Antonín Vaněček

Control of initially unknown plants

Kybernetika, Vol. 13 (1977), No. 6, (371)--420

Persistent URL: http://dml.cz/dmlcz/125069

KYBERNETIKA — VOLUME 13 (1977), NUMBER 6

Control of Initially Unknown Plants

ANTONÍN VANĚČEK

The spectrum of control theory problems is treated as the dynamics synthesis; in all those problems of control, reconstruction, compensation, recursive identification, adaptive filtering, and self-adjusting regulator a problem specification is given by the prescription of a transient process for a relevant error and a problem solution is governed by the principle: those coordinates are best which simplify the problem most.

CONTENTS

PREFACE

Concerning the control of initially unknown plants, we limit ourselves to such problems for which we can both understand the structure of solution and guarantee the finite enumerability of this solution. Our paper evolves inductively: we start with control for known state and parameters, then throwing out the hypothesis of known state we continue with the compensator and finally throwing out even the hypothesis of known parameters, we end with the self-adjusting regulator. The leading principle is to use such a model from the class of equivalent models, which makes the solution most straightforward. By the solution of all problems we understand the construction of such feedback gains (of control, reconstruction, and recursive identification) that the initial error decays with prescribed dynamics.

In spite of the fact that we are concerned mostly with the identification, nowhere – with the exception of preface – there will occur the terms which are on the top in the frequency vocabulary of this area, i.e. partly the terms containing the superlatives (least squares, maximum likelihood, minimum variance), partly the terms referring to the distinguished personalities (bayesian estimation, gaussian noise, Kalman filter).

Our treatise is inductive also with respect to the level of complexity of the controlled plant. We start with the system with single state coordinate, continue to several state coordinates and finish with several inputs and outputs.

PART 1 INTRODUCTION: CONTROL OF INITIALLY UNKNOWN GEOMETRIC SEQUENCE

I can't believe that God plays dice (*Einstein*)

In the introductory part we shall concentrate ourselves on the core of control of initially unknown plants. We shall not be ashamed for dealing with the trivia: our model will be the most simple of the possible ones. Let us consider geometric sequence

(1) $$\alpha,\ \lambda\alpha,\ \lambda^2\alpha,\ \lambda^3\alpha,\ \ldots$$

or recursively

(2) $$x_{t+1} = \lambda x_t\,, \quad x_0 = \alpha \quad (t = 0, 1, 2, \ldots)\,.$$

For all initial states α the sequence (2) with increasing time t and $|\lambda| < 1$ decays – it is stable; for $|\lambda| \geqq 1$, (2) is not stable. For orientation about the characteristic value λ it holds: Let λ be in the vicinity of 1. Then for $\lambda < 1 (\lambda > 1$, resp.) (2) decays to one half (doubles, resp.) approximately at the number of steps equal to the ratio of 70 and the distance of λ to 1 given in %. (Let $\frac{1}{2} = (1 - \varepsilon)^t \cdot 1$ where $0 < \varepsilon \ll 1$. Then $t = \ln \frac{1}{2} / \ln (1 - \varepsilon) \approx 70/\varepsilon\%$.)

There are two sources of (2). At first there is given some empirical sequence for several time indices. (This empirical sequence of course does not exactly fit (2). We shall deal this further.) Second, there is given some deductive theory: velocity of change of some variable is proportional to that variable

(3) $$\frac{\mathrm{d}}{\mathrm{d}t} x_t = \mu x_t \, .$$

Solution of (3) for $x_0 = \alpha$, $t \geqq 0$, is $x_t = \mathrm{e}^{\mu t}\alpha$ which at the times $0, T, 2T \ldots$ becomes α, $\mathrm{e}^{\mu T}\alpha$, $\mathrm{e}^{3\mu T}\alpha, \ldots$ and this for $\lambda = \mathrm{e}^{\mu T}$ gives again (1).

Let us return again to recursive relation (2) and let us further suppose that parameter λ is such that the sequence (1) is not feasible, e.g. its state x_t either grows too quickly ($\lambda \gg 1$) or decays too slowly ($0 \ll \lambda < 1$) or does not oscillate regularly ($\lambda \neq -1$). Let λ_{con} be the parameter of such sequence we want to have:

(4) $$x_{t+1} = \lambda_{con} x_t \, .$$

To change our unfeasible sequence (2) we shall control it by the input u_t:

(5) $$x_{t+1} = \lambda x_t + u_t \, , \quad u_t = -k_{con} x_t \, .$$

Comparing (4) and (5) we obtain the control gain

(6) $$k_{con} = \lambda - \lambda_{con} \, .$$

Using this gain, our unfeasible sequence (2) has changed to the sequence (4) we wanted to have. $|\lambda_{con}| < 1$ makes control (5, 6) stable.

Problem. Model the growing consumption of raw materials and modify it by recycling, i.e. reprocessing of the old products as the part of raw materials.

Control (5, 6) was for stable λ_{con} the control to the zero required state: for all α and $t \to \infty$, $x_t \to 0$. For the stable control to the required nonzero state x_r we enlarge the input of (5):

$$u_t = -k_{con} x_t + k_r x_r \, .$$

Then $x_{t+1} = \lambda_{con} x_t + k_r x_r$ and for $t \to \infty$: $x_t \to (k_r/(1 - \lambda_{con}))\, x_r$ which is equal to x_r for $k_r = 1 - \lambda_{con}$. Further $x_{t+1} - x_r = \lambda_{con}(x_t - x_r)$ and the control error with respect to x_r decays again with given characteristic value λ_{con}.

Control (5) was derived from the current state x_t which we generally do not know, knowing only the measurement of this state, the output

(7) $$y_t = x_t + v_t \, .$$

The output disturbance v_t is such that we resign on the construction on its full mathematical model, e.g. because the dimension of this model is too big for the time we have left to control.

The current state x_t we shall reconstruct from the copy of our model using the difference between the measurement and the old reconstruction, amplified by the reconstruction gain k_{rec}:

$$\hat{x}_{t+1} = \lambda \hat{x}_t + u_t + k_{rec}(y_t - \hat{x}_t) \, .$$

 Comparing the behaviour of reconstruction error

$$\tilde{x}_{t+1} =_{def} x_{t+1} - \hat{x}_{t+1} = (\lambda - k_{rec})\,\tilde{x}_t - k_{rec}v_t$$

with the behaviour we want to have (for $v_t = 0$):

$$\tilde{x}_{t+1} = \lambda_{rec}\tilde{x}_t\,,$$

we shall obtain the reconstruction gain

$$k_{rec} = \lambda - \lambda_{rec}\,.$$

For input derived from the current reconstruction of the state

$$u_t = -k_{rec}\hat{x}_t$$

then

(8) $$x_{t+1} = \lambda_{con}x_t + (\lambda - \lambda_{con})\,\tilde{x}_t\,,$$

(9) $$\tilde{x}_{t+1} = \lambda_{rec}\tilde{x}_t + (\lambda - \lambda_{rec})\,v_t\,.$$

For the discussion of control based on reconstruction, let us first suppose $v_t = 0$. Reconstruction error $\tilde{x}_t$ decaying from $\tilde{x}_0$ according to λ_{rec} drives control error x_t. At least after the decay of $\tilde{x}_t$ there will decay, according to λ_{con}, even x_t. Stable reconstruction and control, i.e. $|\lambda_{rec}|, |\lambda_{con}| < 1$ implies stable control based on reconstruction.

Further let us suppose $v_t \neq 0$. We shall demonstrate that the model*)

(10) $$x_{t+1} = \lambda x_t + (1 - \lambda)\,u_t\,, \quad (0 < \lambda < 1)$$

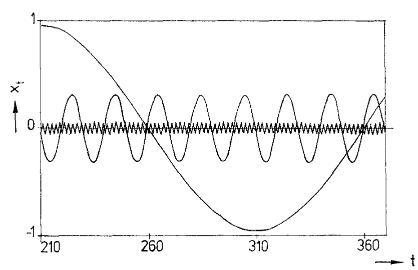

Fig. 1

*) The factor $(1 - \lambda)$ we are introducing for the normalization, compares the control to x_r.

is in some sense the low pass, i.e. it passes (almost without diminution) the slow harmonics of u_t and (almost) rejects the fast harmonics. For $x_0 = 0$, $\lambda = 0{\cdot}9$ we shall drive (10) alternatively by: (i) fast cosinusoid, sampled twice during its period – $u_t = \cos(\pi t/T_s)$, (ii) intermediate cosinusoid $u_t = \cos(\pi t/10T_s)$, (iii) slow cosinusoid $\cos(\pi t/100T_s)$. See the Figure 1.

Let us turn back to (8, 9). The fast components of u_t on their transmission to $\tilde{x}_t$ are almost rejected by the low pass with characteristic value λ_{rec}, and on their transmission to x_t by one another low pass with λ_{con}. (The slow components of u_t transcendent our scalar model and shall be treated in Part 2.)

Finally, in the model (5, 7) we generally do not know the parameter λ: we shall identify λ recursively, simultaneously with control and reconstruction, from the computed input and measured output. In (5, 7) we now have two unknowns: the state and the parameter. To make recursive identification independent of reconstruction (or to pass from the bilinear problem to the linear one), we shall eliminate from (5, 7) the state:

$$(11) \qquad y_{t+1} = x_{t+1} + v_{t+1} = \lambda x_t + u_t + v_{t+1} = \lambda y_t + u_t + v_{t+1} - \lambda v_t\,.$$

We observe that the models (5, 7) – or the internal description, and (11) – or the external description, are equivalent. The internal description has been germane to control and reconstruction, and similarly the external description will be germane to recursive identification. Let us observe that for zero disturbances v we could from the external description (11) identify trivially λ as $(y_{t+1} - u_t)/y_t$. About the disturbance we can usually suppose that it is faster (or in the frequency sense it is high frequency dominant) more than the input and output. (Slow disturbances, e.g. bias and trend, transcend scalar model of Part 1 and will be treated in Part 2.) Previously we demonstrated the properties of low pass (10). Let us use this low pass for the prefiltering of the external variables.

$$(12) \qquad \begin{aligned} y_{pf,t+1} &= \lambda_{pf} y_{pf,t} + (1 - \lambda_{pf})\, y_t\,, \\ u_{pf,t+1} &= \lambda_{pf} u_{pf,t} + (1 - \lambda_{pf})\, u_t \qquad (0 < \lambda_{pf} < 1)\,, \\ v_{pf,t+1} &= \lambda_{pf} v_{pf,t} + (1 - \lambda_{pf})\, v_t\,. \end{aligned}$$

To obtain the external description for the prefiltered external variables let us (i) substitute (11) to (12), (ii) multiply equation ad (i) by $(1 - \lambda_{pf})$, (iii) decompose the equation ad (ii) to the two equations differing by one time unit. Then one of these two equations ad (iii) will be the prefiltered external description

$$(13) \qquad y_{pf,t+1} = \lambda y_{pf,t} + u_{pf,t} + v_{pf,t+1} - \lambda v_{pf,t}\,.$$

So, by the prefiltering the noise/signal ratio have been reduced. Now, the recursive identification of λ will follow the structure of reconstruction. The constancy of parameter λ can be written by the degenerate evolution:

$$(14) \qquad \lambda_{t+1} = \lambda_t\,.$$

We shall identify the parameter λ recursively from the copy of our model using the difference between the measurement and old identification, amplified by the recursive identification gain $k_{id,t}$:

$$\hat{\lambda}_{t+1} = \hat{\lambda}_t + k_{id,t}(y_{pf,t} - \hat{\lambda}_t y_{pf,t-1} - u_{pf,t-1}) . \tag{15}$$

Subtracting (15) from (14) we obtain the evolution of recursive identification error:

$$\tilde{\lambda}_{t+1} =_{def} \lambda - \hat{\lambda}_{t+1} = (1 - k_{id,t} y_{pf,t-1}) \tilde{\lambda}_t - k_{id,t}(v_{pf,t} - \lambda v_{pf,t-1}) .$$

Comparing this with evolution we want to have (for $v = 0$)

$$\tilde{\lambda}_{t+1} = \lambda_{id} \tilde{\lambda}_t$$

we finally obtain the recursive identification gain*)

$$k_{id,t} = (1 - \lambda_{id}) / y_{pf,t-1} .$$

So, our constancy condition on λ is the condition for constancy for the transient time given by λ_{id}. Stability of control based on reconstruction based on recursive identification, or the self-adjusting regulator, is given by the three characteristic values $\lambda_{con}, \lambda_{rec}, \lambda_{id}$.

Problem. The consumption of certain products grows yearly by approx. 5%. It was ascertained that the raw materials (which can not be recycled) will be the limit of this growth and that the raw materials output must remain at the present level. At the same time with improving the estimate both of consumption growth and consumption there should be such consumption control to make this consumption settle at 150% of today state. This new state should be reached smoothly, approx. within 20 years. Control actions and measurements have to be carried monthly. Measurement disturbances are such that the consumption can be estimated approx. from 6 measurements, consumption growth from 12 ones.

Now we shall touch the basic question about modelling. Why do we not pursuit the more subtle analysis of disturbances? Instead of rough frequency description, maybe, we could model them and further identify recursively the parameters (and reconstruct the state) of this model? During such an attempt we should find that not even this identification (and reconstruction) can be solved trivially, i.e. from the minimal number of measurements. So we have to model again the uncertainties of this model and wishing to be consistent, we should identify recursively the parameters (and reconstruct the state) of this model of model. Afterward we should treat the model of model of model, ... because to remain consistent we have no right to stop our modelling. Nevertheless with respect to finite enumerability we have to choose the number of modelling levels. We have chosen the lowest possible number of levels.

*) Evidently under the regularity condition $y_{pf,t-1} \neq 0$. Nontrivial regularity conditions will be met in Parts 2,3.

Sometimes is chosen one more level. Anticipating the Parts 2, 3, yet more can be said on the consequencies of modelling level growth. The lowest possible number of modelling levels leads, for the vector state, to matrices (– or the 2nd order tensors), one more modelling level to the 3rd order tensors, yet one more modelling level to the 4th order tensors, ... So we have one more reason to stop the modelling levels number as soon as possible.

PART 2 CONTROL OF INITIALLY UNKNOWN SINGLE-INPUT SINGLE-OUTPUT PLANT

Things should be explained as simply as possible but not more simply (*Einstein*)

The initially unknown plant which will be controlled by the single input using the measurement of the single output will be characterized by n state coordinates and $n^2 + 2n$ parameters. In Part 1 we were controlling the geometric sequence characterized by the single state coordinate and the single parameter. Nevertheless this shift from 2 to $n^2 + 3n$ indeterminates will not be associated with "the curse of dimensionality". This will be achieved by the introduction of germane (and of course familiar) regularity conditions of reachability and reconstructibility. These will make possible to us to solve each problem in such state coordinates in which the solution is as easy as had been the solution in the scalar case in Part 1. Nevertheless for this ease of the solution we have to pay by the necessity of conversion to the coordinates germane for a problem. In spite of that we are solving the global problem of control of initially unknown plant by the decomposition to the three levels (recursive identification, reconstruction and control) we are able to guarantee the feasible solution not only of separate levels but also of the global solution.

2.1 Model

We are beginning with the internal description (1,2) which couples the external variables (input, output) via the internal one (state). While the external variables which are either applied or measured, are of rational values, the internal variable can have, in addition to this, imaginary component, (1, 2). Similarly as in physics there exists no privileged system of coordinates but such a number of equivalent coordinates how many are there regular (i.e. $1-1$ and onto) transforms of an arbitrary starting coordinates, and similarly for given internal description (1), the equivalent internal description is introduced, (3). State coordinates transform (3) induces the transforms of the state, input, and output matrices, (4). While for the specification of relevant problem (control, reconstruction, recursive identification) for geometric sequence it was sufficient to specify single characteristic value, for the specification of relevant problem for n-dimensional state model we have to specify, e.g., n characteristic values (counting the multiplicites). For the latter specification we shall use the Jordan matrix (5c, 6). This we are using because of ease of computing of its powers, (7, 8). These powers will give us the prescribed evolution of (control, reconstruction, recursive identification) error. Each square matrix is of course equivalent to the Jordan matrix, (9), so the latter specification suffices. Still further we shall limit ourselves to such Jordan matrix with the different blocks corresponding to characteristic values. Because of this we introduce the cyclic matrix, (10 to 15). Under cyclicity condition the Jordan matrix is equivalent to the Frobenius matrix, (16). The latter we also shall call the companion matrix because of its affinity to the characteristic polynomial: the components of the companion matrix are either zeros and ones or the coefficients of (monic, i.e. with unit leading coefficient) characteristic polynomial. Equivalence of cyclic Jordan and Frobenius matrix guarantees, for the stable characteristic values of the Jordan matrix, the stability of error evolution decribed by the powers of the Frobenius matrix, (17). Our solution of the relevant problem (control, reconstruction, recursive identification) will then be as follows: we shall specify the characteristic values $\lambda_1, \lambda_2 \ldots \lambda_n$ of relevant error evolution, from these we shall compute characteristic polynomial $(\lambda - \lambda_1)(\lambda - \lambda_2) \ldots \ldots (\lambda - \lambda_n) = \lambda^n - a_n\lambda^{n-1} \ldots - a_2\lambda - a_1$ and by this also the Frobenius matrix. (Because we are going from the specified rational characteristic values to the polynomial and not take the opposite way, we shall suffice with the algebraically unclosed field of rational numbers.) We shall then correct the original state matrix by the feedback gain in such a way to be equal Frobenius matrix we want to have – trivially by computation of the n differences. This ease of synthesis will be made possible because the state coordinates will be such that the state matrix will be, generally time-variable, version of the Frobenius matrix.

(1) Definition. By the internal description we understand the equations (state and output):

(a) $$x_{t+1} = A_t x_t + b_t u_t\,, \quad y_t = c'_t x_t\,,$$

where x_t is n-dimensional state, u_t is scalar input and y_t is scalar output. State matrix

$$A_t : (\mathbb{Q} + \mathrm{j}\mathbb{Q})^n \to (\mathbb{Q} + \mathrm{j}\mathbb{Q})^n\,,$$

input matrix

$$b_t : \mathbb{Q} \to (\mathbb{Q} + \mathrm{j}\mathbb{Q})^n\,,$$

output matrix

$$c'_t : (\mathbb{Q} + \mathrm{j}\mathbb{Q})^n \to \mathbb{Q}\,,$$

where $\mathbb{Q}$ are rationals, $\mathbb{Q} + \mathrm{j}\mathbb{Q}$ their extension of imaginary components. t is integer time. Initial time x_{t_0}, state matrix A_t, input matrix b_t and output matrix c'_t are such that the output is rational. Instead of (a) we shall write (A_t, b_t, c'_t), too.

(2) Example of internal description for $n = 3$.

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \\ x_{3,t+1} \end{bmatrix} = \begin{bmatrix} \mathrm{j} & 0 & 0 \\ 0 & -\mathrm{j} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} u_t , \quad x_0 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad y_t = [1 \; 1 \; 4] \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \end{bmatrix}.$$

(3) Definition. By the equivalent internal description to the internal description

(a) $$^1x_{t+1} = {}^1A_t\, {}^1x_t + {}^1b_t u_t , \quad y_t = {}^1c_t'\, {}^1x_t$$

we shall understand the internal description

(b) $$^2x_{t+1} = {}^2A_t\, {}^2x_t + {}^2b_t u_t , \quad y_t = {}^2c_t'\, {}^2x_t$$

where

(c) $$^{12}P_t : {}^1x_t \mapsto {}^2x_t = {}^{12}P_t\, {}^1x_t$$

is uniformly regular, i.e. $\left|\det {}^{12}P_t\right| \geqq \varepsilon > 0$ for all t. $({}^iA_t, {}^ib_t, {}^ic_t')$ will be called i-th realization.

(4) Corollary. The transform (3c) induces the transforms

$$\begin{aligned} {}^1A_t &\mapsto {}^2A_t = {}^{12}P_{t+1}\, {}^1A_t\, {}^{12}P_t^{-1} , \\ {}^1b_t &\mapsto {}^2b_t = {}^{12}P_{t+1}\, {}^1b_t , \\ {}^1c_t' &\mapsto {}^2c_t' = {}^1c_t'\, {}^{12}P_t^{-1} \end{aligned}$$

or ${}^1A_t$, ${}^1b_t$, ${}^1c_t'$ are transformed with ${}^{12}P_{t+1}(\cdot)\, {}^{12}P_t^{-1}$, ${}^{12}P_{t+1}(\cdot)$, $(\cdot)\, {}^{12}P_t^{-1}$.

Proof. To (3b) we shall substitute ${}^2x_t$ from (3c). Then by comparison with (3a) we shall obtain the assertion.

(5) Problem. Show that the realizations

(a) $$\left(P \begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix} P^{-1}, \; P \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; [c_1 \; c_2]\, P^{-1} ; \; P \text{ regular} \right),$$

(b) $$\left(\begin{bmatrix} 0 & 1 \\ a_1 & a_2 \end{bmatrix}, \; \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; [c_1 \; c_2] \right),$$

(c) $$\left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \; \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \; \begin{bmatrix} \dfrac{\lambda_1 c_1 - c_2}{\lambda_1 - \lambda_2} & \dfrac{-\lambda_2 c_1 + c_2}{\lambda_1 - \lambda_2} \end{bmatrix} ; \right.$$

$$\left. \lambda_i^2 - a_2 \lambda_i - a_1 = 0 \quad (i = 1, 2) , \quad a_2^2 \neq 4a_1 \right)$$

are equivalent. (The realizations like (b) will be used for control gain synthesis, realizations like (c) will be used for error evolution specification, realizations like (a) will be well dispensed with.)

(6) Definition. The matrix

$$ {}^{1}A = \begin{bmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_r \end{bmatrix} $$

with the dimensions $n \times n$, the blocks of which

$$ J_i = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & 1 \\ & & & \lambda_i \end{bmatrix} \quad (i = 1 \ldots r) $$

are of the dimensions $n_i \times n_i$ $(n_1 + n_2 \ldots + n_r = n)$, is called the Jordan matrix and λ_i are called the characteristic numbers.

(7) Lemma.

$$ J^k = \sum_{j=0}^{K} \binom{k}{j} \begin{bmatrix} \lambda_1^{k-j} D_1^j & & & \\ & \lambda_2^{k-j} D_2^j & & \\ & & \ddots & \\ & & & \lambda_r^{k-j} D_r^j \end{bmatrix}, $$

where

$$ D_i = J_i - \lambda_i I \quad (i = 1, 2 \ldots r) $$
$$ K = \max(k, n_1, n_2 \ldots n_r) . $$

Proof. For every block we shall use the binomial theorem:

$$ J_i^k = (\lambda_i I + D_i)^k $$
$$ = \sum_j \binom{k}{j} \lambda_i^{k-j} I D_i^j \quad (j = 0, 1 \ldots k) $$
$$ = \sum_j \binom{k}{j} \lambda_i^{k-j} D_i^j \quad (j = 0, 1 \ldots k) $$

because $D_i^j = 0$ for $j > n_i$.

(8) Theorem. Let $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_r| < 1$. Then for $k \to \infty$, $J^k \to 0$.

(9) Lemma. Every square matrix over $\mathbb{R} + j\mathbb{R}$ is equivalent with Jordan matrix over $\mathbb{R} + j\mathbb{R}$.

Proof see e.g. [6]. In our paper we shall suffice with Jordan matrix over $\mathbb{Q} + j\mathbb{Q}$ which is cyclic:

(10) Definition. Matrix A is cyclic if there exists such vector g that the vectors $g, Ag, A^2g \dots A^{n-1}g$ are linearly independent.

(11) Example.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is cyclic – it suffices to take

$$g = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

is cyclic – it suffices to take

$$g = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

but

$$\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix}, \quad \begin{bmatrix} J_1 & \\ & J_1 \end{bmatrix}$$

are not cyclic.

(12) Corollary. Let the matrix A be cyclic. Than the vector $A^n g$ is linearly dependent on the vectors $g, Ag, \dots, A^{n-1}g$.

Proof. Any basis in the n-dimensional vector space has n elements. If we take as the basis $g, Ag, \dots, A^{n-1}g$ then any vector, and thus even $A^n g$ must be dependent on this basis.

(13) Theorem. If the blocks of the Jordan matrix and the subblocks of these blocks are mutually different, then the Jordan matrix is cyclic.

Proof. For the orientation we shall start with demonstration for

$$J = \begin{bmatrix} \lambda_1 & 1 & & & \\ & \lambda_1 & 0 & & \\ & & \lambda_2 & 1 & \\ & & & \lambda_2 & 1 \\ & & & & \lambda_2 \end{bmatrix}.$$

Let us choose

$$g = \begin{bmatrix} 0 \\ 1 \\ \hline 0 \\ 0 \\ 1 \end{bmatrix} =_{def} \begin{bmatrix} e_{2,2\times 1} \\ e_{3,3\times 1} \end{bmatrix}.$$

 Then

$$[J^4 g \ldots Jg\ g] = \left[\begin{array}{ccccc} 4\lambda_1^3 & 3\lambda_1^2 & 2\lambda_1 & 1 & 0 \\ \lambda_1^4 & \lambda_1^3 & \lambda_1^2 & \lambda_1 & 1 \\ \hline 6\lambda_2^2 & 3\lambda_2 & 1 & 0 & 0 \\ 4\lambda_2^3 & 3\lambda_2^2 & 2\lambda_2 & 1 & 0 \\ \lambda_2^4 & \lambda_2^3 & \lambda_2^2 & \lambda_2 & 1 \end{array}\right].$$

For $\lambda_1 \neq \lambda_2$ the linear independence of the rows is guaranteed because: (a) for $\lambda_1, \lambda_2 \neq 0$ the rows are either polynomials of the different orders of the same indeterminates (either λ_1 or λ_2) or of the same orders of the different indeterminates, (b) for λ_1 (or λ_2) $= 0$ the rows are of this indeterminate created by the unit vectors in the different coordinates. For the proof let us take

$$g = \begin{bmatrix} e_{n_1,n_1 \times 1} \\ e_{n_2,n_2 \times 1} \\ \cdots\cdots \\ e_{n_r,n_r \times 1} \end{bmatrix}.$$

Then the matrix

$$[J^{n-1} g \ldots Jg\ g] = \begin{bmatrix} B_1 \\ B_2 \\ \cdots \\ B_r \end{bmatrix},$$

where for $i = 1, 2 \ldots r$:

$$B_i = \begin{bmatrix} \binom{n-s_i}{n-1}\lambda_i^{n-s_i} & \binom{n-s_i-1}{n-2}\lambda_i^{n-s_i-1} & \cdots & \binom{0}{n-s_i-1}\lambda_i^0 & \\ \vdots & \vdots & & \ddots & \\ \binom{n-2}{n-1}\lambda_i^{n-2} & \binom{n-3}{n-2}\lambda_i^{n-3} & \cdots\cdots & \binom{0}{1}\lambda_i^0 & \\ \binom{n-1}{n-1}\lambda_i^{n-1} & \binom{n-2}{n-2}\lambda_i^{n-2} & \cdots\cdots & \binom{1}{1}\lambda_i^1 & \binom{0}{0}\lambda_i^0 \end{bmatrix},$$

$s_i = \sum n_j$ ($j = 1, 2 \ldots r$; $j \neq i$). For the block J_i which is not contained in the block J_j the cyclicity is again guaranteed by the reasoning ad (a, b).

(14) Theorem and definition. Let the Jordan matrix ${}^1A$ be cyclic. Then it is equivalent with the Frobenius matrix

$${}^2A = \begin{bmatrix} 0 & \ldots & 0 & a_1 \\ 1 & & & a_2 \\ & \ddots & & \vdots \\ & & 1 & a_n \end{bmatrix} =_{def} \left[\begin{array}{c|c} 0 & \\ \hline I & a \end{array}\right]$$

where

$$-a_k = (-1)^k \sum_{i=1}^{n} \prod_{j=1}^{n-1} \lambda_{i+j(\mathrm{mod}\, n+1)} \quad (k = 1, 2 \dots n)$$

is the coefficient at the $(n - k)$-th power at the characteristic polynomial

$$\text{char }{}^2A = 1 + \sum_{k=1}^{n} -a_k\lambda^{n-k} = \text{char }{}^1A .$$

Frobenius matrix is called also the companion matrix of the characteristic polynomial or just the companion matrix.

Proof. For the orientation we shall start with demonstration for

$${}^1A = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{bmatrix} \quad (\lambda_1 \neq \lambda_2) .$$

Let us take

$$g = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} =_{def} \begin{bmatrix} e_{1,1\times 1} \\ e_{2,2\times 1} \end{bmatrix} .$$

We know already that vectors

$$g,\ {}^1Ag = \begin{bmatrix} \lambda_1 \\ 1 \\ \lambda_2 \end{bmatrix}, \quad {}^1A^2g = \begin{bmatrix} \lambda_1^2 \\ 2\lambda_2 \\ \lambda_2^2 \end{bmatrix}$$

are linearly independent. We shall show that ${}^2A$ is given by the representation of ${}^1A$ in the basis g, ${}^1Ag$, ${}^1A^2g$. Evidently

$${}^1Ag \ = 0 \,.\, g + 1 \,.\, {}^1Ag + 0 \,.\, {}^1A^2g = [g\ {}^1Ag\ {}^1A^2g] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

$${}^1A^2g = 0 \,.\, g + 0 \,.\, {}^1Ag + 1 \,.\, {}^1A^2g = [g\ {}^1Ag\ {}^1A^2g] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

The first column of ${}^2A$ is then

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

the second column of ${}^2A$ is

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

 It remains to determine the third column of ${}^2A$, i.e.

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}.$$

This we shall determine from that, evidently, the equivalent Jordan and Frobenius matrices should have the same characteristic polynomials

$$\det \begin{bmatrix} \lambda - \lambda_1 & & \\ & \lambda - \lambda_2 & -1 \\ & & \lambda - \lambda_2 \end{bmatrix} = \det \begin{bmatrix} \lambda & 0 & -a_1 \\ -1 & \lambda & -a_2 \\ 0 & -1 & \lambda - a_3 \end{bmatrix},$$

$$\lambda^3 - (\lambda_1 + \lambda_2 + \lambda_3)\,\lambda^2 + (\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1)\,\lambda - \lambda_1\lambda_2\lambda_3 =$$
$$= \lambda^3 - a_3\lambda^2 - a_2\lambda - a_1 .$$

For the proof it suffices to take again

$$g = \begin{bmatrix} e_{n_1, n_1 \times 1} \\ e_{n_2, n_2 \times 1} \\ \cdots\cdots \\ e_{n_r, n_r \times 1} \end{bmatrix}.$$

(15) Corollary and definition. Let $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n| < 1$. Then for $k \to \infty$,

$$\left[\begin{array}{c|c} 0 & a \\ \hline I & \end{array}\right]^k \to 0 .$$

We shall say that the realization

$$\left(\left[\begin{array}{c|c} 0 & a \\ \hline I & \end{array}\right], b, c'\right)$$

is stable with the degree of stability given by the characteristic numbers $\lambda_1, \lambda_2 \ldots \lambda_n$. For the all characteristic numbers being zero we shall speak about the maximum degree of stability.

(16) Corollary. Let those characteristic numbers of Jordan cyclic matrix, which have nonzero imaginary part, be adjoined. Then the equivalent Frobenius matrix has rational elements.

(17) Note. We shall use the cyclic Jordan matrix ${}^1A$ for synthesis specification, Frobenius matrix ${}^2A$ for synthesis solution. Because we shall specify ${}^1A$ over $\mathbb{Q} + j\mathbb{Q}$ we shall manage with this algebraically unclosed field.

Starting problem: how, from the initial state, to reach the final state, (1), leads to the regularization condition of reachability, (1 to 3). All the results we recommend the readers to interpret by some linear graphs, e.g. the signal flow graphs, (3): without this the results will look like the bag of tricks. Of more profound importance are the commutative diagrams, (3). The reachability matrix does not serve just to check the regularity but mainly for transform of coordinates, (5 to 9). The central, for the Part 2, are the phase coordinates which are generalizing the notion of cyclicity to time-varying plants. The solution of control in these coordinates is trivial, (10).

(1) Theorem and definition. Let the matrix

$$[A_{t+n-1} \dots A_{t+1} b_t \dots A_{t+n-1} A_{t+n-2} b_{t+n-3}\ A_{t+n-1} b_{t+n-2}\ b_{t+n-1}] =_{def} M_{t+n-1}$$

is uniformly regular. Then for all initial states α and for all final states ω of the realization (A_t, b_t, c'_t) there exists just one future of the input $u_t, u_{t+1} \dots u_{t+n-1}$ by which the final state can be reached:

$$x_t = \alpha\,, \quad x_{t+n} = \omega\,.$$

This future of the input is

$$\begin{bmatrix} u_t \\ u_{t+1} \\ \dots\dots \\ u_{t+n-1} \end{bmatrix} = M_{t+n-1}^{-1}(\omega - A_{t+n-1} \dots A_{t+1} A_t \alpha)\,.$$

The matrix M is called the reachability matrix. The condition of the theorem is called the reachability of (A_t, b_t).

Proof. Iterating the state equation $x_{t+1} = A_t x_t + b_t u_t$ n-times we shall, starting from the initial state α, obtain:

$$\omega = A_{t+n-1} \dots A_t \alpha + M_{t+n-1} \begin{bmatrix} u_t \\ u_{t+1} \\ \dots\dots \\ u_{t+n-1} \end{bmatrix}.$$

(2) Example.

$$\left(\begin{bmatrix} 0 & 1 \\ a_{1,t} & a_{2,t} \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right), \left(\begin{bmatrix} 0 & 1 \\ a_{1,t} & a_{2,t} \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}; a_{1,t} \neq 0\right), \left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}; \lambda_1 \neq \lambda_2\right),$$

(I, b_t; $b_t, b_{t+1} \dots b_{t+n-1}$ linearly independent) are reachable.

(3) Counterexample. $(I, b; n > 1)$, $(0, b_t)$, $(A_t, 0)$ are not reachable.

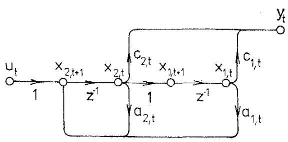

Fig. 2

(4) Note. All our assertions, and their derivations, too, have been based on piecewise linear or row notation. This row notation can be represented by graphs. So, e.g. the equation

$$\begin{bmatrix} x_{1,t+1} \\ x_{2,t+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ a_{1,t} & a_{2,t} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_t , \qquad y_t = \begin{bmatrix} c_{1,t} & c_{2,t} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix}$$

is represented by the signal flow graph at Fig. 2. The other type of graphs are the commutative diagrams. So, e.g. the equivalence of the state coordinates ${}^1x_{t+1} = {}^1A_t\, {}^1x_t$, ${}^2x_{t+1} = {}^2A_t\, {}^2x_t$, ${}^2x_t = {}^{12}P_t\, {}^1x_t$ we shall represent by the commutative diagram:

$$\begin{array}{ccc} {}^1x_t & \xmapsto{{}^1A_t} & {}^1x_{t+1} \\ {}^{12}P_t \downarrow & & \downarrow {}^{12}P_{t+1} \\ {}^2x_t & \xmapsto{{}^2A_t} & {}^2x_{t+1} \end{array}$$

This stands for equality (interchangeability, commutation): ${}^{12}P_{t+1}\, {}^1A_t = {}^2A_t\, {}^{12}P_t$. Meanwhile the signal flow graphs are the language for linear equations, [8], the commutative diagrams suffice, may be, for all abstract mathematics, [6, 7].

(5) Theorem. The reachability matrix ${}^1M_t$ is transformed by ${}^{12}P_{t+1}(\cdot)$.

Proof. To find the transform matrix ${}^{12}P_t =_{def} P_t$ we shall start from the relations:

(a) $$P_{t+1}\, {}^1A_t = {}^2A_t P_t ,$$

(b) $${}^2b_t = P_{t+1}\, {}^1b_t ,$$

the latter of them gives us yet the first linear equation for P. Let us multiply (b) from the left by ${}^2A_{t+1}$:

$${}^2A_{t+1}\, {}^2b_t = {}^2A_{t+1} P_{t+1}\, {}^1b_t .$$

We shall compute ${}^2A_{t+1}P_{t+1}$ from (a) for a time increased by 1:

(c) $${}^2A_{t+1}\,{}^2b_t = P_{t+2}\,{}^1A_{t+1}\,{}^1b_t$$

and we have the second linear equation for P. Let us multiply (c) from the left by ${}^2A_{t+2}$ and let us use again (b), now for the time increased by 2:

$${}^2A_{t+2}\,{}^2A_{t+1}\,{}^2b_t = {}^2A_{t+2}P_{t+2}\,{}^1A_{t+1}\,{}^1b_t = P_{t+3}\,{}^1A_{t+2}\,{}^1A_{t+1}\,{}^1b_t\,.$$

So we obtained the third linear equation for P. Progressing this way we shall obtain finally the n-th linear equation for P:

$${}^2A_{t+n-1}\ldots{}^2A_{t+1}\,{}^2b_t = P_{t+n}\,{}^1A_{t+n-1}\ldots{}^1A_{t+1}\,{}^1b_t\,.$$

Lastly we shall convert all linear equations to the same time, namely for P_{t+1}. So we shall obtain

$${}^2M_t = P_{t+1}\,{}^1M_t\,.$$

(6) Problem. Prove (5) using a commutative diagram.

(7) Corollary. If the realization (A_t, b_t, c_t') is reachable, then is reachable even every equivalent realization. If the realization is not reachable then nor any equivalent realization is reachable.

Proof. I. follows from that the reachability matrix is transformed by ${}^2M_t = {}^{12}P_{t+1}\,{}^1M_t$ where both ${}^1M_t$ and ${}^{12}P_{t+1}$ are regular. II. follows from that now, ${}^1M_t$ in the previous transform is not regular.

(8) Corollary. Let $({}^1A_t, {}^1b_t)$ is reachable. Then the transform matrix is factored, using both reachability matrices, viz.

$${}^{12}P_t = {}^2M_{t-1}\,{}^1M_{t-1}^{-1}\,.$$

(9) Theorem. Let $({}^1A_t, {}^1b_t)$ be reachable with reachability matrix ${}^1M_t$. Then there exists just one transform ${}^{12}P_t$, such that the starting realization $({}^1A_t, {}^1b_t, {}^1c_t')$ is equivalent with the phase realization*) $(a_t', e_n, {}^2c_t')$, where

$$a_{i,t} = e_{n-i+1}'\,{}^1M_{t+i-1}\,{}^1A_{t+i-1}\,{}^1M_{t+i-2}e_1 \quad (i = 1, 2 \ldots n)$$
$${}^2c_t' = {}^1c_t'\,{}^1M_{t-1}\,{}^2M_{t-1}^{-1}\,.$$

Proof. We shall start from the relation (4/par. 2.1):

(a) $${}^2A_t\,{}^{12}P_t = {}^{12}P_{t+1}\,{}^1A_t\,.$$

*) Our most frequently used matrices we are abbreviating still further. Instead of

$$\left(\begin{bmatrix} 0 \mid I \\ \hline a' \end{bmatrix}, \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix}\right)$$

we shall write only (a', e_n).

 Further we shall use the factorization, (8):

(b) $$^{12}P_t = {}^2M_{t-1}\, {}^1M_{t-1}^{-1}\,.$$

Substituting (b) to (a):

$$^2A_t\, {}^2M_{t-1}\, {}^1M_{t-1}^{-1} = {}^2M_t\, {}^1M_t^{-1}\, {}^1A_t$$

or

(c) $$^1M_t^{-1}\, {}^1A_t\, {}^1M_{t-1} = {}^2M_t^{-1}\, {}^2A_t\, {}^2M_{t-1}\,.$$

We know the left-hand side of (c), we know the right-hand side of (c) up to a_t. For this indeterminate a_t we shall compute ${}^2M_t^{-1}\, {}^2A_t\, {}^2M_{t-1}$. By the induction for n, we shall obtain:

$$^1M_t^{-1}\, {}^1A_t\, {}^1M_{t-1} = \left[\begin{array}{c|c} a_{n,t-n+1} & \\ \cdots\cdots & I \\ a_{2,t-1} & \\ \cline{2-2} a_{1,t} & 0 \end{array}\right].$$

Finally

$$a_{1,t} = e_n'\, {}^1M_t^{-1}\, {}^1A_t\, {}^1M_{t-1} e_1\,,$$

$$a_{2,t} = e_{n-1}'\, {}^1M_{t+1}^{-1}\, {}^1A_{t+1}\, {}^1M_t e_1\,,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$a_{n,t} = e_1'\, {}^1M_{t+n-1}^{-1}\, {}^1A_{t+n-1}\, {}^1M_{t+n-2} e_1\,.$$

(10) Theorem. Let $(A_t, b_t) = (a_t', e_n)$, i.e. it is the phase couple. Then the control

$$x_{t+1} = A_t x_t + b_t u_t\,, \quad u_t = -k_{con,t} x_t$$

with the control gain

$$k_{con,t} = a_t - a_{con}$$

is such that the control error evolves as

$$x_{t+1} = \left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a_{con}'} \end{array}\right] x_t\,.$$

Or writing with the help of characteristic polynomials:

$$\operatorname{char}(A_t - b_t k_{con,t}') = \operatorname{char}\left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a_{con}'} \end{array}\right].$$

Proof.

$$\left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a_t'} \end{array}\right] - \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right](a_t - a_{con}) = \left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a_{con}'} \end{array}\right].$$

(11) Corollary. Let

$$u_t = -{}^i k_{con,t}' \, {}^i x_t \quad (i = 1, 2).$$

Then the control gain is transformed by $(\cdot)\,{}^{12}P_t^{-1}$.

2.3 Reconstruction

Starting problem: how to reconstruct the current state from the history of output, (1), leads to the regularization condition of reconstructibility, (1 to 3). The reconstructibility matrix will be used for the transformation to germane coordinates, (4 to 10). In these phase coordinates the synthesis of the feedback reconstruction gain is again trivial, (10). The problems of control and reconstruction enabled the compilation of little dictionary of transforms, (13). Coupling various transforms together, the relations between the external and internal description are enlightned, (13/Fig. 3). Using the feedback reconstruction we can finally reconstruct or control even in the presence of initially unknown spline disturbances, (15 to 17).

(1) Theorem and definition. Let both the matrix A_t and the matrix

$$\begin{bmatrix} c_t' \\ c_{t-1}' A_{t-1}^{-1} \\ c_{t-2}' A_{t-2}^{-1} A_{t-1}^{-1} \\ \dots\dots\dots\dots\dots\dots \\ c_{t-n+1}' A_{t-n+1}^{-1} \dots A_{t-1}^{-1} \end{bmatrix} =_{def} N_t$$

be uniformly regular. Then for all initial states x_{t-n+1} of the realization $(A_t, 0, c_t')$ the resulting current state x_t can be reconstructed from the history of the output $y_{t-n+1}, \dots, y_{t-1}, y_t$. The resulting current state is

$$x_t = N_t^{-1} \begin{bmatrix} y_t \\ y_{t-1} \\ \dots\dots \\ y_{t-n+1} \end{bmatrix}.$$

The matrix N we shall call the reconstructibility matrix. The condition of the theorem we shall call the reconstructibility of (A_t, c_t').

Proof. The equations for the current state

$$
\begin{aligned}
y_t &= c_t' x_t \\
y_{t-1} &= \underbrace{c_{t-1}' x_{t-1}}_{\substack{\| \\ A_{t-1}^{-1} x_t}} \\
y_{t-2} &= \underbrace{c_{t-2}' x_{t-2}}_{\substack{\| \\ A_{t-2}^{-1} x_{t-1} \\ \| \\ A_{t-1}^{-1} x_t}} \\
& \dots\dots\dots\dots\dots \\
y_{t-n+1} &= \underbrace{c_{t-n+1}' x_{t-n+1}}_{\substack{\| \\ \vdots \\ A_{t-1}^{-1} x_t}}
\end{aligned}
$$

can be put together

$$
N_t x_t = \begin{bmatrix} y_t \\ y_{t-1} \\ y_{t-2} \\ \cdots\cdots \\ y_{t-n+1} \end{bmatrix}.
$$

(2) Example.

$$
\left(\begin{bmatrix} 0 & a_{1,t} \\ 1 & a_{2,t} \end{bmatrix}, [0\ 1];\ a_{1,t} \neq 0\right), \left(\begin{bmatrix} 0 & a_{1,t} \\ 1 & a_{2,t} \end{bmatrix}, [1\ 0];\ a_{1,t} \neq 0\right), \left(\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},\right.
$$

$\left.[1\ 1];\ \lambda_1\lambda_2 \neq 0,\ \lambda_1 \neq \lambda_2\right)$, $(I, c_t';\ c_t', c_{t-1}' \dots c_{t-n+1}'$ linearly independent) are reconstructible.

(3) Counterexample. $(A_t, b_t; \det A_t = 0), (I, c'; n > 1), (A_t, 0)$ are not reconstructible.

(4) Theorem. The reconstructibility matrix ${}^1N_t$ is transformed by $(\cdot)\ {}^{12}P_t^{-1}$.

Proof. To find the transform we shall start from the relations $({}^{12}P_t =_{def} P_t)$:

(a) $$P_{t+1}\, {}^1A_t = {}^2A_t P_t\,,$$

(b) $${}^1c_t' = {}^2c_t' P_t\,,$$

the latter of them gives us yet the first linear equation for P_t. Let us substitute in (b) $t-1$ for t and multiply the latter equation from the left by ${}^1A_{t-1}^{-1}$:

(c) $$ {}^1c'_{t-1}\,{}^1A_{t-1}^{-1} = {}^2c'_{t-1}P_{t-1}\,{}^1A_{t-1}^{-1}\,. $$

Further, after the substitution $t-1$ for t let us multiple (a) from the left by ${}^2A_{t-1}^{-1}$ and from the right by ${}^1A_{t-1}^{-1}$:

(d) $$ {}^2A_{t-1}^{-1}P_t = P_{t-1}\,{}^1A_{t-1}^{-1} $$

and further, let us substitute the right-hand side of (d) to (c):

(e) $$ {}^1c'_{t-1}\,{}^1A_{t-1}^{-1} = {}^2c'_{t-1}\,{}^2A_{t-1}^{-1}P_t\,. $$

(e) is already the second equation for P_t. Let us change in (e) $t-1$ for t and multiply the latter equation from the left by ${}^1A_{t-1}^{-1}$:

(f) $$ {}^1c'_{t-2}\,{}^1A_{t-2}^{-1}\,{}^1A_{t-1}^{-1} = {}^2c'_{t-2}\,{}^2A_{t-2}^{-1}P_{t-1}\,{}^1A_{t-1}^{-1}\,. $$

Let us substitute for $P_{t-1}\,{}^1A_{t-1}^{-1}$ from (d):

(g) $$ {}^1c'_{t-2}\,{}^1A_{t-2}^{-1}\,{}^1A_{t-1}^{-1} = {}^2c'_{t-2}\,{}^2A_{t-2}^{-1}\,{}^2A_{t-1}^{-1}P_t $$

(g) is the third equation for P_t. Finally the n-th equation:

$$ {}^1c'_{t-n+1}\,{}^1A_{t-n+1}^{-1}\ldots{}^1A_{t-1}^{-1} = {}^2c'_{t-n+1}\,{}^2A_{t-n+1}^{-1}\ldots{}^2A^{-1}P_t\,. $$

Gathering these n equations:

$$ {}^1N_t = {}^2N_tP_t\,. $$

(5) Corollary. If the realization (A_t, b_t, c'_t) is reconstructible, then is reconstructible even every equivalent realization. If the realization is not reconstructible then nor any equivalent realization is reconstructible.

(6) Corollary. Let $({}^2A_t, {}^2c'_t)$ is reconstructible. Then the transform matrix is factored, using both reconstructibility matrices, viz.

$$ {}^{12}P_t = {}^2N_t^{-1}\,{}^1N_t\,. $$

(7) Theorem. Let $({}^1A_t, {}^1c'_t)$ be reconstructible with the reconstructibility matrix ${}^1N_t$. Then there exists just one transform ${}^{12}P_t$, such that the starting realization $({}^1A_t, {}^1b_t, {}^1c'_t)$ is equivalent with the phase realization*) $(a_t, {}^2b_t, e'_n)$, where

$$ a_{i,t} = e'_1\,{}^1N_{t+n-i+1}\,{}^1A_{t+n-i}\,{}^1N_{t+n-i}^{-1}e_{n-i+1} \quad (i = 1, 2 \ldots n)\,, $$
$$ {}^2b_t = {}^2N_{t+1}^{-1}\,{}^1N_{t+1}\,{}^1b_t\,. $$

*) Abbreviation (a, b, e'_n) stands for

$$ \left(\left[\begin{array}{c|c} 0 & \\ \hline I & a \end{array}\right],\quad b,\quad [\,0\,|1]\right). $$

Proof. We shall start from the relation (4/par. 2.1):

(a) $$^2A_tP_t = P_{t+1}\,{}^1A_t \quad (P_t =_{def} {}^{12}P_t)\,.$$

Further we shall use the factorization (6):

(b) $$P_t = {}^2N_t^{-1}\,{}^1N_t\,.$$

Substituting P from (b) to (a):

$$^2A_t\,{}^2N_t^{-1}\,{}^1N_t = {}^2N_{t+1}^{-1}\,{}^1N_{t+1}\,{}^1A_t\,.$$

The expressions $^1(\cdot)$ we shall transfer to the left-hand side, $^2(\cdot)$ to the right-hand side:

(c) $$^1N_{t+1}\,{}^1A_t\,{}^1N_t^{-1} = {}^2N_{t+1}\,{}^2A_t\,{}^2N_t^{-1}\,.$$

The left-hand side of (c) is known to us, the right-hand side of (c) is known up to a_t. For this indeterminate a_t we shall compute $^2N_{t+1}\,{}^2A_t\,{}^2N_t^{-1}$. By the induction for n, we shall obtain:

(d) $$^1N_t^{-1} = \left[\begin{array}{c|ccccc} & a_{1,t-1} & & & & \\ & a_{2,t-1} & a_{1,t-2} & & & \\ & \vdots & \vdots & \ddots & & \\ & a_{n-2,t-1} & a_{n-3,t-2} & \cdots & a_{1,t-n+2} & \\ & a_{n-1,t-1} & a_{n-2,t-2} & \cdots & a_{2,t-n+2} & a_{1,t-n+1} \\ \hline 1 & & & & & \end{array}\right],$$

$$^1N_{t+1}\,{}^1A_t\,{}^1N_t^{-1} = \left[\begin{array}{c|c} a_{n,t} \;\cdots\; a_{2,t-n+2} & a_{1,t-n+1} \\ \hline I & 0 \end{array}\right].$$

Lastly

$$a_{1,t} = e_1'\,{}^1N_{t+n}\,{}^1A_{t+n-1}\,{}^1N_{t+n-1}^{-1}e_n\,,$$
$$a_{2,t} = e_1'\,{}^1N_{t+n-1}\,{}^1A_{t+n-2}\,{}^1N_{t+n-2}^{-1}e_{n-1}\,,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$a_{n,t} = e_1'\,{}^1N_{t+1}\,{}^1A_t\,{}^1N_t^{-1}e_1\,.$$

(8) Corollary. The state coordinates of the phase realization $(a_t, 0, e_n')$ are the linear combination of the history of output of the lenght n:

$$\begin{bmatrix} x_{1,t} \\ x_{2,t} \\ \cdots \\ x_{n,t} \end{bmatrix} = N_t^{-1} \begin{bmatrix} y_t \\ y_{t-1} \\ \cdots\cdots \\ y_{t-n+1} \end{bmatrix},$$

where for the inverse of reconstructibility matrix see (7d).

Proof. Substitute for x_t, x_{t+1} to the state equation

$$x_{t+1} = \left[\frac{0}{I}\middle| a_t\right] x_t$$

from the assertion of the corollary.

(9) Theorem. The current output of the reconstructible realization $(a_t, 0, e_n')$ is coupled with the history of output of the length n by the autoregression:

(a) $$y_t = a_{n,t-1}y_{t-1} + a_{n-1,t-2}y_{t-2} \dots + a_{1,t-n}y_{t-n}$$

and there exists the equivalent realization

$$({}^2A_t,\ 0,\ {}^2c_t') = \left(\left[\begin{array}{c|c} 0 & I \\ \hline a_{1,t-n+1} & \dots\ a_{n-1,t-1}\ a_{n,t} \end{array}\right],\quad 0,\ [\,0\,|1]\right).$$

The first state coordinates, see (8), are transformed to the second ones which are given by that of output of the length:

$$\begin{bmatrix} {}^2x_{1,t} \\ \dots\dots \\ {}^2x_{n-1,t} \\ {}^2x_{n,t} \end{bmatrix} = \begin{bmatrix} y_{t-n+1} \\ \dots\dots \\ y_{t-1} \\ y_t \end{bmatrix}.$$

The transform matrix is:

$${}^{12}P_t = {}^2A_t \begin{bmatrix} & & 1 \\ & 1 & \\ \cdot^{\cdot^{\cdot}} & & \\ 1 & & \end{bmatrix} {}^1N_t\,.$$

Proof. We shall start by the rewriting of assertion of theorem:

$$\begin{bmatrix} {}^1x_{1,t} \\ {}^1x_{2,t} \\ {}^1x_{3,t} \\ \dots\dots \\ {}^1x_{n-1,t} \\ {}^1x_{n,t} \end{bmatrix} = \begin{bmatrix} 0\ 0\ 0 & \dots & 0\ 0\ a_{1,t-1} \\ 1\ 0\ 0 & \dots & 0\ 0\ a_{2,t-1} \\ 0\ 1\ 0 & \dots & 0\ 0\ a_{3,t-1} \\ \dots\dots & \dots & \dots\dots \\ 0\ 0\ 0 & \dots & 1\ 0\ a_{n-1,t-1} \\ 0\ 0\ 0 & \dots & 0\ 1\ a_{n,t-1} \end{bmatrix} \begin{bmatrix} {}^1x_{1,t-1} \\ {}^1x_{2,t-1} \\ {}^1x_{3,t-1} \\ \dots\dots \\ {}^1x_{n-1,t-1} \\ {}^1x_{n,t-1} \end{bmatrix}, \quad y_t = {}^1x_{n,t}\,.$$

Then progressing from the bottom to the top:

$$
\begin{array}{l}
y_t = {}^1x_{n,t} \\
\quad \| \\
\overbrace{{}^1x_{n-1,t-1} + a_{n,t-1}y_{t-1}} \\
\quad \| \\
\overbrace{{}^1x_{n-2,t-2} + a_{n-1,t-2}y_{t-2}} \\
\quad \| \\
\overbrace{\cdots\cdots\cdots\cdots\cdots\cdots} \\
{}^1x_{1,t-n+1} + a_{2,t-n+1}y_{t-n+1} \\
\quad \| \\
\overbrace{a_{1,t-n}y_{t-n}}
\end{array}
$$

– so we obtained the autoregression. Increasing the time by 1 we shall obtain the n-th row of ${}^2A_t$. The $n - 1 \ldots 2, 1$ rows are then nothing but the trivial equalities

$$
\begin{aligned}
y_t &= y_t\,, \\
&\cdots\cdots\cdots \\
y_{t-n+2} &= y_{t-n+2}\,, \\
y_{t-n+1} &= y_{t-n+1}
\end{aligned}
$$

– so we obtained (a). The transform ${}^{12}P_t$ we shall obtain from the equations

$$
\begin{aligned}
{}^1x_t &= {}^1N_t^{-1} \begin{bmatrix} & & 1 \\ & 1 & \\ \cdot^{\cdot^{\cdot}} & & \\ 1 & & \end{bmatrix} \begin{bmatrix} y_{t-n+1} \\ \vdots \\ y_t \end{bmatrix}, \\
{}^2x_t &= \left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a_{1,t-n+1} \cdots a_{n,t}} \end{array}\right] \begin{bmatrix} y_{t-n+1} \\ \vdots \\ y_t \end{bmatrix}, \\
{}^2x_t &= {}^{12}P_t\, {}^1x_t\,.
\end{aligned}
$$

(10) Theorem. Let us have the reconstructible realization $(A_t, b_t, c'_t) = (a_t, b_t, e'_n)$. Then the reconstruction

$$\hat{x}_{t+1} = A_t\hat{x}_t + b_tu_t + k_{rec,t}(y_t - c'_t\hat{x}_t)$$

with the reconstruction gain

(a) $$k_{rec,t} = a_t - a_{rec}$$

is such that the reconstruction error $\tilde{x}_t =_{def} x_t - \hat{x}_t$ evolves as

(b) $$\tilde{x}_{t+1} = \left[\begin{array}{c|c} 0 & \\ \hline I & a_{rec} \end{array}\right] \tilde{x}_t\,.$$

Or writing with the help of characteristic polynomials:

$$\operatorname{char}(A_t - k_{rec,t}c_t') = \operatorname{char}\left[\begin{array}{c|c} 0 & \\ \hline I & a_{rec} \end{array}\right].$$

Proof. To obtain the dynamics of reconstruction error, we subtract the dynamics of reconstruction from the plant dynamics:

$$\begin{aligned} x_{t+1} &= A_t x_t + b_t u_t\,, \\ \hat{x}_{t+1} &= A_t \hat{x}_t + b_t u_t + k_{rec,t} c_t' \tilde{x}_t \\ \hline \tilde{x}_{t+1} &= x_{t+1} - \hat{x}_{t+1} \\ &= A_t \tilde{x}_t - k_{rec,t} c_t' \tilde{x}_t \\ &= \left[\begin{array}{c|c} 0 & \\ \hline I & a_t - k_{rec,t} \end{array}\right] \tilde{x}_t\,. \end{aligned}$$

Requiring (b) we shall obtain the reconstruction gain (a).

(11) Corollary. Both reconstruction $\hat{x}_t$ and reconstruction error $\tilde{x}_t$ are transformed by ${}^{12}P_t(\cdot)$.

(12) Corollary. Reconstruction gain ${}^1k_{rec,t}$ is transformed by ${}^{12}P_{t+1}(\cdot)$.

(13) Note. We have been led by the principle: Those coordinates are best, which simplify the problem most. Coordinates transforms and the induced transforms of matrices we shall now summarize in the table:*)

${}^1x_t,\ {}^1\hat{x}_t,\ {}^1\tilde{x}_t$	${}^{12}P_t(\cdot)$	
${}^1c_t',\ {}^1N_t,\ {}^1k_{con,t}'$	$(\cdot)\,{}^{12}P_t^{-1}$	
${}^1b_t,\ {}^1M_t,\ {}^1k_{rec,t}$	${}^{12}P_{t+1}(\cdot)$	
${}^1A_t$	${}^{12}P_{t+1}(\cdot)\,{}^{12}P_t^{-1}$	
char ${}^1A$, det ${}^1A$	$1(\cdot)$	(${}^1A$ const.)
$[y_t\ y_{t-1} \cdots y_{t-n+1}]' =_{def} {}^1x_t$	${}^1N_t^{-1}(\cdot)$	$({}^2b = 0)$
$[u_{t-n}\ u_{t-n+1} \cdots u_{t-1}]' =_{def} {}^1x_t$	${}^1M_{t-1}(\cdot)$	$({}^2x_{t-n} = 0)$

The single transforms can be associated. So we shall obtain the commutative diagram (Fig. 3.)

The meaning of injection *inj*, forward shift z, and projection *proj* can be seen directly from the diagram. Extending (4/par. 2.2), by the commuting we understand that for any two nodes connected by two or more different paths, these paths are mutually interchangeable, e.g. for the nodes u_{t-1} and y_{t+1} it holds: $c_{t+1}' A_t b_{t-1} =$

*) The key for reading the table: ${}^1b_t \mapsto {}^2b_t = {}^{12}P_{t+1}\,{}^1b_t$.

 $= c'_{t+1} M_t z\, inj_{t-1}$. At a glance we see that a single commutative diagram writes down otherwise nontransparent tens of equations. We recommend the representation or even the derivation of the results with the help of the commutative diagrams.

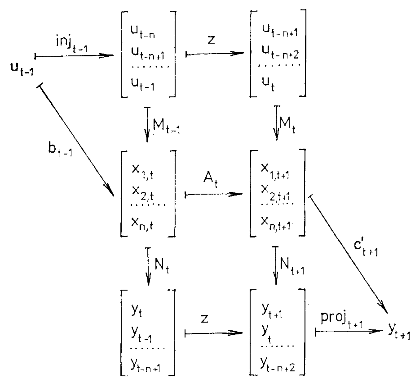

Fig. 3

(14) Problem. Write down injection, forward shift, and projection with the help of matrices.

(15) Theorem. Let the realization (A_t, b_t, c'_t) be perturbed at the output by the polynomial disturbance v_t:

$$x_{t+1} = A_t x_t + b_t u_t\,, \quad y_t = c'_t x_t + v_t\,,$$

$$x_{v,t+1} = A_v x_{v,t}\,, \quad v_t = c'_v x_{v,t}\,,$$

$$A_v = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1 \end{bmatrix}, \quad c'_v = [\,0\,|1] : \mathbb{Q}^r \to \mathbb{Q}$$

Further, let the regularity condition hold:

$$\left(\begin{bmatrix} A_t & 0 \\ 0 & A_v \end{bmatrix}, \; [c'_t \; c'_v]\right)$$

is reconstructible. Then there exists such gain

$$k_{\Sigma,rec,t} = \begin{bmatrix} k_{rec,t} \\ k_{v,t} \end{bmatrix}$$

of the reconstruction

$$\begin{bmatrix}\hat{x}_{t+1}\\ \hat{x}_{v,t+1}\end{bmatrix} = \begin{bmatrix}A_t & 0\\ 0 & A_v\end{bmatrix}\begin{bmatrix}\hat{x}_t\\ \hat{x}_{v,t}\end{bmatrix} = k_{\Sigma,rec,t}(y_t - c'_t\hat{x}_t - c'_v\hat{x}_{v,t}) = \begin{bmatrix}b_t\\ 0\end{bmatrix}u_t$$

that for all output disturbances v_t and for $t \to \infty$:

$$\hat{x}_t \to x_t .$$

Proof.

$$\left(\begin{bmatrix}A_t & 0\\ 0 & A_v\end{bmatrix}, \; [c'_t \; c'_v]\right)$$

is reconstructible, so it is equivalent to the phase couple $(a_{\Sigma,t}, e'_{n+r})$. For the latter we shall construct the gain $k_{\Sigma,rec,t}$ in such a way that characteristic polynomial specifying the decay of

$$\begin{bmatrix}\tilde{x}_t\\ \tilde{x}_{v,t}\end{bmatrix}$$

was equal to the prescribed stable polynomial. Then for $t \to \infty$:

$$\hat{x}_t \to x_t ,$$
$$\hat{x}_{v,t} \to x_{v,t} .$$

(16) Theorem. Let the realization (A_t, b_t, c'_t) is perturbed at the input by the polynomial disturbance w_t:

$$x_{t+1} = A_t x_t + b_t(u_t + w_t) , \quad y_t = c'_t x_t ,$$
$$x_{w,t+1} = A_w x_{w,t} , \quad w_t = c'_w x_{w,t} ,$$
$$A_w = \begin{bmatrix}1 & & & \\ 1 & 1 & & \\ & \ddots & \ddots & \\ & & 1 & 1\end{bmatrix}, \quad c'_w = [\,0\,|1] : \mathbb{Q}^s \to \mathbb{Q} .$$

Further, let the regularity condition hold*): the realization

$$\left(\begin{bmatrix}A_t & 0\\ e_1 \otimes c'_t & A_w\end{bmatrix}, \begin{bmatrix}b_t\\ 0\end{bmatrix}, [\,0\,|1]\right)$$

*)

$$e_1 \otimes c' =_{\text{def}} \begin{bmatrix}1 \,.\, c'\\ 0 \,.\, c'\\ \cdots\\ 0 \,.\, c'\end{bmatrix},$$

so $\otimes$ is the Kronecker product.

is reachable and reconstructible. Then there exists such gain $k'_{\Sigma,con,t} = [k'_{con,t}\ k'_{B,t}]$ of the control

$$\begin{bmatrix} x_{t+1} \\ x_{B,t+1} \end{bmatrix} = \begin{bmatrix} A_t - b_t k'_{con,t} & -k'_{B,t} \\ e_1 \otimes c'_t & A_w \end{bmatrix} \begin{bmatrix} x_t \\ x_{B,t} \end{bmatrix} + \begin{bmatrix} b_t \\ 0 \end{bmatrix} w_t ,$$

that for all input disturbances w_t and for $t \to \infty$:

$$x_t \to 0 .$$

Proof. Putting together the state equations for the plant and for the input disturbance we shall obtain the couple

$$\left(\begin{bmatrix} A_t & 0 \\ e_1 \otimes c'_t & A_w \end{bmatrix}, \begin{bmatrix} b_t \\ 0 \end{bmatrix} \right) .$$

Reachability condition guarantees that the latter couple is equivalent to the phase couple $(a'_{\Sigma,t}, e_{n+s})$: for this we shall find such gain $k'_{\Sigma,con,t}$ that will guarantee that the characteristic polynomial specifying for $w_t = 0$ the decay of control error

$$\begin{bmatrix} x_t \\ x_{B,t} \end{bmatrix}$$

will be equal to the prescribed stable characteristic polynomial. Further, we shall analyze the evolution of the control error for general w_t. From the coordinates

$$\begin{bmatrix} x_t \\ x_{B,t} \end{bmatrix}$$

of the both state of the plant and the state of the disturbance buffer, we shall pass to the coordinates

$$\begin{bmatrix} x_t \\ \tilde{x}_{w,t} \end{bmatrix}$$

where $\tilde{x}_{w,t}$ are the coordinates of the difference between the coordinates of the disturbance $x_{w,t}$ and the pertinent linear combination of the buffer coordinates:

$$\begin{bmatrix} \tilde{x}_{w,1,t} \\ \tilde{x}_{w,2,t} \\ \cdots \\ \tilde{x}_{w,s,t} \end{bmatrix} = \begin{bmatrix} x_{w,1,t} - k_{B,s,t} x_{B,1,t} \\ x_{w,2,t} - k_{B,s-1,t} x_{B,s,t} - k_{B,s,t} x_{B,2,t} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_{w,s,t} - k_{B,1,t} x_{B,1,t} \cdots - k_{B,s,t} x_{B,s,t} \end{bmatrix} .$$

For the dynamics of both control error and disturbance buffering we shall obtain

$$\begin{bmatrix} x_{t+1} \\ \tilde{x}_{w,t+1} \end{bmatrix} = \begin{bmatrix} A_t - b_t k'_{con,t} & e_s \\ -k_{B,t} \otimes c'_t & A_w \end{bmatrix} \begin{bmatrix} x_t \\ \tilde{x}_{w,t} \end{bmatrix} .$$

But the characteristic values of the matrix

$$\begin{bmatrix} A_t - b_t k'_{con,t} & e_s \\ -k_{B,t} \otimes c'_t & A_w \end{bmatrix}$$

are equal to the characteristic values of the matrix

$$\begin{bmatrix} A_t - b_t k'_{con,t} & -k_{B,t} \\ e_1 \otimes c'_t & A_w \end{bmatrix},$$

as follows from the definition of determinant as a sum of product. So our prescription of the dynamics of

$$\begin{bmatrix} x_t \\ x_{B,t} \end{bmatrix}$$

is at the same time the prescription of the dynamics of

$$\begin{bmatrix} x_t \\ \tilde{x}_{w,t} \end{bmatrix}$$

and for $t \to \infty$ it holds:

$$x_t \to 0\,,$$

$$k_{B,s,t} x_{B,1,t} \to x_{w,1,t}\,,$$

$$k_{B,s-1,t} x_{B,1,t} + k_{B,s,t} x_{B,s,t} \to x_{w,2,t}\,,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$k_{B,1,t} x_{B,1,t} + k_{B,2,t} x_{B,2,t} \dots + k_{B,s,t} x_{B,s,t} \to x_{w,s,t}$$

(— or the decay of x_t was enabled by that the pertinent linear combinations of disturbance buffer reconstructed the coordinates of the disturbance state.)

(17) Note. Within the time specified by the characteristic values there is suppressed not only the initial control error but the polynomial input disturbance, too. Control buffers not only the polynomial disturbances but the disturbances composed of polynomial segments (i.e. splines), too. These segments should be of bigger length than the time specified by the characteristic values. Modelling of the disturbances using polynomials is adequate under the slow, with respect of specified time of control error decay, disturbances. Among them belongs an unknown bias ($s = 1$) and an unknown trend ($s = 2$). (Similarly for reconstruction.)

2.4 Control with Reconstruction: Compensator

Natural solution of control in the case of unknown state seems to be the control derived from the reconstructed state, (1). The latter solution enables again the specification of dynamics of both control and reconstruction error, (2 to 4). For easy computation of both gains we compute these gains in two different phase coordinates.

For this ease of computation we pay by the necessity of transform from the first to the second coordinates. For the constant plant we know the second realization immediately, (4).

(1) Definition. Control based on reconstruction, compensator, has the structure:

plant	$x_{t+1} = A_t x_t + b_t u_t\,, \quad y_t = c_t' x_t$	
compensator	reconstruction	$\hat{x}_{t+1} = A_t \hat{x}_t + b_t u_t + k_{rec,t}(y_t - \hat{y}_t),\ \hat{y}_t = c_t' \hat{x}_t$
	control	$u_t = -k_{con,t}' \hat{x}_t$

By the synthesis of compensator we understand computation of gains $k_{con,t}$, $k_{rec,t}$.

(2) Theorem. Let the first realization $({}^1a_t', e_n, {}^1c_t)$ is reconstructible. Then for the compensator gains

$$ {}^1k_{con,t} = {}^1a_t - a_{con}\,, \quad {}^2k_{rec,t} = {}^2a_t - a_{rec}\,, $$

the control and reconstruction errors decay with dynamics

$$ \begin{bmatrix} {}^1x_{t+1} \\ \hline {}^2\tilde{x}_{t+1} \end{bmatrix} = \left[\begin{array}{c|c} \begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a_{con}'} \end{array} & \begin{array}{c} 0 \\ \hline (a_t' - a_{rec}')\,{}^{12}P_t^{-1} \end{array} \\ \hline 0 & \begin{array}{c|c} \begin{array}{c} 0 \\ \hline I \end{array} & a_{rec} \end{array} \end{array}\right] \begin{bmatrix} {}^1x_t \\ \hline {}^2\tilde{x}_t \end{bmatrix} $$

where the second realization is $({}^2a_t, {}^2b_t, e_n')$.

P r o o f. The control gain we compute for the first realization, the reconstruction gain for the second realization. The reconstruction error decays independently of control error, so it suffice investigate the latter. In the second realization:

$$ {}^1x_{t+1} = \left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{{}^1a_t'} \end{array}\right] {}^1x_t + \begin{bmatrix} 0 \\ \hline 1 \end{bmatrix} u_t $$

$$ u_t = -{}^1k_{con,t}'\,{}^1\hat{x}_t = -({}^1a_t' - a_{con}')({}^1x_t - {}^1\tilde{x}_t), \qquad {}^1\tilde{x}_t = {}^{12}P_t^{-1}\,{}^2\tilde{x}_t $$

(3) Corollary. Characteristic values of decay of control and reconstruction errors for the compensator are given by the union of characteristic values of control (specified by a_{con}) and of reconstruction (specified by a_{rec}).

Proof. From the definition of determinant as a sum of products it follows that the characteristic values of

$$\left[\begin{array}{c|c} \begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a'_{con}} \end{array} & \begin{array}{c} 0 \\ \hline (a'_t - a'_{con})\, {}^{12}P_t^{-1} \end{array} \\ \hline 0 & \begin{array}{c|c} \begin{array}{c} 0 \\ \hline I \end{array} & a_{rec} \end{array} \end{array}\right]$$

are identical with the characteristic values of

$$\left[\begin{array}{c|c} \begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a'_{con}} \end{array} & 0 \\ \hline 0 & \begin{array}{c|c} \begin{array}{c} 0 \\ \hline I \end{array} & a_{rec} \end{array} \end{array}\right]$$

which are again identical with the union of characteristic values of

$$\left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a'_{con}} \end{array}\right]$$

and

$$\left[\begin{array}{c|c} \begin{array}{c} 0 \\ \hline I \end{array} & a_{rec} \end{array}\right].$$

(4) Note. After the restriction of (2) to constant ${}^1a$, ${}^1c'$ it holds:

$${}^2A = {}^1A', \quad {}^2b = {}^1c, \quad {}^2c' = {}^1b'.$$

(It can be checked by the substitution to the relation

$${}^2M\,{}^1M^{-1}\,{}^1A = {}^1A'\,{}^2M\,{}^1M^{-1}$$

and similarly for ${}^2b$, ${}^2c'$.) The control gain then can be immediatelly written as

$${}^2k'_{rec} = {}^1a' - a'_{rec}$$

without the necessity of computing

$${}^2A = {}^{12}P\left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{{}^1a'} \end{array}\right]{}^{12}P^{-1}.$$

We know already that, under some regularity conditions, to the realization (A, b, c') with $n^2 + 2n$ parameters there exists equivalent reconstructible phase realization (a, b, e_n') with $2n$ parameters. Because of this, we are concerned with recursive identification of the latter realization parameters a, b. At first, we shall extend our results on reconstruction to the nonzero input and we shall find the relation between the history of input and output, both of the length n and the state, (1). Afterwards, for reconstructible phase realization, we shall pass on to the regression which connects the current output with the history of input and output, both of the length n, (2). This coupling is linear with the coefficients given by the $2n$ parameters of reconstructible phase realization. From the regression we shall pass to prefiltered regression, (3 to 7). By the prefiltering we understand the preliminary processing of input and output – which are in general both disturbed by the additive disturbances – by the two copies of the prefilter. By the prefiltering we want to suppress these additive disturbances at the input and output. E.g. for the disturbances which are fast with respect to plant we can take the low pass from Part 1 as the most simple prefilter. From the considerations which transcend our consistent linear theory follows, [8, 16, 18], that in some quadratic sense the best prefilter would be one equivalent to identified plant. To approximate this best prefilter, we can use for the prefilter some either a priori or current estimate of the identified plant. Lastly we shall convert the recursive identification of $2n$ parameters of the phase realization to the reconstruction of the state of the realization of which the state equation is the rewriting of constancy of these parameters and the output equation is the prefiltered regression, (8, 9). The postulate of constancy of the parameters we introduce to avoid, consistently taken transfinite, recursion (model of parameters changes, model of model of parameters changes, ...). We yet know that constancy is adequate even for the parameters which will change after the decay of recursive identification error. Roughly, all recursive identification will be the corollary of reconstruction.

(1) Corollary. Let (A_t, c_t') be reconstructible. Then for all initial states x_{t-n+1} of the realization (A_t, b_t, c_t') the resulting current state x_t can be reconstructed from the history of output $y_{t-n+1}, \ldots, y_{t-1}, y_t$ and input $u_{t-n+1}, \ldots, u_{t-1}$. The resulting current state is

$$x_t = N_t^{-1} \begin{bmatrix} y_t \\ y_{t-1} \\ \cdots\cdots \\ y_{t-n+1} \end{bmatrix} + H_t \begin{bmatrix} 0 \\ u_{t-1} \\ \cdots\cdots \\ u_{t-n+1} \end{bmatrix}$$

where

$$\begin{aligned} H_{ij,t} &= 0 & &(1 \leqq i \leqq j \leqq n), \\ &= e_{i-j+1}' N_{t-j+1} b_{t-j} & &(1 \leqq j < i \leqq n). \end{aligned}$$

Proof follows the proof of (1/par. 2.3): for x_{t-1} we substitute now $A_{t-1}^{-1}(x_t - b_{t-1}u_{t-1})$ and similarly for x_{t-2} to x_{t-n+1}.

(2) Corollary. The current output of the reconstructible realization (a_t, b_t, e_n') is coupled with the history of output and input of the length n by the regression

$$\text{(a)} \qquad y_t = a_{n,t-1}y_{t-1} + a_{n-1,t-2}y_{t-2} \dots + a_{1,t-n}y_{t-n} + \\ + b_{n,t-1}u_{t-1} + b_{n-1,t-2}u_{t-2} \dots + b_{1,t-n}u_{t-n}\,.$$

Proof follows the proof of (9/par. 2.3): for $x_{n,t}$ we now substitute $x_{n-1,t-1} + a_{n,t-1}y_{t-1} + b_{n,t-1}u_{t-1}$ and similarly for $x_{n-2,t-2}$ to $x_{1,t-n+1}$.

(3) Definition. We shall say that the realization (A_t, b_t, c_t'), where

$$x_{t+1} = A_t x_t + b_t u_t\,, \quad y_t = c_t' x_t\,,$$

is invertible if there exists the realization $({}^IA_t, {}^Ib_t, {}^Ic_t')$, where

$${}^Ix_{t+1} = {}^IA_t\,{}^Ix_t + {}^Ib_t\,{}^Iu_t\,, \quad {}^Iy_t = {}^Ic_t'\,{}^Ix_t \quad (2 \leqq m \leqq n + 1)$$

such that for the tandem connection of both realizations

$${}^Iu_t = y_t\,,$$

the output of the realization $({}^IA_t, {}^Ib_t, {}^Ic_t')$ is equal, for zero initial states of both realization and zero history of u_t of the length m, to the delayed input of the realization (A_t, b_t, c_t')

$$\text{(a)} \qquad {}^Iy_t = u_{t-m}\,.$$

We shall call $({}^IA_t, {}^Ib_t, {}^Ic_t')$ the inverse of (A_t, b_t, c_t').

(4) Corollary. If the realization (A_t, b_t, c_t') is invertible then every realization equivalent with (A_t, b_t, c_t') is invertible. If $({}^IA, {}^Ib_t, {}^Ic_t)$ is inverse of (A_t, b_t, c_t') then the inverse of (A_t, b_t, c_t') is every realization equivalent with $({}^IA_t, {}^Ib_t, {}^Ic_t')$.

(5) Theorem. Let the reconstructible realization (a_t, b_t, e_t') have at least one of n components of b_t uniformly nonzero. Then there exists inverse of this realization.

Proof. It holds the regression (2a). Let the uniformly nonzero component of (2a) be the component $b_{j,t-n-1+j}$, i.e. the coefficient at $u_{t-n-1+j}$. Let us divide (2a) by the latter coefficient. Further let us convert $(1/b_{j,t-n-1+j})\, y_t$ to the right-hand side of the latter equation and u_{t-n} on the left-hand side. The new equation is equivalent with (2a): substituting y_t from (2a) to the new equation, we shall obtain trivial equality. The new recursion should have at the left-hand side (the causal effect side) higher time than at the right-hand side (the cause side). Because of this we should compute the latter recursion with the delay m equal at least $n - j + 2$. Denoting the external variables of this new recursion as ${}^Iu$, ${}^Iy$, we shall obtain instead of the former trivial equality the equality (3a) or our new recursion is inverse to (a).

(6) Theorem and definition. Let the realization $(A_{pf,t}, b_{pf,t}, c'_{pf,t})$ be invertible. Then for reconstructible realization (a_t, b_t, e'_n) it holds

(a) $$y_{pf,t} = a_{n,t-1}y_{pf,t-1} + a_{n-1,t-2}y_{pf,t-2} \dots + a_{1,t-n}y_{pf,t-n} + \\ + b_{n,t-1}u_{pf,t-1} + b_{n-1,t-2}u_{pf,t-2} \dots + b_{1,t-n}u_{pf,t-n}$$

where for zero initial states

$$x_{u,t+1} = \left[\begin{array}{c|c} \frac{0}{I} & a_{pf,t} \end{array}\right] x_{u,t} + b_{pf,t}u_t\,, \quad u_{pf,t} = [\,0\,|1]\, x_{u,t}\,,$$

$$x_{y,t+1} = \left[\begin{array}{c|c} \frac{0}{I} & a_{pf,t} \end{array}\right] x_{y,t} + b_{pf,t}y_t\,, \quad y_{pf,t} = [\,0\,|1]\, x_{y,t}\,.$$

u_{pf}, y_{pf} are called prefiltered input and output. Realization $(a_{pf,t}, b_{pf,t}, e'_n)$ is called the prefilter.

P r o o f. We shall prefilter the prefiltered input and output by the inverse of prefilter. So we shall obtain again the regression for input and output, viz. for the time delayed by m.

(7) Note. In the following we need not construct any inverse. We only need construct the prefilter in such a way to be invertible. We have shown that it suffice to construct it as reconstructible phase realization with at least one of components of b_t uniformly nonzero.

(8) Note. Finally we are reaching both the statement and solution of recursive identification. Let us suppose that the parameters of reconstructible phase realization (a_t, b_t, e'_n), i.e. the coefficients a_i, b_i of the prefiltered regression are constant. Let us take these coefficients for the state. The state equation for the latter state we know: it is trivial one, with $A = I$. Then an output equation is the prefiltered regression. For such introduced realization $(I, 0, c'_{pf,t})$ we are solving, under regularity condition for $(I, c'_{pf,t})$, the recursive identification as the realization's $(I, 0, c'_{pf,t})$ state reconstruction. Finally we also see why we had introduced the time-varying model: it was a.o. because $c'_{pf,t}$ is created by the prefiltered history of input and output. Recursive identification requiring time-varying model is the corollary of reconstruction and the reconstruction is, *cum grano salis*, relabeling of the control.

(9) Corollary. Let for the prefiltered regression (6a) it holds:

(a)
$$^1p_t = \begin{bmatrix} a_1 \\ \dots \\ a_n \\ b_1 \\ \dots \\ b_n \end{bmatrix},$$

(b) $(I, c'_{pf,t})$ where

$$c'_{pf,t} = [y_{pf,t-3n+2} \dots y_{pf,t-2n+1}\, u_{pf,t-3n+2} \dots u_{pf,t-2n+1}]$$

is reconstructible. Then:

(c) to the first realization $(I, 0, c'_{pf,t})$ there exists the equivalent second realization $(^2a_t, 0, e'_{2n})$ where for $i = 1, 2 \dots 2n$:

$$^2a_{i,t} = c'_{pf,t+2n+1-i} \begin{bmatrix} c'_{pf,t+2n-i} \\ \dots\dots \\ c'_{pf,t+1-i} \end{bmatrix}^{-1} e_{2n-i+1}\,,$$

(d) evolution of recursive identification error is

$$^2\tilde{p}_{t+1} = \left[\begin{array}{c|c} 0 & \\ \hline I & a_{id} \end{array}\right] {}^2\tilde{p}_t\,, \quad {}^1\tilde{p}_t = {}^{12}P_t^{-1}\, {}^2\tilde{p}_t$$

where the recursive identification

$$^1\hat{p}_{t+1} = {}^1\hat{p}_t + {}^1k_{id,t}(y_{pf,t-2n+2} - c'_{pf,t}\, {}^1\hat{p}_t)$$

has the gain

$$^1k_{id,t} = {}^{12}P_{t+1}^{-1}(^2a_t - a_{id})\,,$$

$$^{12}P_{t+1} = \left[\begin{array}{c|cccc} & ^2a_{1,t} & & & \\ & ^2a_{2,t} & ^2a_{1,t-1} & & \\ & \dots & \dots & \dots & \dots \\ & ^2a_{2n-1,t} & ^2a_{2n-2,t-1} & \dots & ^2a_{1,t-2n+1} \\ \hline 1 & & & & \end{array}\right] \begin{bmatrix} c'_{pf,t+1} \\ c'_{pf,t} \\ \dots\dots \\ c'_{pf,t-2n+1} \\ c'_{pf,t-2n+2} \end{bmatrix}.$$

Proof. The corollary is the specialization of reconstruction for realization $(I, 0, c'_{pf,t})$.

(10) Note. Both the internal (state) description and the external (regression) description are characterized by n, which is either the dimension of the state or the length of history of input and output. Knowing at least one interpretation of some state coordinates, this situation occurs e.g. in the classical mechanics or for the electrical drives, we know that n is equal to the number of degrees of freedom. Not knowing such interpretation, it remains nothing but return to our regularity conditions of reachability and reconstructibility. When these regularity conditions are fulfilled (are not fulfilled, resp.) the reachability and reconstructibility matrices are regular (are not regular, resp.) and the future of input needed for reaching the new state or the history of output needed to reconstruct the current state are (are not, resp.) of bounded amplitudes. Further, if our linear model originated from the local linearization of nonlinear model, then for adequacy of linearization we require not only the boundeness of inputs and outputs but – in some sense – even sufficient smallness. Because of this need we not only regularity but – in some sense – even good regularity. If the latter is not fulfilled then we have to pass on the model of the smaller dimension n. During this we resign on the control of that part of the original model which led to large inputs and outputs. It can be shown, [2, 9, 16], that the model which control or reconstruction leads to large input or outputs can be characterized by the three, mutually connected ways:

(a) some of the characteristic values of the realization, i.e. the characteristic values of the matrix

$$\left[\begin{array}{c|c} 0 & \\ \hline I & a \end{array}\right]$$

and of the roots of the polynomial $b_1\lambda^{n-1} + b_2\lambda^{n-2} \dots + b_n$ are mutually close;

(b) some of the characteristic values of the realization and of its inverse are mutually close;

(c) some of the stable characteristic values of the realization evolving in the continuous time have the magnitudes of different orders.

2.6 Reconstruction with Recursive Identification: Adaptive Filter

Adaptive filter is composed from recursive identification and reconstruction, (1). For the reconstructible $(I, c'_{pf,t})$ the initial error of recursive identification decays with the time specified by a_{id}. Reconstruction is based on recursively identified parameters. For $x_t, u_t \to 0$ with $t \to \infty$ (this condition will be fulfiled in par. 2.7), decays also the reconstruction error, viz. the time specified by a_{rec}, (2).

(1) Definition. Reconstruction with recursive identification, adaptive filter, has the structure:

plant		$x_{t+1} = \left[\begin{array}{c\|c} 0 & \\ \hline I & a \end{array}\right] x_t + bu_t, \quad y_t = [\,0\,\|1]\, x_t$
adaptive filter	recursive identification	$\begin{bmatrix} \hat{a}_{t+1} \\ \hat{b}_{t+1} \end{bmatrix} = \begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix} + k_{id,t}\left(y_{pf,t-2n+2} - c'_{pf,t}\begin{bmatrix} \hat{a}_t \\ \hat{b}_t \end{bmatrix}\right)$
	reconstruction	$\hat{x}_{t+1} = \left[\begin{array}{c\|c} 0 & \\ \hline I & \hat{a}_t \end{array}\right] \hat{x}_t + \hat{b}_t u_t + k_{rec,t}(y_t - \hat{y}_t), \quad \hat{y}_t = [\,0\,\|1]\, \hat{x}_t$

By the synthesis of adaptive filter we understand the computation of the gains $k_{id,t}$, $k_{rec,t}$.

(2) Theorem. Let $(I, c'_{pf,t})$ be reconstructible. Then for the adaptive filter gain

$$^{1}k_{id,t} = {}^{12}P_{t+1}^{-1}({}^{2}a_t - a_{id}), \quad k_{rec,t} = \hat{a}_t - a_{rec}$$

the evolution of recursive identification and reconstruction errors is

$$^{12}P_{t+1}\begin{bmatrix}\tilde{a}_{t+1}\\ \tilde{b}_{t+1}\end{bmatrix} = \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} a_{id}\right] {}^{12}P_t \begin{bmatrix}\tilde{a}_t\\ \tilde{b}_t\end{bmatrix},$$

$$\tilde{x}_{t+1} = \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} a_{rec}\right]\tilde{x}_t + \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} \tilde{a}_t\right] x_t + \tilde{b}_t u_t .$$

Proof. Evolution of recursive identification error is the same as for recursive identification without reconstruction, (9d/par. 2.5). Because of this we shall investigate only the evolution of reconstruction error

$$x_{t+1} = \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} a\right] x_t + b u_t, \quad y_t = [\,0\,|1]\, x_t,$$

$$\hat{x}_{t+1} = \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} \hat{a}_t\right]\hat{x}_t + \hat{b}_t u_t + k_{rec,t}(y_t - [\,0\,|1]\,\hat{x}_t),$$

$$\begin{aligned}
\tilde{x}_{t+1} &=_{def} x_{t+1} - \hat{x}_{t+1}, \\
&= \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} a\right] x_t - \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} a - \tilde{a}_t\right](x_t - \tilde{x}_t) + \tilde{b}_t u_t \\
&\quad - (a - \tilde{a}_t - a_{rec})[\,0\,|1]\,\tilde{x}_t \\
&= \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} \tilde{a}_t\right] x_t + \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} a_{rec}\right]\tilde{x}_t + \tilde{b}_t u_t .
\end{aligned}$$

2.7 Control with Adaptive Filter: Self-Adjusting Regulator

Self-adjusting regulator is composed from the adaptive filter and control, (1). Under the reconstructibility of the state

$$\begin{bmatrix} a \\ b \end{bmatrix}$$

from the output given by the prefiltered regression and under the reconstructibility and reachability of the state x_t of the phase realization (a, b, e'_n), the initial errors of recursive identification, reconstruction, and control decay with the time specified by a_{id}, a_{rec}, a_{reg}, (2). Control of initially unknown plant has four levels: for the specification of them all we are using always the phase coordinates, (3).

(1) Definition. Control based on adaptive filtering, self-adjusting regulator, has the structure:

plant (1/par. 2.6)		
self-adjusting regulator	adaptive filter (1/par. 2.6)	
	control	$u_t = -k'_{con,t}\hat{x}_t$

By the synthesis of self-adjusting regulator we understand the computation of the gains $k_{id,t}$, $k_{rec,t}$, $k_{con,t}$.

(2) Theorem. Let:

(a) realization (a, b, e'_n) be reachable and reconstructible,

(b) realization $(I, 0, c'_{pf,t})$ be reconstructible.

Let us denote:

(A) transform from the realization (a', e_n, b') to the realization (a, b, e'_n) as ${}^{12}P : {}^{1}x_t \mapsto {}^{2}x_t$,

(B) phase realization equivalent to the realization $(I, 0, c'_{pf,t})$ as $({}^{4}a_t, 0, e'_{2n})$,

(C) transform from the realization $(I, 0, c'_{pf,t})$ to the realization $({}^{4}a_t, 0, e'_{2n})$ as

$$ {}^{34}P_t : \begin{bmatrix} a \\ b \end{bmatrix} =_{def} {}^{3}x \mapsto {}^{4}x_t \,. $$

Then for the gain of self-adjusting regulator

$$ {}^{3}k_{id,t} = {}^{34}P^{-1}_{t+1}({}^{4}a_t - a_{id}) \,, \quad {}^{2}k_{rec,t} = \hat{a}_t - a_{rec} \,, \quad {}^{1}k_{con,t} = \hat{a}_t - a_{con} \,, $$

the evolution of recursive identification, reconstruction and control error is:

$$ \begin{array}{|c|} \hline {}^{34}P_{t+1}\begin{bmatrix} \tilde{a}_{t+1} \\ \tilde{b}_{t+1} \end{bmatrix} \\ \hline {}^{2}\tilde{x}_{t+1} \\ \hline {}^{1}x_{t+1} \\ \hline \end{array} = \begin{array}{|c|c|c|} \hline \begin{array}{c|c} 0 & \\ \hline I & \end{array} a_{id} & & \\ \hline & \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} a_{rec}\right] + \\ & + \tilde{b}_t\, {}^{1}k'_{con,t}\, {}^{12}P^{-1} & \left[\begin{array}{c|c} 0 & \\ \hline I & \end{array} \tilde{a}_t\right] - \\ & & - \tilde{b}_t\, {}^{1}k'_{con,t} \\ \hline & \dfrac{0}{-\,{}^{1}k'_{con,t}\, {}^{12}P^{-1}} & \dfrac{0 \mid I}{a'_{con} + \tilde{a}'_t} \\ \hline \end{array} \begin{array}{|c|} \hline {}^{34}P_t\begin{bmatrix} \tilde{a}_t \\ \tilde{b}_t \end{bmatrix} \\ \hline {}^{2}\tilde{x}_t \\ \hline {}^{1}x_t \\ \hline \end{array} $$

Proof. Evolution of control error for the first realization:

$$
{}^1x_{t+1} = \left[\begin{array}{c|c} 0 & I \\ \hline \multicolumn{2}{c}{a'} \end{array}\right] {}^1x_t + \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right] u_t
$$

$$
u_t = -\,{}^1k'_{con,t}\;{}^1\hat{x}_t, \qquad {}^1k'_{con,t} = \hat{a}'_t - a'_{con}, \qquad {}^1\hat{x}_t = {}^1x_t - {}^1\tilde{x}_t = {}^{12}P^{-1}\,{}^2\tilde{x}_t
$$

Evolution of reconstruction error for the second realization is the same as the evolution of reconstruction error for adaptive filter, (2/par. 2.6) – under the constraint that the control is derived from ${}^2x_t$:

$$
{}^2\tilde{x}_{t+1} = \left[\begin{array}{c|c} 0 & \\ \hline I & a_{rec} \end{array}\right] {}^2\tilde{x}_t + \left[\begin{array}{c|c} 0 & \\ \hline I & \tilde{a}_t \end{array}\right] {}^2x_t + \tilde{b}_t u_t
$$

$$
{}^2x_t = {}^{12}P\,{}^1x_t, \qquad \tilde{b}_t u_t = -\,{}^2k'_{con,t}\;{}^2\hat{x}_t, \qquad {}^2\hat{x}_t = {}^2x_t - {}^2\tilde{x}_t, \qquad {}^2x_t = {}^{12}P\,{}^1x_t, \qquad {}^2k'_{con,t} = {}^1k_{con,t}\,{}^{12}P^{-1}
$$

Evolution of recursive identification error is – up to numbering of realizations – the same as for adaptive filter, (2/par. 2.6).

(3) Note. The Fig. 4 summarizes self-adjusting regulator from the point of view of dynamics prescription.

2.8 Time Quantizing: Input Pulse Amplitude Modulation and Output Sampling

The possible origin of the model

(1) $$x_{t+1} = Ax_t + bu_t, \quad y_t = c'x_t,$$

evolving in the discrete time $t = 0, 1, 2 \ldots$ will be discussed. Let us consider the model

(2) $$\frac{\mathrm{d}}{\mathrm{d}t} x_t = Fx_t + gu_t, \quad y_t = c'x_t.$$

evolving in the continuous time $t \geqq 0$. (In this paragraph an exception will be made: the field will be either that of reals of that or complex numbers.) Having control (2) by digital computer we have limit ourselves to both output sampling and input

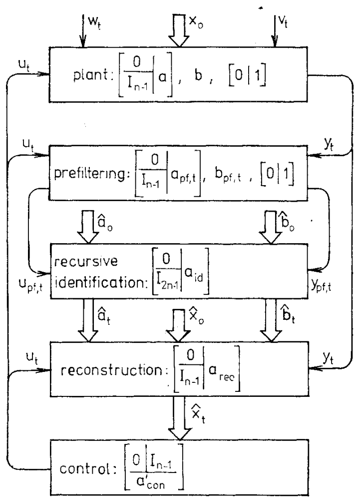

Fig. 4

computation in the discrete times. For the simplicity, let us suppose that both the output sampling and the input computation are done with the sampling period just in the times $t = 0, T, 2T \ldots (T > 0)$. By the input computed at the latter discrete times $t = 0, T, 2T \ldots$ we are acting to the plant (2) evolving in the continuous time $t \geqq 0$. To stay within the linear model, let us suppose that the input on the interval $(kT, (k + 1) T]$ is equal to the input at time kT. Let us drive the plant (2) by this pulse amplitude modulated input. Then

$$x_{t+T} = e^{FT} x_t + \left(\int_t^{t+T} e^{F\tau} g \, d\tau \right) u_t$$

$$=_{def} A x_t + b u_t$$

and after the time normalization $t \mapsto t/T$ we finally obtain (1).

Which are the limitations to the sampling period T? The basic postulate is that the discrete time model should be adequate to continuous time model. Or if we know the conversion

$$(F, g, c') \mapsto \left(e^{FT}, \int_0^T e^{F\tau} g \, d\tau, c'\right)$$

we have to know also the conversion

$$(A, b, c') \mapsto (?, ??, c') . \tag{4}$$

For this it is necessary that the matrix exponential have the inverse. Let us start from the Jordan state matrix, for the simplicity with different characteristic values. For given stable real characteristic value λ_1 we see immediately $e^{\lambda_1 T} \to 0$, for $T \to \infty$. So for sampling period e.g. by order bigger than time constant $-1/\lambda_1$ conversion is impossible. Further let $\lambda_2 = -\lambda_3 = j\omega$ $(\omega > 0)$, i.e.

$$\frac{d}{dt}\begin{bmatrix} x_{2,t} \\ x_{3,t} \end{bmatrix} = \begin{bmatrix} j\omega & 0 \\ 0 & -j\omega \end{bmatrix} \begin{bmatrix} x_{2,t} \\ x_{3,t} \end{bmatrix}$$

or in the discrete time

$$\begin{bmatrix} x_{2,t+T} \\ x_{3,t+T} \end{bmatrix} = \begin{bmatrix} e^{j\omega T} & 0 \\ 0 & e^{-j\omega T} \end{bmatrix} \begin{bmatrix} x_{2,t} \\ x_{3,t} \end{bmatrix} .$$

But $e^{j\omega T} = e^{j(\omega T + k2\pi)}$ $(k = 0, 1, 2 \ldots)$ so from the discrete time state matrix we can find unique continuous time state matrix e.g. only for $\omega < 2\pi T$. For too long sampling period T, with respect to the period of the plant oscillation modes $2\pi/\omega$ is unicity of conversion again lost. The limit $\omega < 2\pi T$ holds even for $L_2 = \alpha + j\omega$, $L_3 = \alpha - j\omega$.

Let us further postulate that the conversion from (2) to (1) would preserve reachability. Let us again consider $L_2 = -L_3 = j\omega$ $(\omega > 0)$. For

$$\left(\begin{bmatrix} j\omega & 0 \\ 0 & -j\omega \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$$

the determinant of reachability matrix: $\det M = 2j\omega \neq 0$. After the conversion to the discrete time model

$$\left(\begin{bmatrix} e^{j\omega T} & 0 \\ 0 & e^{-j\omega T} \end{bmatrix}, \begin{bmatrix} (e^{j\omega T} - 1)/j\omega \\ (e^{-j\omega T} - 1)/(-j\omega) \end{bmatrix}\right)$$

we shall obtain the determinant $\det M = 2(\cos \omega T - 1)/j\omega$, so for $T = 2\pi/\omega, 4\pi/\omega, 6\pi/\omega, \ldots$ the reachability is lost because of resonance between the sampling period with the period of the plant oscillation modes. (Similarly for $L_2 = \alpha + j\omega$, $L_3 = \alpha - j\omega$ and for the whole (A, b).)

Our synthesis, beginning with control gain computation and ending with the self-adjusting regulator gains computation is linear over the field of rationals $\mathbb{Q}$. Let us explain why to use the rationals $\mathbb{Q}$ and not the reals $\mathbb{R}$. The digital computer with standard arithmetic does not make the realization over $\mathbb{R}$ possible, e.g. because of roundoff. Standard arithmetic of digital computer is algebraically unclosed, unstable. The discipline which should offer to us at least the analysis of the nonlinear perturbations which are the reason of numerical inaccuracies for computation over $\mathbb{R}$, namely the numerical mathematics, is unable to give us such analysis. Because of this, in the former discipline there is no consensus about the choice e.g. several tens of algorithms for matrix inversion. Because of this we shall take the full advantage of the field of rationals $\mathbb{Q}$ for the numerical realization of our synthesis.

Let us remind that the data of our synthesis are both the numbers obtained from the measurement and the prescribed characteristic polynomial coefficients. By the scaling, our rational data can be converted to the integer data. E.g. before the matrix inverse computation, the scaling is done by the integer denominator of a matrix determinant. (Some more subtle approaches are possible.)

As the further step, we shall take the full advantage of the fact that for our synthesis we do not need all integers $\mathbb{Z}$ but we can manage e.g. with nonnegative integers*) smaller than the n-th power of a prime p, say q. Those numbers form the ring $\mathbb{F}_q$ of q elements with arithmetic operations taken modulo $q = p^n$. We shall convert our, now already integer, data to the data modulo q. Conversion $\mathbb{Z} \to \mathbb{F}_q$ is the mapping of straight line (alternatively of spiral) equidistant point onto the circle equidistant points. If we take q less than the number of data levels, the former conversion will not be invertible. For the large number of data levels we should take large q.

Because of this, as still another step, instead of single conversion $\mathbb{Z} \to \mathbb{F}_q$ we shall take several, say r, conversions $\mathbb{Z} \to F_{q_1}$, $\mathbb{Z} \to \mathbb{F}_{q_2}, \ldots, \mathbb{Z} \to \mathbb{F}_{q_r}$. These conversions are now the mappings onto r circles. Let**) m be positive integer. Then it is factorizable as

$$m = p_1^{n_1} p_2^{n_2} \ldots p_r^{n_r} =_{def} q_1 q_2 \ldots q_r\,.$$

We shall call the conversion

$$x \longmapsto (x \bmod q_1,\, x \bmod q_2, \ldots, x \bmod q_r) =_{def} (x_1, x_2, \ldots, x_r)$$

the residual representation of x. For arithmetic in residual representation it holds

$$x + y \longmapsto (x_1 + y_1 \bmod q_1,\ x_2 + y_2 \bmod q_2, \ldots, x_r + y_r \bmod q_r)\,,$$

$$x - y \longmapsto (x_1 - y_1 \bmod q_1,\ x_2 - y_2 \bmod q_2, \ldots, x_r - y_r \bmod q_r)\,,$$

*) Both positive and negative integers we obtain by one more conversion by the affine shift.
**) The following survey is based on [11].

$$x \,.\, y \mapsto (x_1 \,.\, y_1 \bmod q_1,\ x_2 \,.\, y_2 \bmod q_2, \ldots, x_r \,.\, y_r \bmod q_r),$$

$$x^{-1} \mapsto x^{f(m)-1} \bmod m \quad (x, m \text{ relatively prime}),$$

where $f(m)$ is the cardinality of such numbers from $\{1, 2 \ldots m - 1\}$ which are relatively prime with m. For the backward conversion from the residual representation it holds

$$(x_1, x_2 \ldots x_r) \mapsto x = a_1 + q_1 a_2 + q_1 q_2 a_3 \ldots + q_1 q_2 \ldots q_r a_r$$

where

$$a_1 = x_1 \bmod q_1,$$

$$a_2 = \frac{x_2 - a_1}{q_1} \bmod q_2,$$

$$a_3 = \frac{\dfrac{x_3 - a_1}{q_1} - a_2}{q_2} \bmod q_3,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_r = \frac{\dfrac{\dfrac{\dfrac{\dfrac{x_r - a_1}{q_1} - a_2}{q_2} - a_3}{q_3} - a_4}{\ddots} - a_r}{q_{r-1}} \bmod q_r .$$

Thus far the theory, as surveyed by [11]. The residual arithmetic (without x^{-1}) was invented by M. Valach in the fifties, with respect to the synthesis of carry-free arithmetic unit. (This unit was then implemented in Czech EPOS computer – again without x^{-1}.)

Now we shall sketch how to use residual representation for numerical realization (in the sense of exact, not loosing even mere least significant bit) of our synthesis. Let us discuss the most involved operation which is the matrix inverse computation for coordinate transform for recursive identification. Let us choose some finite root-free algorithm, say of $L_K \ldots L_2 L_1 = L$ arithmetic operations. Then for the numerical realization it suffices that the diagram

$$\text{(1)} \qquad \begin{array}{ccccccc} \mathbb{Q}^N_{restr} \subset \mathbb{Q}^N & \xrightarrow{m} & \mathbb{Z}^N & \xrightarrow{m} & (\mathbb{F}^N_{q_1} \times \ldots \times & \mathbb{F}^N_{q_r}) \\ \downarrow L & & & & \downarrow L \bmod q_1 \ldots & \downarrow L \bmod q_r \\ \mathbb{Q}^M_{restr} \subset \mathbb{Q}^M & \xleftarrow{m} & \mathbb{Z}^M & \xleftarrow{m} & (\mathbb{F}^M_{q_1} \times \ldots \times & \mathbb{F}^M_{q_r}) \end{array}$$

commutes. (N is the dimension of data vector, M is the dimension of the results vector, $\subset$ is the restriction of rationals to sufficiently small rationals, $\xrightarrow{m}$ is monomorphism, i.e. operations preserving mapping which has the left inverse.) For the commuting of (1) it suffices that the diagram

$$
(2)\qquad
\begin{array}{lllll}
\mathbb{Q}^{N}_{restr} \subset \mathbb{Q}^{N} & \xrightarrow{m} & \mathbb{Z}^{N} & \xrightarrow{m} & (\mathbb{F}^{N}_{q_1} \times \dots \times \mathbb{F}^{N}_{q_r}) \\
\quad\quad \downarrow L_1 & & & & \downarrow L_1 \bmod q_1 \dots \downarrow L_1 \bmod q_r \\
\mathbb{Q}^{N_1}_{restr} \subset \mathbb{Q}^{N_1} & \xleftarrow{m} & \mathbb{Z}^{N_1} & \xleftarrow{m} & (\mathbb{F}^{N_1}_{q_1} \times \dots \times \mathbb{F}^{N_1}_{q_r}) \\
\dots\dots & \dots & \dots & \dots & \dots\dots \\
\mathbb{Q}^{N_{K-1}}_{restr} \subset \mathbb{Q}^{N_{K-1}} & \xleftarrow{m} & \mathbb{Z}^{N_{K-1}} & \xleftarrow{m} & (\mathbb{F}^{N_{K-1}}_{q_1} \times \dots \times \mathbb{F}^{N_{K-1}}_{q_r}) \\
\quad\quad \downarrow L_K & & & & \downarrow L_K \bmod q_1 \dots \downarrow L_K \bmod q_r \\
\mathbb{Q}^{M}_{restr} \subset \mathbb{Q}^{M} & \xleftarrow{m} & \mathbb{Z}^{M} & \xleftarrow{m} & (\mathbb{F}^{M}_{q_1} \times \dots \times \mathbb{F}^{M}_{q_r})
\end{array}
$$

commutes. I.e. that in none of K operations there is an overflow (with respect to restriction) and the inverse element is always defined. To prove the existence of $m = q_1 q_2 \dots q_r$ such that the latter diagram commutes, it is the simplest to use single division expression for the matrix inverse. So we have

(3) Theorem. There exists such $m = q_1 q_2 \dots q_r$ that (1) commutes.

PART 3 CONTROL OF INITIALLY UNKNOWN MULTI-INPUT MULTI-OUTPUT PLANT: SKETCH

No problem is so big or so complicated that is can't be run away from (*Linus*)

For the control of initially unknown plant with n-dimensional state we have now at our disposal r inputs and q outputs:

$$x_{t+1} = A_t x_t + B_t u_t\,, \quad y_t = C_t x_t\,,$$

where the matrices A_t, B_t, C_t have the dimensions $n \times n$, $n \times r$, $q \times n$. Yet we know that almost all control of initially unknown single-input single-output plant had been built on the reachability matrix which served us to tranform a generic realization to phase realization for which the control synthesis was trivial. (Reconstruction had been masquerade control and recursive identification special case of reconstruction.) Because of this we shall concentrate ourselves to introduction of reachability using several inputs, (1). For reachable $(A_t, [b_{1,t}\, b_{2,t} \dots b_{r,t}])$ we shall introduce – this trick had been used for the first time at [3] – such feedback $K_{1,t}$ derived from the state that is reachable even $(A_t - B_t K_{1,t}, b_{1,t})$, (2). But to this couple, i.e. to the control using single input, we had dedicated (incl. the variations for reconstruction) the whole Part 2. So we converted the Part 3 to the Part 2.

(1) Theorem and definition. Let the matrix

$$[M_{p,t+\nu-1} \dots M_{2,t+\nu-1} M_{1,t+\nu-1}] =_{def} \mathscr{M}_{t+\nu-1}$$

where for $j = 1,2 \dots p$

$$M_{j,t+\nu-1} = [A_{t+\nu-1} A_{t+\nu-2} \dots A_{t+\nu-\nu_j+1} b_{j,t+\nu-\nu_j} \dots A_{t+\nu-1} b_{j,t+\nu-2}\, b_{j,t+\nu-1}],$$

$$B_t = [b_{1,t}\, b_{2,t} \dots b_{r,t}],$$

$$\nu = \max\{\nu_j\}, \quad 1 \leqq \nu_j, \quad 1 \leqq p \leqq r, \quad \nu_1 + \nu_2 \dots + \nu_p = n$$

is uniformly regular. ($\mathscr{M}$, M_j, A, B are of dimensions $n \times n$, $n \times \nu_j$, $n \times n$, $n \times r$.) Then for

(a) all initial states α,

(b) all final states ω,

(c) all inputs $u_{j,t}, u_{j,t+1} \dots u_{j,t+\nu-\nu_j-1}$ and $u_{J,t}, u_{J,t+1} \dots u_{J,t+\nu-1}$ $(J = p+1, p+2 \dots r)$ there exists just one future of input $u_{j,t+\nu-\nu_j}, u_{j,t+\nu-\nu_j+1}, \dots u_{j,t+\nu-1}$ such that

$$x_t = \alpha, \quad x_{t+\nu} = \omega.$$

This future of input is

$$\begin{bmatrix} u_{p,t+\nu-\nu_p} \\ \cdots\cdots \\ u_{p,t+\nu-1} \\ \cdots\cdots \\ \cdots\cdots \\ u_{1,t+\nu-\nu_1} \\ \cdots\cdots \\ u_{1,t+\nu-1} \end{bmatrix} = \mathscr{M}^{-1}_{t+\nu-1}(\omega - F_{t+\nu-1,t}\alpha - \sum F_{t+k+1,t+i+1} b_{k,t+1} u_{k,t+i})$$

where

$$F_{m,n} =_{def} A_m A_{m-1} \dots A_n, \quad F_{m,m} =_{def} I$$

and we are summing over i, k according (c). The matrix $\mathscr{M}$ is called the reachability matrix, the index ν is called the reachability index. For $p = 1$, i.e. for reachable $(A_t, b_{1,t})$ we shall say that (A_t, B_t) is reachable by single input $u_{1,t}$. (The input components are numbered in such a way that the scalar input which suffices to reachability is labeled as number one.)

Proof. Iterating the state equation $x_{t+1} = A_t x_t + B_t u_t$ beginning with $x_t = \alpha$ we shall obtain

$$x_{t+v} = A_{t+v-1} \dots A_{t+1} A_t \alpha + \\ + \left[A_{t+v-1} \dots A_{t+2} A_{t+1} B_t \dots A_{t+v-1} B_{t+v-2} \; B_{t+v-1}\right] \begin{bmatrix} u_t \\ \dots\dots \\ u_{t+v-2} \\ u_{t+v-1} \end{bmatrix}.$$

Further we shall decompose the vector of input into p components and consequently even the input matrix into p columns. We shall compose the reachability matrix of dimensions $n \times n$ from the submatrices of dimensions $n \times v_1$, $n \times v_2, \dots, n \times v_p$. The first submatrix, belonging to the first component of vector of input, we are composing like in Part 2. Single component of input generally does not suffices for reaching the final state. Because of this – and at the difference of Part 2 – the first submatrix is generally not a square one. So we shall append to it the second submatrix, in such a way that the columns of the first and the second submatrices are linearly independent on the previous (i.e. on the left) columns. Finally we shall append the p-th submatrix to make columns of now yet square matrix linearly independent. During the composition of submatrices from the single columns there can be left*) the columns which correspond to those input values at the beginning of reaching, which can be made – together with initial and final states – the object of specification.

$$x_{t+v} = A_{t+v-1} \dots A_{t+1} A_t \alpha + \\ + \left[A_{t+v-1} \dots A_{t+v-1-v_p} b_{p,t+v-v_p} \dots A_{t+v-1} b_{p,t+v-2} \; b_{p,t+v-1} \vdots \right. \\ \vdots A_{t+v-1} \dots A_{t+v+1-v_2} b_{2,t+v-v_2} \dots A_{t+v-1} b_{2,t+v-2} \; b_{2,t+v-1} \\ \left. A_{t+v-1} \dots A_{t+v+1-v_2} b_{1,t+v-v_1} \dots A_{t+v-1} b_{1,t+v-1} \; b_{1,t+v-1}\right]$$

$$\times \begin{bmatrix} u_{p,t+v-v_p} \\ \dots\dots \\ u_{p,t+v-2} \\ u_{p,t+v-1} \\ \dots\dots \\ u_{2,t+v-v_2} \\ \dots\dots \\ u_{2,t+v-2} \\ u_{2,t+v-1} \\ u_{1,t+v-v_1} \\ \dots\dots \\ u_{1,t+v-2} \\ u_{1,t+v-1} \end{bmatrix} +$$

*) $\sigma_i = 1$ for $i = 1, 2 \dots p$, $v_i < v$ else $\sigma_i = 0$.

$$+ \sigma_1 \sum_{i=\nu_1+1}^{\nu} A_{t+\nu-1} \dots A_{t+\nu+1-i} b_{1,t+\nu-i} u_{1,t+\nu-i} +$$

$$+ \sigma_2 \sum_{i=\nu_2+1}^{\nu} A_{t+\nu-1} \dots A_{t+\nu+1-i} b_{2,t+\nu-i} u_{2,t+\nu-i} +$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ \sigma_p \sum_{i=\nu_p+1}^{\nu} A_{t+\nu-1} \dots A_{t+\nu+1-i} b_{p,t+\nu-i} u_{p,t+\nu-i} +$$

$$+ b_{p+1,t} u_{p+1,t} + \sum_{i=1}^{\nu-1} A_{t+\nu-1} \dots A_{t+\nu+1-i} b_{p+1,t+\nu-i} u_{p+1,t+\nu-i} +$$

$$+ b_{p+2,t} u_{p+2,t} + \sum_{i=1}^{\nu-1} A_{t+\nu-1} \dots A_{t+\nu+1-i} b_{p+2,t+\nu-i} u_{p+2,t+\nu-i} +$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+ b_{r,t} u_{r,t} + \sum_{i=1}^{\nu-1} A_{t+\nu-1} \dots A_{t+\nu+1-i} b_{r,t+\nu-i} u_{r,t+\nu-i} .$$

The latter equation we finally solve for

$$\begin{bmatrix} u_{p,t+\nu-\nu_p} \\ \dots\dots \\ u_{1,t+\nu-1} \end{bmatrix}.$$

(2) Theorem. Let (A_t, B_t) be reachable. Then there exists such preliminary control gain $K_{1,t}$ that even $(A_t - B_t K_{1,t}, b_{1,t})$ is reachable. It holds:

$$K_{1,t} = S \mathscr{M}_t^{-1}$$

where the selector

$$\begin{aligned} S &= [s_n \dots s_2 \; s_1] , \\ s_j &= -e_j \quad \text{for} \quad j = \nu_1, \nu_1 + \nu_2, \dots, \nu_1 + \nu_2 \dots + \nu_{p-1} \\ &= 0 \quad \text{for remaining} \quad j . \end{aligned}$$

(Matrices $K_{1,t}$, S, s_j, e_j are of dimensions $r \times n$, $r \times n$, $r \times 1$, $r \times 1$.)

Proof. We shall prove that the column vectors of reachability matrix $\mathscr{M}_{t+\nu-1}$ of the couple $(A_t - B_t K_{1,t}, b_{1,t})$ are uniformly linear independent. To this, we shall prove the relation

$$K_{1,t+\nu-1} \mathscr{M}_{t+\nu-1} = S .$$

Gradually we shall obtain for the column vectors of the reachability matrix for single input in the dependence of the column vectors of reachability matrix for p

 inputs:

$$b_{1,t+\nu-1} = b_{1,t+\nu-1},$$

$$(A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})\, b_{1,t+\nu-2} = A_{t+\nu-1}b_{1,t+\nu-2}$$

because of $K_{1,t+\nu-1}b_{1,t+\nu-2} = 0$.

$$(A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})(A_{t+\nu-2} - B_{t+\nu-2}K_{1,t+\nu-2})\, b_{1,t+\nu-3} =$$
$$= (A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})\, A_{t+\nu-2}b_{1,t+\nu-3},$$

because of $K_{1,t+\nu-2}b_{1,t+\nu-3} = 0$ and further equals to

$$A_{t+\nu-1}A_{t+\nu-2}b_{1,t+\nu-3}$$

because of $K_{1,t+\nu-1}A_{t+\nu-2}b_{1,t+m-3} = 0$

....

$$(A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})(A_{t+\nu-2} - B_{t+\nu-2}K_{1,t+\nu-2}) \dots$$
$$\dots (A_{t+\nu-\nu_j+1} - B_{t+\nu+\nu_j+1}K_{1,t+\nu-\nu_1+1})\, b_{1,t+\nu-\nu_1} =$$
$$= (A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})\, A_{t+\nu-2}A_{t+\nu-3} \dots A_{t+\nu-\nu_1+1}b_{1,t+\nu-\nu_1} =$$
$$= A_{t+\nu-1}A_{t+\nu-2} \dots A_{t+\nu-\nu_1+1}b_{1,t+\nu-\nu_1},$$

because of $K_{1,t+\nu-1}A_{t+\nu-2} \dots b_{1,t+\nu-\nu_1} = 0$. These ν_1 vectors compose matrix $M_{1,t+\nu-1}$ which is part of reachability matrix $\mathcal{M}_{t+\nu-1}$ for p inputs, which is uniformly regular: consequently the latter ν_1 vectors are uniformly linearly independent. Further:

$$(A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})(A_{t+\nu-2} - B_{t+\nu-2}K_{1,t+\nu-2}) \dots$$
$$\dots (A_{t+\nu-\nu_1} - B_{t+\nu-\nu_1}K_{1,t+\nu-\nu_1})\, b_{1,t+\nu-\nu_1-1} =$$
$$= (A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})\, A_{t+\nu-2}A_{t+\nu-3} \dots A_{t+\nu-\nu_1}b_{1,t+\nu-\nu_1-1} =$$
$$= A_{t+\nu-1} \dots A_{t+\nu-\nu_1}b_{1,t+\nu-\nu_1-1} + B_{t+\nu-1}e_2 =$$
$$= b_{2,t+\nu-1} + \text{linear combinations of previous vectors},$$
$$(A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})(A_{t+\nu-2} - B_{t+\nu-2}K_{1,t+\nu-2}) \dots$$
$$\dots (A_{t+\nu-\nu_1-1} - B_{t+\nu-\nu_1}K_{1,t+\nu-\nu_1})\, b_{1,t+\nu-\nu_1-2} =$$
$$= (A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})(b_{2,t+\nu-2} + \text{lin. comb. of prev. vect.}) =$$
$$= A_{t+\nu-1}b_{2,t+\nu-2} + \text{linear combinations of previous vectors}$$

....

$$(A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1})(A_{t+\nu-2} - B_{t+\nu-2}K_{1,t+\nu-2}) \dots$$
$$\dots (A_{t+\nu-n+1} - B_{t+\nu-n+1}K_{1,t+\nu-n+1})\, b_{1,t+\nu-n} =$$

$$= \left(A_{t+\nu-1} - B_{t+\nu-1}K_{1,t+\nu-1}\right)\left(A_{t+\nu-2} \cdots A_{t+\nu-\nu_r+1}b_{r,t+\nu-\nu_r} + \right.$$

$$\left. + \text{ linear combinations of previous vectors}\right) =$$

$$= A_{t+\nu-1} \cdots A_{t+\nu-\nu_r+1}b_{r,t+\nu-\nu_r} +$$

$$+ \text{ linear combinations of previous vectors.}$$

(3) Note. Our approach to control of initially unknown plants was based on three equivalent problem-oriented models. Still further equivalent models and their applications are treated in [17].

SYMBOLS

x, A	state, state matrix
u, b, M, k'_{con}	input, input matrix, reachability matrix, control gain
y, c', N, k_{rec}	output, output matrix, reconstructibility matrix, reconstruction gain
v, w	output disturbance, input disturbance
t	time
a, b	phase realization parameters, regression parameters
${}^{12}P$	transform matrix from the first to the second state coordinates
n	dimension of state, length of regression
a_{con}, a_{rec}, a_{id}	specified coefficients of characteristic polynomials of decay of initial error of control, reconstruction, recursive identification
I	unit matrix
$e_1, e_2 \ldots e_n$	first, second, …, n-th column of unit matrix
$\left[\begin{array}{c\|c} 0 & \\ \hline I & a \end{array}\right], \left[\begin{array}{c\|c} 0 & I \\ \hline \multicolumn{2}{c}{a'} \end{array}\right]$	$\begin{bmatrix} 0 & \ldots & 0 & a_1 \\ 1 & \ddots & & \vdots \\ & & 1 & a_n \end{bmatrix}, \begin{bmatrix} 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \\ a_1 & \ldots & & a_n \end{bmatrix}$
$(a, b, e'_n), (a', e_n, c')$	$\left(\left[\begin{array}{c\|c} 0 & \\ \hline I & a \end{array}\right], b, [\,0\,\|1]\right), \left(\left[\begin{array}{c\|c} 0 & I \\ \hline \multicolumn{2}{c}{a'} \end{array}\right], \left[\begin{array}{c} 0 \\ \hline 1 \end{array}\right], c'\right)$
λ, char A	characteristic value, characteristic polynomial of matrix A
$\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{F}_q$	reals, rationals, integers, integers modulo q
$\hat{x}, \tilde{x}$	reconstruction of x, error of reconstruction of x
a'	transposition of a
$=_{def}$	equal by definition, denotes
$\rightarrow, \mapsto$	mapping, rule of mapping

(Received July 15, 1977.)

[1] S. Barnett: A matrix circle in linear control theory. Bull. Inst. Math. Appl. *12* (June 1976), 173—177.

[2] S. Barnett: Introduction to mathematical control theory. Clarendon Pr., Oxford, 1975.

[3] M. Heymann: Comments "On pole assignment in multi-input controllable linear systems". IEEE Trans. Aut. Control *AC-13* (Dec. 1968), 748—749.

[4] R. E. Kalman, Y. C. Ho, K. S. Narenda: Controllability of linear dynamical systems. Contributions to Differential Equations, *1*, Interscience, N.Y., 1963, 189—213.

[5] R. E. Kalman, P. L. Falb, M. A. Arbib: Topics in the Mathematical Control Theory. McGraw-Hill, N.Y. 1969.

[6] S. MacLane: Categories for the Working Mathematician. Springer, N.Y. 1971.

[7] S. MacLane, G. Birkhoff: Algebra. MacMillan, L. 1967.

[8] S. J. Mason, H. J. Zimmermann: Electric Circuits, Signals, and Systems. Wiley, N.Y. 1960.

[9] L. Padulo, M. A. Arbib: System Theory. Saunders, Philadelphia, 1974.

[10] V. Peterka, M. Ullrich: Matematické modelování stochastických systémů (Mathematical modelling of stochastic systems). Zborník 3. Symp. SKS, Kybernetické aspekty problémov identifikácie, Vys. Tatry, Oct. 1971, 53—67.

[11] J. P. Tremblay, R. Manohar: Discrete Mathematical Structures with Application to Computer Sciences. McGraw-Hill, N.Y. 1975.

[12] A. Vaněček: LQG problem of estimation and control in the tensor space. Kybernetika *11* (1975), 4, 287—300.

[13] A. Vaněček: Maximally stable identification and control, and self-adjusting regulators. Preprints IFAC 4th Symp. Identification and System Parameter Estimation, Tbilisi (Sept. 1976), Vol. a, 339—408.

[14] A. Vaněček: Self-adjusting regulators realizable by microcomputers and microprocessors, Vol. II, Res. Rep. ČSAV-ÚTIA No. 720, Praha, No. 1977. (In Czech.)

[15] A. Vaněček: On compensators, adaptive filters, and self-adjusting regulators. Res. Rep. ČSAV-ÚTIA No. 766, Praha, Apr. 1977.

[16] A. Vaněček: Identifiability of time constants with magnitudes different by orders. Automatizace *20* (1977), 2, 34—7. (In Czech.)

[17] A. Vaněček: On Control of initially unknown plants and problem-oriented equivalent models. Rep. Istituto di Automatica, Università di Bologna, Oct. 1977.

[18] P. Young: Some observation on instrumental variable methods of time-series analysis. Int. J. of Control *23* (1976), 5, 593—612.

Ing. Antonín Vaněček, CSc., Ústav teorie informace a automatizace ČSAV (Institute of Information Theory and Automation — Czechoslovak Academy of Sciences), Pod vodárenskou věží 4, 182 08 Praha 8. Czechoslovakia.