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REMARKS ON THE THEORY OF IMPLICIT LINEAR CONTINUOUS–TIME SYSTEMS[1]

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Given a (not necessarily regular or square) linear implicit system $Ex'(t) = Fx(t)+Gu(t)$, we study the space of admissible initial conditions and the controllable space of the system. Distributional trajectories are considered.

0. INTRODUCTION

We shall consider implicit linear continuous-time systems described by a differential equation of the form

$$Ex'(t) = Fx(t) + Gu(t). \tag{1}$$

It may happen that for some initial conditions the above differential equation has no solution. For such reason, it is reasonable to extend the concept of solution to include a class of distributions as solutions. It will require introducing a distributional version of the considered differential equation, as explained in [17], and also in [4,5,7].

In the paper we sketch the distributional theory of equation (1). In particular, we determine the space of admissible initial conditions and the controllable spaces.

Let us note that some aspects of the distributional theory of equation (1) have been previously studied in [4,5,7,13,17], and also in other references. The above mentioned papers treat the same circle of problems we are considering but only for the case when the pencil $(\lambda E - F)$ is regular (cf. [9]). This restrictive assumption is not used in our paper. When the first version of the paper (i.e. [18]) has been completed, some new results concerning not necessarily regular systems have also been obtained by other researches; the results are briefly reported in [10] and [16]. More results can also be found in [11,12]. The framework of [10,11,12] is very similar to that of our paper since the above-mentioned references study the same class of solutions of a distributional counterpart of (1). In consequence of this fact, some results of [10,11,12] are very close to the results of our Sections 2 and 3. The presentation of the theory of implicit systems given in [16] is different from that of

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our paper. The starting point of [16] are some concepts from [4]. More importantly, the class of distributions considered in [16] is different from that considered in our paper. Unfortunately, no proofs of presented results are provided in [16].

1. BASIC NOTATION

We now recall some standard notation from the theory of distributions we shall use in the sequel. As usual, we write $\mathcal{D}$ and $\mathcal{D}'$ for the space of test functions $\mathcal{D}(\mathbb{R})$ and the space of distributions $\mathcal{D}'(\mathbb{R})$, respectively. If $f \in \mathcal{D}'$, then $f^{(0)} := f$ and, for $k = 1, 2, \ldots$ $f^{(k)} :=$ the kth distributional derivative of f. Let us recall that every locally integrable function on $\mathbb{R}$ can be canonically identified with an element of $\mathcal{D}'$. (Such distributions are called *regular*.) A prominent member of $\mathcal{D}'$ is δ (Dirac measure at 0).

The symbol $\mathcal{D}'_+$ will stand for the space of distributions with support contained in $\mathbb{R}_0^+$ ($\mathbb{R}_0^+ := [0, \infty)$.) It is well known that $\mathcal{D}'_+$ is a convolution algebra; in particular, if f and g belong to $\mathcal{D}'_+$ then their convolution product $f * g$ exists and belongs to $\mathcal{D}'_+$. Let us observe that $\delta^{(i)} \in \mathcal{D}'_+$, and $\delta^{(i)} * f = f * \delta^{(i)} = f^{(i)}$, for $i = 0, 1, \ldots$ and $f \in \mathcal{D}'_+$ (of course, the formula is also valid for all $f \in \mathcal{D}'$).

Let f be a function $\mathbb{R} \to \mathbb{R}$. We say that f is smooth on $\mathbb{R}_0^+$ iff $f|(-\infty, 0) = 0$ ('$|$' means restriction) and there exist $\varepsilon > 0$ and an infinitely differentiable function $\tilde{f} : (-\varepsilon, \infty) \to \mathbb{R}$ such that $f|\mathbb{R}_0^+ = \tilde{f}|\mathbb{R}_0^+$. It is obvious that every function smooth on $\mathbb{R}_0^+$ defines a regular distribution which belongs to $\mathcal{D}'_+$. Following [14], we denote by $\mathcal{C}_{\text{imp}}$ the smallest subalgebra of $\mathcal{D}'_+$ containing δ, $\delta^{(1)}$ and all f smooth on $\mathbb{R}_0^+$. The elements of $\mathcal{C}_{\text{imp}}$ are called *impulsive-smooth* distributions. It is well known and not difficult to prove that for any $f \in \mathcal{C}_{\text{imp}}$ there exist a nonnegative integer k and a function f_{sm} smooth on $\mathbb{R}_0^+$ such that

$$f = f_{\text{sm}} + \sum_{i=0}^{\infty} f_{-i} \delta^{(i)}$$

for some real numbers f_{-i}, which vanish for $i > k$. Let us note that the smooth (on $\mathbb{R}_0^+$) function f_{sm} and numbers f_{-i} are uniquely defined by f. In particular, $f_{\text{imp}} := \sum_{i=0}^{\infty} f_{-i} \delta^{(i)}$ is also uniquely defined by f. For any $f \in \mathcal{C}_{\text{imp}}$, f_{sm} and f_{imp} are respectively called the *smooth* and *impulsive* part of f. We say that an impulsive-smooth distribution f is smooth iff $f = f_{\text{sm}}$, and f is impulsive iff $f = f_{\text{imp}}$. For any $f \in \mathcal{C}_{\text{imp}}$, $f(0+) := f_{\text{sm}}(0)$.

At this point it is reasonable to note that from the control-theoretic point of view the space $\mathcal{C}_{\text{imp}}$ has some disadvantages one of them being the fact that the space is not invariant with respect to right translations.

Let $f \in \mathcal{D}'$ be a regular distribution satisfying the following conditions:

(a) $f|(-\infty, 0) = 0$ (so $f \in \mathcal{D}'_+$), and

(b) $f|(0, \infty)$ is (equivalent to) a locally absolutely continuous function for which $f(0+) := \lim_{t \to 0+} f(t)$ exists.

Then the (classical) derivative of f, to be denoted by f', exists almost everywhere (on $\mathbb{R}$), is locally integrable, and $f^{(1)}$, the distributional derivative of f, is related to f' in the following way:

$$f^{(1)} = f' + f(0+)\,\delta. \tag{2}$$

Of course, (2) can be also applied to smooth distributions from $\mathcal{C}_{\text{imp}}$.

In the rest of the paper we use standard notion and elementary results of distribution theory which are presented e.g. in [15, 19]. We are assuming that the reader is familiar with basic facts from the theory of distributions; as a consequence we are using many results from [15, 19] without quoting them explicitly.

2. IMPULSIVE-SMOOTH SOLUTIONS OF IMPLICIT LINEAR DIFFERENTIAL EQUATIONS

We begin by formulating a distributional version of equation (1). We shall assume that E, $F \in \mathbb{R}^{q\times n}$ and $G \in \mathbb{R}^{q\times m}$. Let $x : \mathbb{R} \to \mathbb{R}^n$ be locally absolutely continuous on $\mathbb{R}_0^+$ and $u : \mathbb{R} \to \mathbb{R}^m$ be locally integrable on $\mathbb{R}_0^+$. Let $x|(-\infty,0) = 0$ and $u|(-\infty,0) = 0$ (so $x \in (\mathcal{D}'_+)^n$ and $u \in (\mathcal{D}'_+)^m$). Assume also that $x|\mathbb{R}_0^+$ and $u|\mathbb{R}_0^+$ satisfy almost everywhere on $\mathbb{R}_0^+$ equation (1). It is immediate that then x and u satisfy (1) almost everywhere on $\mathbb{R}$, i.e. $Ex' = Fx + Gu$. Put $\overline{x} := x(0+)$ and use formula (2). It follows that x and u, treated as (vector-valued) distributions, satisfy the following equation over $\mathcal{D}'_+$:

$$\boxed{Ex^{(1)} = Fx + Gu + E\overline{x}_0\delta.} \tag{3}$$

The equation can also be written in the following equivalent form:

$$\left(E\delta^{(1)} - F\delta\right) * x = Gu + E\overline{x}_0\delta. \tag{4}$$

In the rest of the paper we shall concentrate on properties of equation (3) (or (4)). We call x a *solution* of (3) (or (4)) *with initial condition* $\overline{x}_0$, *corresponding to input* u, iff x, u and $\overline{x}_0$ satisfy (3). Unless otherwise stated we assume that x and u are impulsive-smooth, i.e. $x \in (\mathcal{C}_{\text{imp}})^n$ and $u \in (\mathcal{C}_{\text{imp}})^m$.

Let $x \in (\mathcal{C}_{\text{imp}})^n$ such that

$$x = x_{\text{sm}} + x_{\text{imp}} = x_{\text{sm}} + \sum_{i=0}^{\infty} x_{-i}\delta^{(i)} \tag{5}$$

be a solution of (4), corresponding to an initial condition $\overline{x}_0$ and an input $u \in (\mathcal{C}_{\text{imp}})^n$, where u is given by

$$u = u_{\text{sm}} + u_{\text{imp}} = u_{\text{sm}} + \sum_{i=0}^{\infty} u_{-i}\delta^{(i)}. \tag{6}$$

We determine the smooth part of x and the coefficients x_{-i}. The left-hand side of (4) can be rewritten in the following form:

$$\begin{aligned}\left(E\delta^{(1)} - F\delta\right) * x &= \left(E\delta^{(1)} - F\delta\right) * x_{\rm sm} + \left(E\delta^{(1)} - F\delta\right) * x_{\rm imp} = \\ &= (Ex'_{\rm sm} + Ex_{\rm sm}(0)\,\delta - Fx_{\rm sm}) + \left(\sum_{i=0}^{\infty} Ex_{-i}\delta^{(i+1)} - \sum_{i=0}^{\infty} Fx_{-i}\delta^{(i)}\right) = \\ &= (Ex'_{\rm sm} + Ex_{\rm sm}(0)\,\delta - Fx_{\rm sm}) + \left(-Fx_0\delta + \sum_{i=1}^{\infty}(Ex_{-i+1} - Fx_{-i})\,\delta^{(i)}\right).\end{aligned}$$

The right-hand side of (4) is equal to

$$\begin{aligned}Gu + E\overline{x}_0\delta &= Gu_{\rm sm} + Gu_{\rm imp} + E\overline{x}_0\delta = \\ &= Gu_{\rm sm} + \sum_{i=0}^{\infty} Gu_{-i}\delta^{(i)} + E\overline{x}_0\delta.\end{aligned}$$

It follows that the smooth part of x satisfies the following differential equation:

$$\boxed{Ex'_{\rm sm} = Fx_{\rm sm} + Gu_{\rm sm}.} \tag{7}$$

The coefficients x_{-i} of the impulsive part of x are related to the coefficients u_{-i} of the impulsive part of u by the following difference equation:

$$\boxed{Ex_{-i+1} = Fx_{-i} + Gu_{-i},} \tag{8}$$

where $i = 1, 2, \ldots$. Let us recall that both x_{-i} and u_{-i} vanish for large i.

A relation between $x_{\rm sm}(0)$, x_0, u_0 and $\overline{x}_0$ is given by

$$\boxed{Ex_{\rm sm}(0) = Fx_0 + Gu_0 + E\overline{x}_0.} \tag{9}$$

Reversing the above argument one can easily check that the following result holds true.

Lemma 1. Let $x \in (\mathcal{C}_{\rm imp})^n$ and $u \in (\mathcal{C}_{\rm imp})^m$ be given by (5) and (6). Then x is a solution of (4) with initial condition $\overline{x}_0$, corresponding to the input u, if and only if $x_{\rm sm}$, x_{-i}, $u_{\rm sm}$ and u_{-i} satisfy the relations (7), (8) and (9).

We end this section with the following remark concerning equation (7).

Remark 1. The equality $Ex'_{\rm sm} = Fx_{\rm sm} + Gu_{\rm sm}$, i.e. equation (7) is to be satisfied in the distributional sense. However both sides of the equality are regular and even smooth on $\mathbb{R}_0^+$. It follows that (7) holds true if and only if the 'classical' differential equation $E\left(\frac{\rm d}{{\rm d}t}\, x_{\rm sm}(t)\right) = Fx_{\rm sm}(t) + Gu_{\rm sm}(t)$ is satisfied for $t \in \mathbb{R}_0^+$. This property allows to determine the subspace of $\mathbb{R}^n$ to which the initial values of $x_{\rm sm}$ should belong. More precisely, let $\boldsymbol{V}_{\rm cl} := \{\xi \in \mathbb{R}^n \,|\, \exists$ smooth $x_{\rm sm}$ and $u_{\rm sm}$ satisfying (7), with $x_{\rm sm}(0) = \xi\}$. Then a slight modification of the proof of Thm. 5.1 of [20] leads to the equality $\boldsymbol{V}_{\rm cl} = \boldsymbol{V}(E, F, G)$, where the space $\boldsymbol{V}(E, F, G)$ (which is defined in the next section) can be determined explicitly by algebraic means.

3. SPACES OF ADMISSIBLE INITIAL CONDITIONS

As it was previously mentioned, the existence of solutions of equation (1) is not guaranteed for some initial conditions. The same may happen also for the distributional version of equation (1), i.e. for equation (3). We find now the space of admissible initial conditions. For any initial condition from this space there exists at least one impulsive-smooth solution of equation (3).

Let the spaces $\boldsymbol{V}_i(E,F,G) \subset \mathbb{R}^n$ be defined by the following formula:

$$\boldsymbol{V}_0(E,F,G) := \mathbb{R}^n,$$
$$\boldsymbol{V}_{i+1}(E,F,G) := F^{-1}\left(E(\boldsymbol{V}_i(E,F,G)) + \mathrm{Im}(G)\right),$$

for $i = 0, 1, \ldots$.

Similarly, let the spaces $\boldsymbol{R}_i(E,F,G) \subset \mathbb{R}^n$ be defined as follows

$$\boldsymbol{R}_0(E,F,G) := \{0\} \text{ (i.e., the zero subspace of } \mathbb{R}^n\text{)},$$
$$\boldsymbol{R}_{i+1}(E,F,G) := E^{-1}\left(F(\boldsymbol{R}_i(E,F,G)) + \mathrm{Im}(G)\right),$$

for $i = 0, 1, \ldots$.

It easy to observe (and well known, see e.g. [2]) that the sequence $(\boldsymbol{V}_i(E,F,G))$ is decreasing and the sequence $(\boldsymbol{R}_i(E,F,G))$ is increasing. It follows that for $i \geq n$, the sequences $(\boldsymbol{V}_i(E,F,G))$ and $(\boldsymbol{R}_i(E,F,G))$ are constant. We shall denote respectively by $\boldsymbol{V}(E,F,G)$ and $\boldsymbol{R}(E,F,G)$, the limits of those sequences.

Let

$$\boldsymbol{V} := \{\overline{x}_0 \in \mathbb{R}^n \mid \exists \text{ smooth } x_{\mathrm{sm}} \text{ and } u_{\mathrm{sm}} \text{ satisfying (3)}\}.$$

In other words, $\boldsymbol{V}$ is the space of all such $\overline{x}_0 \in \mathbb{R}^n$, for which one can find $x \in (\mathcal{C}_{\mathrm{imp}})^n$ and $u \in (\mathcal{C}_{\mathrm{imp}})^m$ such that $x = x_{\mathrm{sm}}$, $u = u_{\mathrm{sm}}$ (so that they are smooth), and $Ex^{(1)} = Fx + Gu + E\overline{x}_0\delta$.

Proposition 1. $\boldsymbol{V} = \boldsymbol{V}(E,F,G) + \mathrm{Ker}\, E$.

Proof. In view of Lemma 1, it is obvious that smooth x_{sm} and u_{sm} satisfy (3) if and only if they satisfy (7) and (cf. (9)) $Ex_{\mathrm{sm}}(0) = E\overline{x}_0$. Since the space of values of $x_{\mathrm{sm}}(0)$ coincides with $\boldsymbol{V}(E,F,G)$ (cf. Remark 1 in Section 1), the equality $Ex_{\mathrm{sm}}(0) = E\overline{x}_0$ holds for some $x_{\mathrm{sm}}(0)$ if and only if $E\overline{x}_0 \in E\boldsymbol{V}(E,F,G)$. Therefore the result is immediate. □

Let

$$\boldsymbol{R} := \{\overline{x}_0 \in \mathbb{R}^n \mid \exists \text{ impulsive } x_{\mathrm{imp}} \text{ and } u_{\mathrm{imp}} \text{ satisfying (3)}\}.$$

In other words, $\boldsymbol{R}$ is the space of all such $\overline{x}_0 \in \mathbb{R}^n$, for which one can find $x \in (\mathcal{C}_{\mathrm{imp}})^n$ and $u \in (\mathcal{C}_{\mathrm{imp}})^m$ such that $x = x_{\mathrm{imp}}$, $u = u_{\mathrm{imp}}$ (so that they are impulsive), and $Ex^{(1)} = Fx + Gu + E\overline{x}_0\delta$.

Proposition 2. $\boldsymbol{R} = \boldsymbol{R}(E, F, G)$.

Proof. Let x_{imp} and u_{imp} be impulsive distributions given by

$$x_{\text{imp}} = \sum_{i=0}^{\infty} x_{-i}\delta^{(i)} \quad \text{and} \quad u_{\text{imp}} = \sum_{i=0}^{\infty} u_{-i}\delta^{(i)},$$

respectively. In view of Lemma 1, x_{imp} and u_{imp} satisfy (3) if and only if the sequences (x_{-i}) and (u_{-i}) satisfy equation (8) and (cf. (9)) $E(-\overline{x}_0) = Fx_0 + Gu_0$. It follows that $(-\overline{x}_0)$ can be characterized as the 'final' value of an 'x-trajectory' of (8), with i running through nonnegative integers. Now it is sufficient to note that (in view of results of Section 2 of [2]) these space coincides with $\boldsymbol{R}(E, F, G)$ so that $\boldsymbol{R} \subset \boldsymbol{R}(E, F, G)$. The inclusion $\boldsymbol{R}(E, F, G) \subset \boldsymbol{R}$ can be shown similarly, by constructing an appropriate x_{imp} and u_{imp}. □

Let

$$\boldsymbol{W} := \{\overline{x}_0 \in \mathbb{R}^n \mid \exists \text{ impulsive-smooth } x \text{ and } u \text{ satisfying (3)}\}.$$

Proposition 3. $\boldsymbol{W} = \boldsymbol{V}(E, F, G) + \boldsymbol{R}(E, F, G)$.

Proof. The inclusion $\boldsymbol{V}(E, F, G) + \boldsymbol{R}(E, F, G) \subset \boldsymbol{W}$ is a direct consequence of Propositions 1 and 2, and linearity of (3). To prove the reverse inclusion let us observe that there exist impulsive-smooth x and u satisfying (3) if and only if the equalities (7), (8) and (9) hold, with x and u given by (5) and (6). Using the argument presented in the proofs of Propositions 2 and 3, we can easily check that $x_0 \in \boldsymbol{R}(E, F, G)$ and $x_{\text{sm}}(0) \in \boldsymbol{V}(E, F, G)$. It follows that

$$\overline{x}_0 \in E^{-1}(E\boldsymbol{V}(E, F, G) + F\boldsymbol{R}(E, F, G) + \operatorname{Im} G) = \boldsymbol{V}(E, F, G) + \boldsymbol{R}(E, F, G),$$

where the last equality is a consequence of the modular distributive law, the inclusion (cf. [2]) $E\boldsymbol{R}(E, F, G) \subset F\boldsymbol{R}(E, F, G) + \operatorname{Im} G$ and the obvious relation $\operatorname{Ker} E \subset \boldsymbol{R}(E, F, G)$. □

Remark 2. Let us note that $\boldsymbol{V}(E, F, G) + \boldsymbol{R}(E, F, G)$ is the largest almost invariant subspace of a suitably defined implicit discrete-time system (cf. [1]). In the case when the pencil $(\lambda E - F)$ is regular the space $\boldsymbol{W}$ coincides with the whole space $\mathbb{R}^n$ since then $\mathbb{R}^n = \boldsymbol{V}(E, F, 0) \oplus \boldsymbol{R}(E, F, 0)$.

4. CONTROLLABLE SPACE

Let

$$\boldsymbol{C}(E, F, G) := \boldsymbol{V}(E, F, G) \cap \boldsymbol{R}(E, F, G)$$

and

$$\boldsymbol{C}_{i-s} := \{\xi \in \mathbb{R}^n \mid \exists \text{ impulsive-smooth } x \text{ and } u \text{ satisfying (3) with } \overline{x}_0 = 0 \text{ and } x(0+) = \xi\}.$$

Proposition 4. $\boldsymbol{C}_{i-s} = \boldsymbol{C}(E, F, G)$.

Proof. Since $x(0+) = x_{\mathrm{sm}}(0)$, $\boldsymbol{C}_{i-s}$ is contained in $\boldsymbol{V}(E, F, G)$ (cf. Remark 1). On the other hand, the inclusion $\boldsymbol{C}_{i-s} \subset \boldsymbol{R}(E, F, G)$ can be easily proved using the relation (cf. (9)) $Ex_{\mathrm{sm}}(0) = Fx_0 + Gu_0$. Hence $\boldsymbol{C}_{i-s} \subset \boldsymbol{C}(E, F, G)$. The reverse inclusion can be proved in the same way. □

We can now prove the following result about the uniqueness of solutions of equation (3) (see also [20]):

Proposition 5. $x = 0$ is the only impulsive-smooth solution of (3), with $u = 0$ and $\overline{x}_0 = 0$, if and only if $\boldsymbol{C}(E, F, 0) = 0$.

Proof. ($\Rightarrow$) Suppose $\boldsymbol{C}(E, F, 0) \neq 0$. Then Proposition 4 ensures the existence of a solution x of (3), with $u = 0$ and $\overline{x}_0 = 0$, and satisfying the condition $x(0+) \neq 0$. Such x cannot be equal to 0.

($\Rightarrow$) Denote respectively by $\tilde{E}$ and $\tilde{F}$ the restriction of E and F to the subspace $\boldsymbol{V}(E, F, 0)$. It is well known (see e.g. [2]) that the equality $\boldsymbol{C}(E, F, 0) = 0$ holds true if and only if $\tilde{E}$ injective. In this case there exists a left inverse, to be denoted by L, of $\tilde{E}$. Multiplying by L both sides of (3), with $u = 0$ and $\overline{x}_0 = 0$, we see that x satisfies $x^{(1)} = L\tilde{F}x$, i.e. a standard homogeneous differential equation, with $\overline{x} = 0$. Hence $x = 0$. □

Let $\boldsymbol{S}$ be a subspace of $\mathbb{R}^n$. We say that *values of a distribution* $x \in (\mathcal{D}')^n$ lie in $\boldsymbol{S}$ iff for every $\varphi \in \mathcal{D}$, $x(\varphi) \in \boldsymbol{S}$ (of course, $x(\varphi)$ denote the vector from $\mathbb{R}^n$ whose ith coordinate is equal to the value of the ith coordinate of x on φ). Let $\boldsymbol{C}_{\mathrm{dv}}$ denote the smallest subspace of $\mathbb{R}^n$ in which lie the *values of all impulsive-smooth solutions of equation* (3), with $\overline{x}_0 = 0$.

To prove our next result we shall need the following lemma being an immediate consequence of [3, Thm. 1].

Lemma 2. There exist subspaces $\boldsymbol{X} \subset \mathbb{R}^n$ and $\boldsymbol{Z}_1$, $\boldsymbol{Z}_2 \subset \mathbb{R}^q$ such that $\mathbb{R}^q = \boldsymbol{Z}_1 \oplus \boldsymbol{Z}_2$, $\mathbb{R}^n = \boldsymbol{C}(E, F, G) \oplus \boldsymbol{X}_2$ and, with respect to this decomposition, the linear mappings E, F and G can be represented respectively by

$$\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \quad \begin{bmatrix} F_{11} & F_{12} \\ 0 & F_{22} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} G_1 \\ 0 \end{bmatrix},$$

with the property that $\boldsymbol{C}(E_{11}, F_{11}, G_1) = \boldsymbol{C}(E, F, G)$ and $\boldsymbol{C}(E_{22}, F_{22}, 0) = 0$.

Proposition 6. $\boldsymbol{C}_{\mathrm{dv}} = \boldsymbol{C}_{i-s}$.

Proof. The inclusion $\boldsymbol{C}_{\mathrm{dv}} \subset \boldsymbol{C}_{i-s}$ is an easy consequence of the decomposition of Lemma 2, which together with Proposition 5 ensures that the values of any impulsive-smooth x satisfying (3), with $\overline{x}_0 = 0$, lie in $\boldsymbol{C}(E_{11}, F_{11}, G_1) = \boldsymbol{C}(E, F, G) = \boldsymbol{C}_{i-s}$.

To show that $C_{i-s} \subset C_{\mathrm{dv}}$ consider any η belonging to the orthogonal complement of C_{dv} so that $\eta^{\mathrm{T}}x(\varphi) = 0$, for all $\varphi \in \mathcal{D}$ and all impulsive-smooth solutions x of (3), with $\overline{x}_0 = 0$. Then also $\eta^{\mathrm{T}}x_{\mathrm{sm}}(\varphi) = 0$ so that the regular distribution $\eta^{\mathrm{T}}x_{\mathrm{sm}}$ is zero. Since $\eta^{\mathrm{T}}x_{\mathrm{sm}}$ is smooth on $\mathbb{R}_0^+$, we obtain the equality $\eta^{\mathrm{T}}(0+) = 0$ and hence (because x is an arbitrary solution of (3), with $\overline{x}_0 = 0$) $\eta^{\mathrm{T}}\xi = 0$, for all $\xi \in C_{i-s}$. □

Let $\mathcal{C}_+^\infty$ denote the space of all infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$ with support contained in $\mathbb{R}_0^+$. It is obvious that $\mathcal{C}_+^\infty \subset \mathcal{C}_{\mathrm{imp}}$. Let us also observe that for every $f \in \mathcal{C}_{\mathrm{imp}}$, $f * \varphi$ is a well defined element of $\mathcal{C}_+^\infty$, whenever $\varphi \in \mathcal{D}$ and the support of φ lies in $\mathbb{R}_0^+$.

Consider

$$C := \{\xi \in \mathbb{R}^n \mid \exists\, x \in (\mathcal{C}_+^\infty)^n \text{ and } u \in (\mathcal{C}_+^\infty)^m \text{ satisfying (3) with } \overline{x}_0 = 0 \text{ and such that } x(T) = \xi, \text{ for some } T > 0\}.$$

Since $\mathcal{C}_+^\infty$ is invariant with respect to right translations, C is a linear subspace of $\mathbb{R}^n$.

Proposition 7. $C = C_{i-s}$.

Proof. In view of Propositions 6 and 4, we always have $C \subset C_{\mathrm{dv}} = C_{i-s} = C(E,F,G)$. It follows that to prove the equality $C = C_{i-s}$ it is sufficient to show that the intersection of C_{i-s} and the orthogonal complement of C is 0. For this, consider an arbitrary η belonging to the orthogonal complement of C. It is obvious that there exist impulsive-smooth x and u satisfying (3) with $\overline{x}_0 = 0$ and $x(0+) = \eta$. Let $\varphi \in \mathcal{D}$ has its support contained in $\mathbb{R}_0^+$ and $x_\varphi = x * \varphi$, $u_\varphi := u * \varphi$. Then $x_\varphi \in (\mathcal{C}_+^\infty)^n$, $u_\varphi(\mathcal{C}_+^\infty)^m$ and they satisfy (3) with $\overline{x}_0 = 0$. It is immediate that $\eta^{\mathrm{T}}x_\varphi = 0$. Consider now a family of $\varphi \in \mathcal{D}$ such that $0 \le \varphi$, $\int \varphi = 1$ and $\mathrm{supp}(\varphi) \to \{1\}$. Then $x_\varphi \to x * \delta_1$ in $\mathcal{D}'$; δ_1 stands for the Dirac measure at 1. It is obvious that $0 = \eta^{\mathrm{T}}x_\varphi \to \eta^{\mathrm{T}}(x * \delta_1)$. It follows that $\eta^{\mathrm{T}}x = 0$; in particular $\eta^{\mathrm{T}}x_{\mathrm{sm}} = 0$. Hence $\eta^{\mathrm{T}}x(0+) = 0$, and therefore $\eta^{\mathrm{T}}\eta = 0$. Hence $\eta = 0$, and we have proved that $C = C_{i-s}$, as required. □

Remark 3. The above proposition improves various results (see e. g. Thm. 4.4 of [8]) on controllability of implicit systems by considering only "very" regular solutions of (3).

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