Kybernetika

Tran Quoc Chien

Perturbation theory of duality in vector nonconvex optimization via the abstract duality scheme

Kybernetika, Vol. 25 (1989), No. 3, 169--174

Persistent URL: http://dml.cz/dmlcz/125086

KYBERNETIKA — VOLUME 25 (1989), NUMBER 3

PERTURBATION THEORY OF DUALITY IN VECTOR NONCONVEX OPTIMIZATION VIA THE ABSTRACT DUALITY SCHEME

TRAN QUOC CHIEN

In the present paper the nonconvex perturbation duality of Lindberg (see [1] §8) is extended for vector optimization by means of the abstract duality scheme.

1. INTRODUCTION

The perturbation duality theory is known for its generality and nice properties of symmetry. However, the traditional approach via the theory of conjugate functionals can hardly be converted to vector optimization for the difficulties connected with the high-dimensional structure.

In Tran Quoc Chien [3] the perturbation duality is successfully established for vector convex optimization by the abstract duality scheme. This work can be regarded as the nonconvex analogy of the mentioned one.

Throughout this work we suppose that all spaces in question are real and V is an ordered linear space if no other requirements are added. The positive cone V_+ is assumed to have nonempty core ($\operatorname{cor} V_+ \neq \emptyset$).

All concepts and notations appearing in the paper (e.g. the binary relations $>$, $\geqq$, V_+-quasiinterval, supremal, infimal $\operatorname{Sup}_\Omega A$, $\operatorname{Inf}_\Omega A$ etc.) are introduced in Tran Quoc Chien [2, 4]. The general duality principles used in this paper are referred to [2, 4], too.

2. NONCONVEX PERTURBATIONAL DUALITY

Given a set X and a function $f\colon X \to \bar{V}$, we shall be concerned with the problem

$$\text{(P)} \qquad \operatorname{Max-Sup} f(X)\,.$$

Chosen a set Q and a suitable function $\phi\colon X \times Q \to \bar{V}$ with $\phi(x, 0) = f(x)$ for all

$x \in X$, a family of problems

$$(\mathrm{P}_q) \qquad \text{Max-Sup}\ \phi(X, q)$$

will be taken into consideration. The function ϕ resp. the problem (P_q) are termed the perturbed objective resp. the perturbed primal problem.

Now in order to develop a duality theory, we shall extend the concept upper ψ-regularity of Lindberg (see [1] §8) for multivalued functions.

Let Y be an arbitrary set and ψ: $Q \times Y \to V$ be a given function, termed the basic function. Let g: $Q \to V$ be a multivalued function, one sets

$$M(g) = \{(y, v) \in Y \times V |\, g(q) \gtreqqless \psi(q, y) + v \quad \forall q \in Q\}.$$

2.1. Definition. A function $g(q)$ is said to be *upper ψ-regular at $q \in Q$* if

$$\operatorname{Sup} g(q) \subset \operatorname{Inf}\{\psi(q, y) + v \mid (y, v) \in M(g)\}.$$

Following the abstract duality scheme we set

$$\mathscr{P} = Q \times V, \quad P(v) = E \cap E_v,$$

where

$$E = \bigcup_{x \in X} \{(q, v) \in \mathscr{P} \mid v \leqq \phi(x, q)\}$$

and

$$E_v = \{0\} \times (v + V_+).$$

The function P: $V \to \mathscr{P}$ evidently satisfies the primal availability. Put

$$\begin{aligned} \mathscr{P}_0 &= \{(q, v) \in \mathscr{P} \mid \exists v' \in V: (q, v) \in P(v')\} \\ &= \{(0, v) \in \mathscr{P} \mid \exists x \in X: v \leqq \phi(x, 0) = f(x)\} \end{aligned}$$

and

$$\mu(0, v) = v - V_+ \quad \forall (0, v) \in \mathscr{P}_0.$$

Obviously

$$\mu(\mathscr{P}_0) = f(X) - V_+,$$

hence the problem (P) is equivalent to the following

$$(\mathrm{AP}) \qquad \text{Max-Sup}\ \mu(\mathscr{P}_0).$$

In the sequel we require that the basic function $\psi(q, y)$ satisfies the following assumption:

$$\psi(0, y) = 0 \quad \forall y \in Y.$$

Now set

$$\mathscr{D} = Y,$$

$$D(v) = \{y \in Y |\, v + \psi(q, y) \gtreqless \phi(x, q)\ \forall (x, q) \in X \times Q\},$$

$$\nu(y) = \{v \in V |\, y \in D(v)\}$$

and

$$\mathscr{D}_0 = \{y \in Y |\, \nu(y) \neq \emptyset\}.$$

The problem

(AD) $$\text{Min-Inf}\, v(\mathscr{D}_0)$$

is then called the (ϕ, ψ)-*dual* to the problem (P) (the dual availablility is evidently satisfied).

2.2. Theorem (weak duality). The weak duality condition is satisfied, hence

$$f(X) \not> v(\mathscr{D}_0)\,.$$

Proof. Let $v \in V$ with $D(v) \neq \emptyset$. If, on the contrary, the weak duality condition is not satisfied, there exists a $v' \in V$ such that $v' > v$ and $P(v') \neq \emptyset$. Then there exists an $x \in X$ with

$$v' \leqq \phi(x, 0) = f(x)\,. \tag{2.2.1}$$

On the other side we have

$$v + \psi(q, y) \not< \phi(x, q) \quad \forall (x, q) \in X \times Q\,.$$

Particularly, for $q = 0$

$$v \not< \phi(x, 0) = f(x) \quad \forall x \in X$$

which contradicts (2.2.1). Hence the weak duality condition must be satisfied. □

Now we set

$$h(q) = \text{Sup}\, \phi(X, q) \quad \forall q \in Q\,.$$

2.3. Lemma. If $V_{++} = \text{cor}\, V_+$, then

$$D(v) = \{y \in Y \mid (y, v) \in M(h)\}$$

for all $v \in V$.

Proof. This lemma follows from the following equivalences

$$\begin{aligned} & v + \psi(q, y) \not< \phi(x, q) && \forall (x, q) \in X \times Q \\ \Leftrightarrow\ & v + \psi(q, y) \not< \text{Sup}\, \phi(X, q) && \forall q \in Q \\ \Leftrightarrow\ & v + \psi(q, y) \not< h(q) && \forall q \in Q \end{aligned}$$ □

2.4. Theorem (strong duality). Suppose that $h(q)$ is ψ-regular at $0 \in Q$ and $V_{++} = \text{cor}\, V_+$, then

$$\text{Sup}\, f(X) = \text{Inf}\, v(\mathscr{D}_0)\,.$$

Proof. Since $h(q)$ is inside stable and ψ-regular at 0 one has

$$\begin{aligned} h(0) = \text{Sup}\, h(0) &\subset \text{Inf}\,\{\psi(0, y) + v \mid (y, v) \in M(h)\} \\ &= \text{Inf}\,\{v \mid \exists y\colon (y, v) \in M(h)\} \\ &= \text{Inf}\, v(\mathscr{D}_0) \quad \text{by Lemma 2.3}\,. \end{aligned}$$

Let $v' \in V$ be such that $P(v) = \emptyset\ \forall v > v'$. Fix a $v > v'$. Choose an arbitrary $v_+ \in \text{cor}\, V_+$, then

$$S = (f(X) - V_+) \cap \{v + tv_+ \mid t \leqq 0\} \neq \emptyset\,.$$

Put $t_0 = \sup \{t \mid v + tv_+ \in S\}$, then obviously $v_0 = v + t_0 \cdot v_+ \in h(0)$ with $v_0 < v$. Then by the inclusion above there exists $y \in Y$ such that $(y, v) \in M(h)$. Hence $D(v) \neq \emptyset$ by Lemma 2.3. The Sup-Inf strong duality condition is thus satisfied and by the Sup-Inf strong duality principle (see [2, 4]) we obtain

$$\operatorname{Sup} f(X) = \operatorname{Inf} v(\mathscr{D}_0) . \qquad \square$$

2.5. Definition. Let Q be endowed with a topology. A multivalued function $g: Q \to V$ is called *upper semicontinuous at* $q_0 \in Q$ if $\emptyset \neq g(q_0) \subset V$ and

$$\forall v_+ \in \operatorname{cor} V_+ \ \exists \text{ a neighbourhood } U \text{ of } q_0 \text{ such that}$$

$$\forall q \in U \quad \forall v \in g(q) \quad \exists v_0 \in g(q_0): v \leqq v_0 + v_+ .$$

2.6. Definition. A function $\psi: Q \times Y \to V$ is *sharp at* $0 \in Q$ if

$$\forall y_0 \in Y \quad \forall v \in V \quad \exists v_+ \in \operatorname{cor} V_+ \quad \forall \text{ 0-neighbourhood } U \text{ in } Q,$$

$$\exists y_1 \in Y \quad \exists \text{ 0-neighbourhood } U_0 \subset U:$$

$$\psi(q, y_1) \geqq \psi(q, y) + v \quad \forall q \in Q \setminus U_0$$

$$\psi(q, y_1) \geqq -v_+ \qquad \forall q \in U_0 .$$

2.7. Theorem. Suppose that $\psi(q, y)$ is sharp at $0 \in Q$, $h(q)$ is upper semicontinuous at 0 and $M(h) \neq \emptyset$. Then $h(q)$ is upper ψ-regular at 0.

Proof. Fix $v_* \in \operatorname{Sup} h(0)$, $v_+ \in \operatorname{cor} V_+$. Choose an arbitrary pair $(y_0, v_0) \in M(h) \neq \emptyset$ with

$$\psi(q, y_0) + v_0 \not\leqq h(q) \quad \forall q \in Q . \tag{2.7.1}$$

Put

$$v = v_0 - v_* - 2v_+ .$$

Since $h(q)$ is upper semicontinuous at 0 there exists a 0-neighbourhood U such that Definition 2.5 is satisfied for v_+. By virtue of the sharpness of ψ at $0 \in Q$ there exists a 0-neighbourhood $U_0 \subset U$ and a $y_+ \in Y$ such that

$$\psi(q, y_+) \geqq \psi(q, y_0) + v \quad \forall q \in Q \setminus U_0$$

and

$$\psi(q, y_+) \geqq -v_+ \quad \forall q \in U_0 .$$

Putting $w_+ = v_* + 2v_+$ we have

$$\psi(q, y_+) + w_+ = \psi(q, y_+) + v_* + 2v_+ \geqq$$

$$\geqq \psi(q, y_0) + v_0 \not\leqq h(q) \quad \forall q \in Q \setminus U_0$$

and

$$\psi(q, v_+) + w_+ \geqq -v_+ + v_* + 2v_+ = v_* + v_+ \not\leqq h(q) \quad \forall q \in U_0 .$$

The last inequality follows from the upper semicontinuity of $h(q)$ at 0. Indeed, if there is a $v' \in h(q)$ with $v_* + v_+ < v'$ then by the upper semicontinuity there exists a $v'' \in h(0)$ such that $v' \leqq v'' + v_+$. Consequently $v_* < v''$ which contradicts $v_* \in \operatorname{Sup} h(0)$.

Hence we obtain $(y_+, w_+) \in M(h)$ and $w_+ \to v_*$ when $v_+ \to 0$ on a line passing 0. The assertion is thus proved. $\square$

2.8. Theorem. Suppose that X is a reflexive Banach space, Q is endowed with a topology, $\phi(x, q)$ is weakly (with respect to X) upper semicontinuous at $(x, 0)$ for all $x \in X$ and functions $\phi_q(x) = \phi(x, q)$ are upper coercive on X uniformly on some 0-neighbourhood in Q, i.e.

$$\forall v \in V\colon \bigcup_{q \in U} \{x \mid \phi_q(x) \gtrneqq v\} \quad \text{is bounded} .$$

Then $h(q)$ is upper semicontinuous at 0.

Proof. Suppose, on the contrary, that there exist a $v_+ \in \operatorname{cor} V_+$ and a net $\{q_\lambda\}$ converging to 0 such that

$$\forall \lambda \quad \exists v_\lambda \in h(q_\lambda)\colon v_\lambda \gtrneqq h(0) + v_+ .$$

Since $v_\lambda \in h(q_\lambda) = \operatorname{Sup} \phi(X, q_\lambda)$, one has

$$\forall \lambda \quad \exists x_\lambda \in X\colon \phi(x_\lambda, q_\lambda) \gtrneqq h(0) + v_+/2 .$$

In virtue of the reflexiveness of X there exists a cluster point $\bar{x}$ of $\{x_\lambda\}$. Now since $(x_\lambda, q_\lambda) \to (\bar{x}, 0)$ and $\phi(x, q)$ is upper semicontinuous at $(\bar{x}, 0)$, we have

$$h(0) \gtrneqq \phi(\bar{x}, 0) \gtrneqq h(0) + v_+/2 ,$$

which is the desired contradiction. $\square$

2.9. Theorem (Inf-Sup formulation). If V_+ is reproducing, i.e. $V_+ - V_+ = V$ and $V_{++} = \operatorname{cor} V_+$, then

$$\operatorname{Inf} v(\mathscr{D}_0) = \operatorname*{Inf}_{\mathscr{D}_0} \operatorname{Sup} \{\phi(x, q) - \psi(q, y) \mid (x, q) \in X \times Q\} .$$

Proof. The proof is similar to that of Theorem 2.13 [7]. $\square$

Now we introduce the so-called Lagrangian function

$$L(y, x) = \operatorname{Sup} \{\phi(x, q) - \psi(q, y) \mid q \in Q\} .$$

From Proposition 1.1 we obtain

2.9. Proposition. If $L(y, x)$ is sup-stable to the with respect to the set

$$\bigcup_{q \in Q} \{\phi(x, q) - \psi(q, y)\}$$

V_+ is reproducing and $V_{++} = \operatorname{cor} V_+$, then

$$\operatorname{Inf} v(\mathscr{D}_0) = \operatorname*{Inf}_{\mathscr{D}_0} \operatorname*{Sup}_{x} L(y, x) .$$

(Received March 24, 1987.)

REFERENCES

[1] J. V. Outrata and J. Jarušek: Duality Theory in Mathematical Programming and Optimal Control. Supplement to Journal Kybernetika, Volume *20* (1984).

[2] Tran Quoc Chien: Fenchel-Lagrange duality in vector fractional programming via the abstract duality scheme. Kybernetika *22* (1986), 4, 299—319.

[3] Tran Quoc Chien: Perturbation theory of duality in vector optimization via the abstract duality scheme. Kybernetika *23* (1987), 1, 67—81.

[4] Tran Quoc Chien: Unification of the abstract duality scheme. Kybernetika *23* (1987), 5, 360—364.

RNDr. Tran Quoc Chien CSc., Department of Mathematics — Danang Polytechnic Institute, Danang. Vietnam.