stochastic star plans using our example of a robot looking for a box.

Example 2. Let us simplify the situation from Example 1 by considering only one operator φ ("going to the next room") and a goal "finding the box". Our aim will be to examine in which extent the expression $(\varphi)^*$ may represent an ε -stochastic star plan for this problem.

First, we construct an image space I appropriate for this purpose. Let the core theory T_1 include the axioms

$$\exists i \text{ ROBOT-IN}(i)$$

(i.e., "the robot is in one of the rooms") and

$$\exists j \text{ BOX-IN}(j)$$

(“the box is in one of the rooms”). The problem $\langle X, Y \rangle$ is expressed by the formulas

$$\begin{aligned} X &\equiv \text{ROBOT-IN}(1), \\ Y &= \exists i(\text{ROBOT-IN}(i) \ \& \ \text{BOX-IN}(i)). \end{aligned}$$

An intuitive way of defining the operator φ would be to set

$$C_\varphi \equiv \text{ROBOT-IN}(k), \quad R_\varphi \equiv \text{ROBOT-IN}(k+1).$$

(“ $k+1$ ” means “the $(k+1)$ st room”) but this would make φ dependent on a parameter k which is not essential to the ‘going-to-the-next-room’, operator, and moreover, it would not yield a convenient randomization of the transfer from one room to another. For our purposes it is therefore advantageous to assume $C_\varphi \equiv R_\varphi \equiv \text{true}$ and to treat pairs $\langle \text{ROBOT-IN}(k), \text{ROBOT-IN}(k+1) \rangle$ as incidental phenomena of φ . In addition, we can represent the fact that the position of the box does not change when the robot moves (a frame property) by incidental phenomena of the type $\langle \text{BOX-IN}(k), \text{BOX-IN}(k) \rangle$; in our case we use their complementary form $\langle \neg \text{BOX-IN}(k), \neg \text{BOX-IN}(k) \rangle$. To make easier our task let us describe (and later randomize) only those incidental phenomena which will be explicitly used in the analyzed stochastic plan corresponding to $(\varphi)^*$. Define for $k \geq 1$

$$\begin{aligned} A_k &\equiv \text{ROBOT-IN}(k) \ \& \ \bigwedge_{i=1}^k \neg \text{BOX-IN}(i), \\ B_k &\equiv \text{ROBOT-IN}(k+1) \ \& \ \bigwedge_{i=1}^k \neg \text{BOX-IN}(i); \end{aligned}$$

let $\tau_k = \langle \varphi, A_k, B_k \rangle$ and denote $\alpha_0 = \Lambda$ and $\alpha_k = \tau_1 \tau_2 \dots \tau_k$ ($k \geq 1$). We suggest the set

$$\Gamma = \{ \alpha_k \mid k \geq 0 \}$$

as a natural and appropriate solution to our problem. Indeed, it is just a formal way of expressing the following reasoning of the planning subject: “suppose that the k -th room is entered (after $k-1$ repetitions of φ) and the box has not yet been found in any of the previous rooms (i.e., B_{k-1} holds). Now, the box is either in room k and then the goal is achieved (Y holds) or the box is not in room k (A_k holds) and then the only possibility is to go to the next, $(k+1)$ st room.”

Let us consider the following stochastic dependence function

$$(12) \quad P_{\varphi^i}(A_k, B_{k+i-1}) = 1 - \frac{\eta}{i^2}$$

($i \geq 1$) its values for other pairs of formulas will be irrelevant. Here η is an arbitrary constant, $0 \leq \eta < 1$. If $\eta \neq 0$, then (12) indicate an increase of the probability that repetitions of φ actually causes the change, say, from A_1 to B_i (the possibility of a failure to move to a next room is considered).

Let us now introduce the goal assignment Y as follows.

$$\begin{aligned} Y(\alpha_0) &\equiv Y(\Lambda) \equiv X; \\ Y(\alpha_k) &\equiv Y(\tau_1 \dots \tau_k) \equiv B_k, \quad k \geq 1. \end{aligned}$$

In this way $Y(\alpha_k)$ represents the planner's a priori knowledge of the world after the sequence α_k has been realized. The probability that the goal has been achieved in such a world is measured by the function $P_\Lambda(Y(\alpha_k), Y)$. Let us choose

$$(13) \quad P_\Lambda(Y(\alpha_k), Y) = 1 - p^{k+1}, \quad (k \geq 0)$$

where p is a constant, $0 < p < 1$ (such a function would be obtained, e.g., from an assumption that the box was placed in a room randomly according to Poisson distribution with parameter p).

We are now in the position to verify that the sets $\bar{K} = \{\varphi^*\}$ and $\Gamma = \{\alpha_k \mid k \geq 0\}$ satisfy Definition 8, resp. 9. Clearly,

$$\mathcal{S}(\bar{K}) = \{\varphi^k \mid k \geq 0\} = \text{op}(\Gamma).$$

To show that Γ is a quasi-plan for X with the goal assignment Y we observe that (i) and (ii) from Definition 4 obtain the respective forms

$$T_1[X] \vdash A_1 \vee X$$

and

$$T_1[B_k] \vdash A_{k+1} \vee B_k,$$

both trivially valid.

Let $\varepsilon > 0$ and $\delta > 0$. Let us ask whether there is a number $n = n_{\varepsilon, \delta}$ for which the sequence α_n satisfies (i) and (ii) of Definition 8. We distinguish two cases for (i). First, if $R = RS$ then by (12) (for $i = 1$)

$$R(\alpha_n) = \prod_{j=1}^n P_p(A_j, B_j) = (1 - \eta)^n$$

and we obtain the inequality

$$(14) \quad (1 - \eta)^n \geq 1 - \varepsilon.$$

Second, if $R = RP$, then we obtain analogously the inequality

$$(15) \quad \prod_{i=1}^n \left(1 - \frac{\eta}{i^2}\right) \geq 1 - \varepsilon$$

Now to satisfy (ii) we obtain by (13) the inequality $1 - p^{n+1} \geq 1 - \delta$, i.e.,

$$(16) \quad \delta \geq p^{n+1}.$$

To satisfy (16) for arbitrarily small δ we need arbitrarily large n .

In the case $R = RS$ the conditions (i) and (ii) can be both satisfied only if $\eta = 0$. On the other hand, for $R = RP$, it is enough to take $\varepsilon \geq 1 - \prod_{i=1}^{\infty} (1 - \eta/i^2) \geq 1 - \eta\pi^2/6$.

Thus we can conclude that $(\varphi)^*$ is an ε -star plan for $\langle X, Y \rangle$ with respect to the path-dependent reliability RP iff $\varepsilon \geq \eta\pi^2/6$ and with respect to the simple reliability RS iff $\eta = 0$ and $\varepsilon \geq 0$. $(\varphi)^*$ is a star plan for $\langle X, Y \rangle$ (with respect to RS or RP) iff $\eta = 0$. Note that, in agreement with Theorem 3, $\eta = 0$ implies $RS(\alpha_n) = 1$ for every n .

5. ASYMPTOTIC PROPERTIES OF STAR EXPRESSIONS

In the last section we have introduced the concept of a stochastic star plan based on the idea that considering *all repetitions* (of an operator or a sequence of operators) *as one single entity* (indicated by a star occurrence) enables goal-oriented planning even in cases when a lack of important knowledge of the world would make deterministic planning unfeasible. A closer look at the main definitions (Definitions 8 or 9) shows that the concept of a stochastic star plan has two aspects. The first concerns the branching structure of a plan (represented by the concept of a quasi-plan), the second is related to its approximating qualities — the reliability of transition sequences and the degree in which they approach the goal. In this section we shall be concerned with the second aspect.

First, we shall consider the question of reliability. Let $\bar{a} \in \text{St}(\Sigma)$ be an arbitrary but fixed star expression. It may be desirable to associate directly with \bar{a} its “reliability” say $\varrho(\bar{a}) \in \langle 0, 1 \rangle$. Two difficulties arise, however. First, star expressions consist of operators, while the reliability function R was defined for transition sequences. Therefore we shall always assume that an analyzing function $f: \mathcal{S}(\bar{a}) \rightarrow \Sigma_+^*$ is available; recall that $\text{op } f(a) = a$ for every $a \in \mathcal{S}(\bar{a})$. Let us define

$$\varrho_f(a) = R(f(a))$$

for each $a \in \mathcal{S}(\bar{a})$.

Second, star expressions represent possibly infinite sets of sequences. There are several intuitively plausible ways how to define $\varrho_f(\bar{a})$ on the basis of $\varrho_f: \mathcal{S}(\bar{a}) \rightarrow \langle 0, 1 \rangle$. For instance,

$$\varrho_f^{(1)}(\bar{a}) = \inf \{ \varrho_f(a) \mid a \in \mathcal{S}(\bar{a}) \}$$

or

$$\varrho_f^{(2)}(\bar{a}) = \sup \{ \varrho_f(a) \mid a \in \mathcal{S}(\bar{a}) \}.$$

The former reflects a pessimistic approach, the latter an optimistic one. There are, however, strong intuitive reasons for a definition of an asymptotic nature. Our

motivation for introducing stars was to use them as indicators of loops with unbounded number of repetitions. In the execution time this number is, of course, finite (maybe small) but the planning system is unable to make any predictions of its value. One is therefore inclined to derive the reliability of a plan asymptotically from the reliability of its partial realizations with the number of repetitions growing unboundedly.

The following definition represents a number of possibilities according to the manner how the repetitions grow and whether we follow the best or the worst cases.

Definition 10. Let $\bar{a} \in \text{St}(\Sigma)$ be a star expression, let $n \geq 0$ be the number of star occurrences in \bar{a} , let $g : \mathbb{N}^n \rightarrow \mathbb{N}$ and let f be an analyzing function for \bar{a} . The *upper* (resp. *lower*) *asymptotic reliability of \bar{a}* (with respect to f and g), denoted by $\varrho_{f,g}^+(\bar{a})$ (resp. $\varrho_{f,g}^-(\bar{a})$), is defined as follows. If $n = 0$ (i.e. if $\bar{a} = a \in \Sigma^*$) then

$$\varrho_{f,g}^+(\bar{a}) = \varrho_{f,g}^-(\bar{a}) = \varrho_f(a).$$

If $n > 0$ then

$$\varrho_{f,g}^+(\bar{a}) = \limsup_{g(\bar{k}) \rightarrow \infty} \varrho_f(\bar{a}[\bar{k}])$$

and

$$\varrho_{f,g}^-(\bar{a}) = \liminf_{g(\bar{k}) \rightarrow \infty} \varrho_f(\bar{a}[\bar{k}]).$$

The function g determines the character of the limit, that is the topology with respect to which the limits are computed (any set $\{\bar{k} \mid g(\bar{k}) > t\}$ is a neighbourhood of the infinity).

Thus, for instance, to say that the upper (resp. lower) asymptotic reliability of \bar{a} is greater than $1 - \varepsilon$ ($\varepsilon > 0$) is the same as to say that for all (resp. for a sufficiently large) $m > 0$, infinitely many (resp. all) uniform repetition instances $\bar{a}[\bar{k}]$ satisfying $g(\bar{k}) > m$ have reliability greater than $1 - \varepsilon$. Two natural examples of g are $g = \min$ (yielding a topology equivalent to the ordinary Euclidean one) and $g = \max$. In practice we would not, in general, compute the exact values of the asymptotic reliabilities. It is usually sufficient to know their lower estimates which may be often easy to obtain. For instance, to show that $\varrho_{f,\max}^+(\bar{a}) \geq 1 - \varepsilon$ it is enough to choose an arbitrary infinite sequence of distinct repetition instances a_1, a_2, \dots , from $\mathcal{S}_u(\bar{a})$ such that

$$\lim_{n \rightarrow \infty} R(f(a_n)) \geq 1 - \varepsilon.$$

Example 3. Continuing the case study from Example 2 we shall compute the asymptotic reliability of the expression $(\varphi)^*$ with respect to the analyzing function $f(\varphi^k) = \alpha_k$ and to an arbitrary increasing function, say, $g(k) = k$. For the case $R = \text{RS}$ we have

$$\varrho_f(\varphi^k) = \text{RS}(\alpha_k) = (1 - \eta)^k$$

and thus

$$\varrho_{f,\varrho}^+((\varphi)^*) = \varrho_{f,\varrho}^-((\varphi)^*) = \lim_{k \rightarrow \infty} (1 - \eta)^k = \begin{cases} 0 & \text{if } \eta > 0, \\ 1 & \text{if } \eta = 0. \end{cases}$$

For the case $R = RP$

$$\varrho_{f,\varrho}(\varphi^k) = RP(\alpha_k) = \prod_{i=1}^k \left(1 - \frac{\eta}{i^2}\right)$$

and thus

$$\varrho_{f,\varrho}^+((\varphi)^*) = \varrho_{f,\varrho}^-((\varphi)^*) = \lim_{k \rightarrow \infty} \prod_{i=1}^k \left(1 - \frac{\eta}{i^2}\right) \geq 1 - \frac{\pi^2}{6} \eta.$$

We shall turn now to the question of degree in which a transition sequence ensures the goal. Once a plan is constructed it is not important which particular form of the goal assignment Y was used – what matters is just the proximity to the goal. We shall introduce a new concept of a δ -suitability of transition sequences.

Definition 11. Let $\alpha \in \Sigma_1^*$ be a transition sequence, $\langle X, Y \rangle$ a problem, and $\delta \geq 0$. We say that α is δ -suitable for $\langle X, Y \rangle$ iff there is a formula Y' such that $P_\Lambda(Y', Y) \geq 1 - \delta$ and if $\alpha = \Lambda$ then

$$T_i[X] \vdash Y'$$

else if the last transition in α is $\langle \varphi, A, B \rangle$ then

$$T_i[R_\varphi \& B] \vdash Y'.$$

Clearly, if α is δ -suitable for $\langle X, Y \rangle$ then it is δ' -suitable for any $\delta' \geq \delta$.

It appears that for representing the asymptotic properties of star expressions in a plan it is convenient to formalize a property that a given star expression admits repetition instances which have the corresponding transition sequences δ -suitable with δ arbitrarily close to zero, and at the same time, of reliability arbitrarily close to one (the case of stochastic star plans), or above a certain predetermined level (the case of ε -stochastic star plans). We arrive at the following definition.

Definition 12. Let $\bar{a} \in \text{St}(\Sigma)$ be a star expression with an analyzing function f , let $\langle X, Y \rangle$ be a problem and let $\varepsilon > 0$. We say that \bar{a} is *asymptotically suitable* for $\langle X, Y \rangle$ with the reliability level $1 - \varepsilon$ iff for every $\delta > 0$ there exists $a \in \mathcal{S}_\omega(\bar{a})$ such that

- (i) $f(a)$ is δ -suitable for $\langle X, Y \rangle$ and
- (ii) $\varrho_f(a) \geq 1 - \varepsilon$.

Note that this definition admits a trivial “nonasymptotic” case, when some particular repetition instance $a \in \mathcal{S}_\omega(\bar{a})$ is δ -suitable for arbitrary $\delta > 0$. If in addition

$q_f(a) \geq 1 - \varepsilon$, then the defining property is fulfilled independently of other repetition instances of \bar{a} . When this does not happen we shall talk about asymptotic suitability *in proper sense*.

The following theorem relates the asymptotic suitability to ε -stochastic star plans.

Theorem 4. Let $\langle X, Y \rangle$ be a problem, let $\varepsilon > 0$, let $\Gamma \in \Sigma_t^*$ be a quasi-plan for X and let $\bar{K} \subseteq \text{St}(\Sigma)$ be such that $\mathcal{S}(\bar{K}) = \text{op}(\Gamma)$ and each $\bar{a} \in \bar{K}$ is asymptotically suitable for $\langle X, Y \rangle$ with the reliability level $1 - \varepsilon$ (and with respect to some analyzing function $f: \mathcal{S}_u(\bar{a}) \rightarrow \Gamma$). Then \bar{K} is an ε -stochastic star plan (and Γ the corresponding analyzed ε -stochastic star plan) for $\langle X, Y \rangle$.

Proof. Under the assumptions of the theorem let Y be the goal-assignment that makes Γ a quasi-plan. By Definitions 11 and 12 (and denoting $f(a)$ by α), for each $\bar{a} \in \bar{K}$ and each $\delta > 0$ there exists $\alpha = f(a) \in \Gamma$ such that

$$\begin{aligned} \text{op}(\alpha) &= a \in \mathcal{S}_u(\bar{a}), \\ \mathbf{R}(\alpha) &= q_f(a) \geq 1 - \varepsilon \end{aligned}$$

and there exists Y'_α such that $\mathbf{P}_\Lambda(Y'_\alpha, Y) \geq 1 - \delta$ and

$$\begin{aligned} \alpha = \Lambda \quad \text{implies} \quad T_i[X] \vdash Y'_\alpha, \\ \alpha = \alpha \langle \varphi, A, B \rangle \quad \text{implies} \quad T_i[R_\varphi \ \& \ B] \vdash Y'_\alpha. \end{aligned}$$

Now to show that \bar{K} and Γ satisfy the conclusion of the theorem it is enough to define for any $\alpha \in \Sigma_t^*$

$$Y'(\alpha) = \begin{cases} Y'_\alpha & \text{if } \alpha \text{ satisfies the four conditions above,} \\ Y(\alpha) & \text{otherwise.} \end{cases}$$

Clearly, Γ is a quasi plan with a goal assignment Y' . The theorem now follows directly from Definition 8 (rephrased according to Definition 9). Q.E.D.

It is interesting to note that Theorem 4 provides a stronger property than Definition 8 (resp. 9). The difference is in the condition of δ -suitability of α which requires that, upon termination of α , Y' is provable, while in the definition of quasi-plan, there is assumed that $Y(\alpha)$ possibly in disjunction with other formulas (representing the continuing branches of the plan) is provable. Consequently, in the latter case the decision whether to determinate after executing $\text{op}(\alpha)$ has to be left for the execution stage.

It is natural to investigate possibilities of practical testing of a given star expression whether it is asymptotically suitable with respect to a given reliability level (or to determine such a level when it is not given).

A direct use of Definition 12 may appear rather difficult and we may look for cases where a specific knowledge of real properties of involved operators may be

helpful. A typical such case occurs when we are able to identify certain parts of a given expression whose repetitions influence essentially the suitability (through the parameter δ) and distinguish them from another parts influencing the reliability.

Consider, for instance, the expression $((\varphi)^* \psi^*)$ (think about our example of φ striking a nail and ψ hanging a picture; the above expression may be a plan for the task of "hanging all pictures"). Assume that the reliability of $(\varphi^n \psi^m)$ decreases with increasing m but increases with increasing n (as in the case discussed earlier) and that, on the other hand, the suitability increases with m (more pictures are hanged) and does not depend on n . In such a case it is quite natural first to study the suitability for $m \rightarrow \infty$ ignoring the reliability (this corresponds to taking the reliability level 0, i.e. $\varepsilon = 1$ in Definition 12), and then to work on n in order to obtain some higher reliability level.

In general, this idea of splitting the two components of asymptotic properties of star expressions may be often helpful. The computation may be even further simplified if we dispose of a value of some type of the asymptotic reliability from Definition 10.

Let us make our above considerations more specific. Consider a star expression \bar{a} with an analyzing function f and assume that we already know that \bar{a} is asymptotically suitable for $\langle X, Y \rangle$ with the reliability level zero. This means that we can construct an infinite sequence of (not necessarily distinct) tuples $\bar{k}_1, \bar{k}_2, \dots$ such that for each $\delta > 0$ there exists $i \geq 1$ such that $f(\bar{a}[\bar{k}_i])$ is δ -suitable for $\langle X, Y \rangle$. Assume, moreover, that we know, independently, that

$$(17) \quad e_{f,\delta}^-(\bar{a}) > 1 - \varepsilon$$

for some function g . How combine these two facts? There may exist a procedure (hopefully an efficient one) associating with every \bar{k}_i another tuple \bar{k}'_i such that $\bar{a}[\bar{k}'_i]$ is δ -suitable for the same δ as $\bar{a}[\bar{k}_i]$ (without caring for the reliability) and the new sequence \bar{k}'_i has the property that for arbitrarily large $m > 0$ there exists $i \geq 1$ such that $g(\bar{k}'_i) > m$. If this happens we can conclude that \bar{a} is asymptotically suitable for $\langle X, Y \rangle$ with the reliability level $1 - \varepsilon$.

Indeed, by Definition 10 and (17) there exists $m_0 > 0$ such that for all $\bar{k}, g(\bar{k}) > m_0$ implies $f(\bar{a}[\bar{k}]) > 1 - \varepsilon$. By the property of the sequence (\bar{k}_i) there is then $i \geq 1$ such that

$$e_f(\bar{a}[\bar{k}_j]) > 1 - \varepsilon$$

for all $j \geq i_0$. The conclusion then follows from the asymptotic suitability of (\bar{k}_i) and hence also of (\bar{k}'_i) .

The situation may be simplified by an appropriate choice of the function g in (17) which determines the limiting character of asymptotic reliability. Considering, for instance, $g = \max$ (which enables using $(\bar{k}'_i) = (\bar{k}_i)$) we arrive at the following result which, as a particular case of our above reasoning does not require a separate proof.

Theorem 5. Let $\langle X, Y \rangle$ be a problem and let a star expression \bar{a} (with an analyzing function f) be asymptotically suitable for $\langle X, Y \rangle$ in proper sense, with the reliability level zero. Let

$$q_{f, \max}^-(\bar{a}) > 1 - \varepsilon$$

for some $\varepsilon > 0$. Then \bar{a} is asymptotically suitable for $\langle X, Y \rangle$ with the reliability level $1 - \varepsilon$.

Let us apply this theorem to our running example.

Example 4. We claim that for the case of the robot looking for the box (cf. Example 2) the expression $(\varphi)^*$ (with the analyzing function $f(\varphi^n) = \alpha^n$ and with $R = RP$) is asymptotically suitable for the problem $\langle X, Y \rangle$, with the reliability level $1 - 2\eta$. For any $\delta > 0$ let n be large enough so that $p^{n+1} \leq \delta$. Then, by (13),

$$P_{\wedge}(Y(\alpha_n), Y) \geq 1 - \delta,$$

i.e., due to the particular form of $Y(\alpha_n)$, $\alpha_n = f(\varphi^n)$ is δ -suitable for $\langle X, Y \rangle$. At the same time, α_n is not δ' -suitable for $\delta' < p^{n+1}$ (by assumption $p > 0$). Thus $(\varphi)^*$ is asymptotically suitable in proper sense for $\langle X, Y \rangle$ with the reliability level 0. Now, since

$$q_{f, \max}^-((\varphi)^*) = \lim_{n \rightarrow \infty} RP(\alpha_n) \geq 1 - \eta \frac{\pi^2}{6} > 1 - 2\eta$$

(cf. Example 3), our claim follows directly from Theorem 5.

6. GENERATING STAR PLANS

In the preceding sections we have developed an extensive formalism involving such concepts as (ε, δ) -stochastic plan and ε -stochastic star plan, which could serve as "approximate" solutions of problems when deterministic plans are difficult to obtain, or even when they do not exist. We have analyzed these concepts qualitatively from various aspects, including the reliability and suitability of a plan for a given problem. So far we have not, however, discussed the question how such approximating plans can be effectively constructed. A theory, no matter how powerful, which would merely evaluate the performance of plans would be of little use if not accompanied by an effective procedure for plan formation.

The main subject of this section is to outline a general method, called STAR-PLANNER, which converts a given set of transition sequences gradually into a set of star-expressions asymptotically suitable for a given problem (with a given reliability level). It is understood that the input set of STAR-PLANNER is obtained from a deterministic problem solver generating a (branching) plan or, in a general case,

a potentially infinite quasi-plan for the problem in question. This problem solver, called GENERATOR, combined with STAR-PLANNER, may thus serve as a promising tool for generating ε -stochastic star plans.

To elucidate this conception we shall first discuss deterministic plan formation. Let us start with a concrete, but rigorously tractable case of planning in situation calculus (cf. [17]). A problem $\langle X, Y \rangle$ obtains the form of an implicative formula

$$(18) \quad X[[s_0]] \rightarrow \exists s Y[[s]]$$

which expresses the statement “if X holds in the initial situation (s_0) then there exists a goal situation (s) in which Y holds”. This formula together with an axiomatic representation of the environment and of the robot’s abilities, is put into an automatic theorem prover supplemented by an answer-extraction routine. If the formula (18) is provable the answer-extraction routine produces as its output a finite set of operator sequences which is a deterministic (branching) plan for $\langle X, Y \rangle$.

It is obvious that the described way of generating plans is practically applicable only when the cardinality of the obtained set of sequences is rather small. In fact, the artificial intelligence literature often tacitly assumes plans consisting only of a single operator sequence (straight-line plans). The situation is quite different when the generated sets are untractably large or even infinite. This is exactly the point where our stochastic approach enters the scene.

It is important to view properly the both mentioned cases: the generated set is large but finite (a deterministic plan exists) or the generated set is infinite (there is no deterministic plan in the conventional sense). We argue that there are no practical reasons for distinguishing these two cases*). In the situation calculus the latter case occurs when the formula (18) is not provable: the theorem prover may in certain cases work without halting forever all the time producing new and new operator sequences (this possibility is inevitable due to the undecidability of the first-order predicate calculus). In spite of that the infinite set of successively generated transition sequences may have properties that qualify it intuitively as a genuine plan for the original problem (cf. Example 1).

Our approach is based on two main theses. According to the first, even small finite fragments of large finite, or infinite, outputs of deterministic problem solvers are representative enough to be taken as a point of departure for approximate planning. According to the second thesis, even infinite set may be treated as an executable plan, provided it is presented in a finitistic way.

The first thesis concern GENERATOR. We shall abstract from its particular form, may it be a resolution theorem prover in situation calculus, an image space search method, or any other problem solver, provided it uses incidental phenomena. We assume that, with input $\langle X, Y \rangle$, it generates successively transition sequences $\alpha_1, \alpha_2, \dots$:

*) Making no distinction between large finite and infinite objects is characteristic for heuristic approaches in artificial intelligence.

each sequence α_i representing one branch in a would-be deterministic analyzed branching plan. What is essential is the requirement that GENERATOR produces *analyzed* operator sequences (elements of Σ_i^*) where the incidental phenomena actually used during the computation are explicitly indicated. (Note that the incidental phenomena are here understood in the deterministic sense as the system Inc_i ; in STAR-PLANNER they will be replaced by the randomized system \mathbf{P}_i .) Formally, we could enforce our first thesis by the assumption that GENERATOR produces a quasi-plan, but we shall not need further such explicit requirement.

Our second thesis leads us to the conception of STAR-PLANNER as a procedure constructing a finite set of star expressions. The rough idea is that STAR-PLANNER alternately (i) asks GENERATOR for new sequences and (ii) makes guesses how to obtain asymptotically suitable star expressions such that some (preferably all) so far accumulated sequences are their repetition instances. Thus STAR-PLANNER may stop independently of the fact that GENERATOR does not have finished its work (this is the point where the distinction between the large finite and the infinite is dispensed with).

We shall present STAR-PLANNER in a semiformal way, purposely in the form of a highly nondeterministic algorithm. Every nondeterministic step is conceived as a possible point for implementing various heuristic hints. Some ideas for such heuristics as well as possible default decisions when no heuristics are available are mentioned at appropriate places.

The inputs to STAR-PLANNER, besides GENERATOR (which is called by STAR-PLANNER as its subroutine), are the problem $\langle X, Y \rangle$, a reliability level $1 - \varepsilon$, a reliability function R (RS, RM or RP) and some other parameters which will be explained later. The procedure consists of eleven steps.

Step 1. Initialize $\Gamma := \emptyset$; $\tilde{K} := \emptyset$; $i := 0$.

[$\Gamma \subseteq \Sigma_i^*$ will hold for all the transition sequences produced by GENERATOR up to a current step, $\tilde{K} \subseteq \text{St}(\Sigma)$ will be the set of star expressions under construction (eventually the output) and i counts the sequences α_i produced by GENERATOR.]

Step 2. Set $i := i + 1$, $\alpha := \alpha_i$, $\Gamma := \Gamma \cup \{\alpha\}$. If the GENERATOR refuses to give α_i or if i exceeds a previously set limit go to Step 11.

Step 3. If $R(\alpha) \geq 1 - \varepsilon_0$ and α is δ_0 -suitable for $\langle X, Y \rangle$ go to Step 4. Otherwise go back to Step 2.

[Here $\varepsilon_0, \delta_0 \in \langle 0, 1 \rangle$ are parameters which select candidates for the starring procedure in Steps 4–8. In practice Step 3 should be understood as only a partial test: unless the answer is easy to obtain it is considered to be negative.]

Step 4. Find $a \in \text{Rp}(\Sigma)$ such that $\text{op}(\alpha) = |a|$.

[This is a nondeterministic step since there may be several repetition expressions for the same sequence (example: $|(\varphi\psi)^3\varphi| = |\varphi(\psi\varphi)^3|$). An informed heuristic would suggest the natural repetitions; in absence of such heuristics there is a simple default algorithm: search for shortest repeated subsequences from the left to the right, replace them by repetition expressions and iterate this procedure for thus obtained expression.]

Step 5. Let k be the number of occurrences of repetition indices in a . Initialize*) $\mathcal{W} := \mathcal{P}\{1, 2, \dots, k\}$.

Step 6. If \mathcal{W} is empty go back to Step 2. Else select $Q \in \mathcal{W}$ and construct the star expression \bar{b} as follows: for each $j \in Q$ replace formally the j -th occurrence of a repetition index in a by a star occurrence (in particular, if Q is empty, then $\bar{b} = a$).

[The nondeterminism of this step is due to the phrase “select $Q \in \mathcal{W}$ ”. This appears to be one of the most important places where a suitable heuristic should be chosen. An ideal heuristic would recognize those particular repetitions which indicate natural and/or necessary loops in the constructed plan and distinguish them from accidental repetitions. Due to the difficulty of finding such an ideal heuristic a practical implementation may follow a more realistic “syntactically” controlled strategy. Such a strategy may be, for instance, based on the following suggestions:

- (1) give the preference to Q with small cardinality (this yield star expressions with small number of stars);
- (2) give the preference to inner, resp. outer, occurrences of repetition indices (preference to small, resp. large, loops);
- (3) give the preference to large repetition indices (an important strategy based on the intuitively sound argument that a larger number of repetitions more likely indicates a loop);
- (4) use the experience from the previous passes through Step 6 (this step is a part of a loop, cf. Step 8). In particular, there may be cases when the computation expenses for Steps 7 and 8 may be decreased, say, by selecting Q with the “previous Q ” as a subset.]

Step 7. Specify an analyzing function $f: \mathcal{S}_v(\bar{b}) \rightarrow \Sigma_t^*$ such that $f(|a|) = \alpha$.

[By “specify” we mean to give rules how to evaluate f when its value is needed (in Step 8). In general, Step 7 may appear rather difficult; however, in concrete cases the knowledge of α may be quite sufficient for extrapolating f to higher (or lower) repetitions.]

*) \mathcal{P} denotes the power set, i.e. the set of all subsets.

Step 8. Test whether \bar{b} is asymptotically suitable for $\langle X, Y \rangle$ with the reliability level $1 - \varepsilon$ (and with respect to analyzing function f). If not, either set $\mathcal{W}' := \mathcal{W} - \{Q\}$ and return to Step 6 or (give up and) return to Step 2.

[The question of effective testing whether a given star expression is asymptotically suitable with a given reliability level was thoroughly analyzed in Section 4. The either-or-nondeterminism of Step 7 enables to leave the loop of Step 6 sooner than \mathcal{W} is emptied, thus avoiding the maximum number of 2^k passes. The heuristics for this nondeterminism may be combined with the heuristics for Step 6 and may be either static, e.g., by admitting the Q 's of only certain prespecified type, or dynamic, by learning from the previous passes.]

Step 9. Set $\bar{K} := \bar{K} \cup \{\bar{b}\}$ and for all $\beta \in \Gamma$ check whether $\text{op}(\beta) \in \mathcal{S}(\bar{K})$. If not return to Step 2.

[Since \bar{K} is a set of regular expressions the standard automata-theoretic techniques can be used for this purpose.]

Step 10. Halt. Output is \bar{K} .

Step 11. Halt. Announce a failure.

It is clear that the set \bar{K} produced by STAR-PLANNER consists entirely of star expressions asymptotically suitable for $\langle X, Y \rangle$ with the reliability level $1 - \varepsilon$ (and with respect to appropriate analyzing function constructed in Step 7). However, in order that such \bar{K} be an ε -stochastic star plan, a necessary and sufficient condition would be, according to Theorem 4, that the corresponding set of analyzed sequences

$$\Gamma_{\bar{K}} = \bigcup_{\bar{a} \in \bar{K}} f_{\bar{a}}(\mathcal{S}_v(\bar{a}))$$

(different $f_{\bar{a}}$ for different $\bar{a} \in \bar{K}$, in general) is a quasi-plan for X .

Let us illustrate by an example how STAR-PLANNER works.

Example 5. Let us return to Example 1 with two operators

φ : "go to the next room"

ψ : "paint the box"

and the task to paint the box. C_φ and R_φ were described in Example 2, let

$$C_\psi \equiv \exists i(\text{ROBOT-IN}(i) \ \& \ \text{BOX-IN}(i)),$$

$$R_\psi \equiv \text{BOX-PAINTED}.$$

Consider a GENERATOR generating successively transition sequences $\alpha_1, \alpha_2, \dots$

where α_i is

$$(19) \quad \langle \varphi, A_1, B_1 \rangle \langle \varphi, A_2, B_2 \rangle \dots \langle \varphi, A_{n(i)}, B_{n(i)} \rangle \langle \psi, A_{n(i)+1}, B_{n(i)+1} \rangle$$

with $A_k, k \leq n(i) + 1$ and $B_k, k \leq n(i)$, being the same as in Example 2, $B_{n(i)+1}$ is “ROBOT-IN($n(i)$) \vee BOX-PAINTED”. We do not require the sequences α_i to be of monotonously increasing length (if i increases), suppose only, that any sequence of such a type will, eventually, occur (more formally, we suppose that for any $j \in \mathbb{N}$ there is $i \in \mathbb{N}$ such that $j = n(i)$). Clearly, $\text{op}(\alpha_i) = \varphi^{n(i)}\psi$ and the analyzing function is supposed to ascribe to any operator sequence of the form $\varphi^j\psi$ the transition sequence as in (19) (hence, it is just the case when the values of f for other sequences than those already generated can be easily extrapolated).

Suppose that (unlike as in Example 2)

$$P_{\varphi^k}(A_1, B_k) = 1 - \frac{\eta}{2^i}; \quad 0 \leq \eta < 1, \quad k \geq 1,$$

$$P_{\varphi^k\psi}(A_1, B_{k+1}) = 1 - \frac{1}{k+1}, \quad k \geq 0.$$

Taking $R = RP$ we obtain

$$(20) \quad R(\alpha_i) = \left[\prod_{j=1}^{n(i)} \left(1 - \frac{\eta}{2^j} \right) \right] \left(1 - \frac{1}{n(i)+1} \right).$$

Denoting by $C = C(\eta)$ the limit value of the right-hand side of (20) for $n(i) \rightarrow \infty$ we can easily see (an exercise in elementary analysis), that $C \geq 1 - \eta$. Considering, now, the star expression $(\varphi)^*\psi \in \text{St}(\Sigma)$ and using the definition of ϱ_f , we obtain that $\varrho_f((\varphi)^*\psi) = C \geq 1 - \eta$. As will be shown later, $(\varphi)^*\psi$ is the only star expression for which the value ϱ_f must be computed or at least minorized. Assume that $P_\Lambda(B_{i+1}, Y) = 1 - 1/(i+1)$ and recall that $Y \equiv \text{BOX-PAINTED}$. Let $0 \leq \varepsilon, \varepsilon_0, \delta_0 (< \frac{1}{2})$ be given such that $\eta < \min(\varepsilon_0, \varepsilon)$. Other objects occurring in STAR-PLANNER will be specified later.

Step 1 initializes $\Gamma := \emptyset, \tilde{K} := \emptyset, i := 0$. Step 2 offers, successively, operator sequences of the type $\varphi^i\psi, i \geq 0$ not necessarily in the order of increasing length. As

$$(21) \quad \lim_{i \rightarrow \infty} \left[\prod_{j=1}^i \left(1 - \frac{\eta}{2^j} \right) \right] \left[1 - \frac{1}{i+1} \right] = C \geq 1 - \eta > 1 - \varepsilon_0,$$

there is an index $n_0 = n_0(\eta, \varepsilon_0)$ such that, for any $n(i) < n_0$, $R(\alpha_i) \geq 1 - \varepsilon_0$ holds (if $\eta = 0$ then clearly $n_0 = \lceil 1/\varepsilon - 1 \rceil$). At the same time, $P_\Lambda(B_{i+1}, Y)$ tends to 1 if $n(i)$ increases and, for $n(i) > \lceil 1/\delta_0 - 1 \rceil$, $P_\Lambda(B_{n(i)+1}, Y) \geq 1 - \delta_0$ holds. Hence, sooner or later, the condition of Step 3 will not be satisfied and STAR-PLANNER will continue by Step 4. Let it happen, for the first time, for α_i ; set $n = n(i), \alpha = \alpha_i, \varepsilon_0, \delta_0 < \frac{1}{2}$ gives that $n > 1$.

Now, $\text{op}(\alpha) = \varphi^n \psi$. Considering, e.g., the heuristics trying to introduce as large repetition indices as possible, we obtain the expression $(\varphi)^n \psi \in \text{Rp}(\Sigma)$ with the property that $\text{op}(\alpha) = (\varphi)^n \psi$, so Step 4 is executed. Trying to apply Step 5, we have $k = 2$, $\mathcal{W} = \mathcal{P}\{1, 2\}$. Selecting $Q \in \mathcal{W}$ we may use a heuristic combining viewpoints (1) and (3) mentioned after Step 5 when STAR-PLANNER described, i.e. we prefer to replace by the star *small* number of *high* repetition indices, so we select $\{Q\} = 1$. Executing Step 6 for $a = \text{op}(\alpha)$ we obtain, hence, $\bar{b} = (\varphi)^* \psi$. Step 7 establishes an analyzing function f according to (19). As $\lim_{i \rightarrow \infty} (1 - 1/(i + 1)) = 1$, (21) holds and both the limited functions are monotonously increasing, we can easily conclude that $(\varphi^*) \psi$ is asymptotically suitable with the level $1 - \varepsilon$, so the answer to Step 8 is positive and Step 9 follows. We set $\bar{K} = \{\bar{b}\}$ and see that any $\beta \in \Gamma$ is such that $\text{op}(\beta) = \varphi^i \psi$ for an $i \in \mathbb{N}$, so $\text{op}(\beta) \in \Phi(\bar{K}) = \mathcal{A}(\bar{b})$. Hence, STAR-PLANNER halts after Step 10 giving $\bar{K} = \{(\varphi)^* \psi\}$.

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