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ON A GAME THEORETICAL MODEL OF COOPERATIVE MARKET

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The general coalition-game concept is used for the modelling of cooperative situations in a market. Some connections between the market equilibrium and strong cooperative solution of a game are shown. A weaker form of market equilibrium is suggested and briefly investigated, and some possibilities of dynamical changes of the considered market are discussed.

0. INTRODUCTION

The presented paper is subjected to some problems concerning the mutual relations between a market model and certain type of coalition-games. This connection between markets and games was investigated and some interesting results were presented e.g. in [1], [8] or [10]. However, there exists certain disproportion between the market model and the game theoretical tools. One of its sources is a different attitude to the cooperation in both, market and game theoretical, models.

The cooperation in markets investigated e.g. in [1] or [8] is limited to the exchange of goods respecting given prices, and the optimization concerns the individual profit only. On the other hand, the cooperation in coalition-games is rather stronger. The connections between markets and coalition-games are investigated by means of the theory of coalition-games with side-payments (cf. [2], [7], [8], [9]). In those games, coalitions aim to maximize their common profit, and the final income is distributed among their members in accordance with their agreement. Moreover, the achievement of certain profit of an agent in a market depends on the willingness and ability of other agents to accept the suggested exchange of the profit. The profit, of a coalition in a coalition-game with side-payments is usually interpreted as a guaranteed one, independent on the behaviour of other players.

It could be useful to apply the game theoretical, coalitional model of cooperation also to the market models. An attempt to construct such model of cooperative market was presented in [6]. The existence of coalitions with common stock of goods is assumed there. The common goods are exchanged, and the increased profit of the given coalition is distributed among its members. The application of the coalition-games with side-payments in this model implied the necessary existence of a linear representative of transferable utility common for all agents. This assumption is rather strong and unrealistic for usual applications.

This discrepancy can be removed if we apply another more general model of co-operation represented here by general coalition-games. In the following sections, we briefly recall some basic notions of the general coalition-games theory introduced and investigated in [3] and [4]. The cooperative market model investigated in this paper is described in Section 2, and the connection between general coalition-games and markets is investigated in further sections. It is shown there that the properties of the general coalition-games allow to derive strong relations between game theoretical solution and market equilibrium. Moreover, there can be suggested a weaker modification of the cooperative market connected with a corresponding equilibrium, which is also investigated in this paper. The last section contains a model of certain type of dynamic evolution of the considered market, and its relations to the usual static one.

1. GENERAL COALITION-GAMES

The coalition-games concept represents a wide scale of specialized types of games with different forms of cooperation among players. The coalition-games applied to the market models in [1] and [8] and in other works of that type are the coalition-games with side-payments (cf. [2], [7], [8], [9]). It was already mentioned in the preceding introductory section that the coalition-games with side-payments can cause some problems if they are applied to the cooperative marked models.

The general coalition-games model was suggested in [3] and investigated also in some other papers (cf. [4], [5]). It represents an attempt to consider only the essential features of cooperation in games. Their main properties, motivation of definitions and relations to some other more special coalition-games are discussed in [3]. Here, we recall the main notions which will be used in the following sections.

Let us consider a non-empty and finite set I, let us denote by R the set of all real numbers, and let us consider a mapping $\boldsymbol{V}$ from the class, 2^I into the class of subsets of R^I such that for any $K \in 2^I$ the set $\boldsymbol{V}(K) \subset R^I$ fulfils the following conditions

$$(1.1) \qquad \boldsymbol{V}(K) \textit{ is closed};$$

$$(1.2) \qquad \textit{if} \quad \boldsymbol{x} = (x_i)_{i \in I} \in \boldsymbol{V}(K), \quad \boldsymbol{y} = (y_i)_{i \in I} \in R^I, \quad y_i \leqq x_i \quad \textit{for all} \quad i \in K, \quad \textit{then} \quad \boldsymbol{y} \in \boldsymbol{V}(K);$$

$$(1.3) \qquad \boldsymbol{V}(K) \neq \emptyset;$$

$$(1.4) \qquad \boldsymbol{V}(K) = R^I \Leftrightarrow K = \emptyset.$$

Then the pair (I, V) is called a *general coalition-game* (or briefly a *game*), elements of the set I are *players*, sets $K \subset I$ are *coalitions*, and the mapping V is a *general characteristic function*. Any real-valued vector $x \in R^I$ is called an *imputation*, and each partition $\mathscr{K}$ of the set I into non-empty and disjoint coalitions is a *coalition structure*.

If $K \subset I$ then it is useful to define the set

(1.5) $$V^*(K) = \{y = (y_i)_{i\in I} : y \in R^I \text{ and there is no } x = (x_i)_{i\in I} \in V(K) \text{ such that } x_i \geqq y_i \text{ for all } i \in K \text{ and } x_j > y_j \text{ for some } j \in K\} = \{y = (y_i)_{i\in I} \in R^I : \text{for all } x \in V(K) \text{ is either } y_i > x_i \text{ for some } i \in K \text{ or } y_i = x_i \text{ for all } i \in K\}.$$

The mapping V^* from 2^I into the class of subsets of R^I is called a *superoptimum* of the game (I, V).

In order to simplify the notations used below we denote

$$V(\mathscr{M}) = \bigcap_{M\in\mathscr{M}} V(M), \quad V^*(\mathscr{M}) = \bigcap_{M\in\mathscr{M}} V^*(M),$$

for any non-empty set of coalitions $\mathscr{M} \subset 2^I$, $\mathscr{M} \neq \emptyset$.

Remark 1. It is not difficult to verify that for any $K \in 2^I$, $K \neq \emptyset$, and any $x \in V(K) - V^*(K)$ there exists $y \in V(K)$ such that $y_i \geqq x_i$ for all $i \in K$ where the inequality is strict for at least one $i \in K$. If $V^*(K)$ is a closed subset of R^I then the imputation y can be found in $V(K) \cap V^*(K)$.

The solution of a general coalition-game will contain some stable imputations. i.e. imputations undominable by any coalition and any achievable imputation. The domination concept is formulated by means of the superoptimum mapping. As the general coalition-games are not necessarily superadditive, it is useful to note not only the stable imputations but also the coalitions and coalition structures in which they can be achieved.

An imputation $x \in R^I$ is *strongly stable* iff there exists a coalition structure $\mathscr{K} \subset 2^I$ such that

$$x \in V(\mathscr{K}) = \bigcap_{K\in\mathscr{K}} V(K) \quad \text{and} \quad x \in \bigcap_{K\subset I} V^*(K).$$

A coalition structure $\mathscr{K}$ is strongly stable iff there exists a strongly stable imputation $x \in R^I$ such that $x \in V(\mathscr{K})$.

Remark 2. It is obvious that a coalition structure $\mathscr{K}$ is strongly stable iff

$$V(\mathscr{K}) \cap \left(\bigcap_{K\subset I} V^*(K)\right) \neq \emptyset.$$

The definition of the general coalition-games includes a lot of different types of games. Some of them are superadditive, subadditive or additive. Such games are investigated in [4], and it is useful to mention here some ideas which are applicable to the solution of some market problems.

A general coalition-game (I, V) is called *superadditive* iff for every pair of disjoint coalitions $K, L \subset I$, $K \cap L = \emptyset$, is

$$V(K \cap L) \supset V(K) \cap V(L),$$

it is called *subadditive* iff for every pair of disjoin coalitions $K, L \subset I$, $K \cap L = \emptyset$, is

$$V^*(K \cap L) \supset V^*(K) \cap V^*(L),$$

and it is called *additive* iff it is superadditive and subadditive.

Lemma 1. If $V(K \cap L) \subset V(K) \cap V(L)$ for all $K, L \subset I$, $K \cap L = \emptyset$, then the game (I, V) is subadditive.

Proof. The statement was proved in [3], Lemma 13. □

Theorem 1. A coalition structure $\mathscr{K}$ in a subadditive general coalition-game is strongly stable if and only if

$$V(\mathscr{K}) \cap (\bigcap_{i \in I} V^*(\{i\})) \neq \emptyset.$$

Proof. The subadditivity definition implies for any coalition $K \subset I$ the inclusion

$$V^*(K) \subset \bigcap_{i \in I} V^*(\{i\}),$$

where $\{i\}$ are one-player coalitions for $i \in I$. It means that

$$\bigcap_{K \subset I} V^*(K) = \bigcap_{i \in I} V^*(\{i\}),$$

and the statement is proved. □

Corollary. The coalition structure formed by exactly all one-player coalitions in a subadditive game is always strongly stable. It means that in subadditive games always exist strongly stable imputations.

Theorem 2. If the game (I, V) is additive then all coalition structures are strongly stable.

Proof. The theorem was proved in [3], Theorem 8. □

Theorem 3. If the game (I, V) is additive then the real-valued vector $z = (z_j)_{j \in I}$ for which

$$V(\{i\}) = \{x = (x_j)_{j \in I} : x_i \leqq z_i\} \quad \text{for all} \quad i \in I$$

is the single strongly stable imputation in the game.

Proof. As for all $i \in I$ is

$$V(\{i\}) = \{x = (x_j)_{j \in I} : x_i \leqq z_i\},$$
$$V^*(\{i\}) = \{y = (y_j)_{j \in I} : y_i \geqq z_i\},$$

then $z \in V^*(\{i\})$ for all $i \in I$, and $z \in V(\mathscr{I})$ for the coalition structure $\mathscr{I}$ of exactly all one-player coalitions. It means that z is strongly stable as follows from Theorem 1, Let us suppose, now, that there exists $x \in R^I$, $x \neq z$, such that x is strongly stable. It means that $x \in V^*(\{i\})$ for all $i \in I$, i.e. $x_i \geqq z_i$ for all $i \in I$, and $x_j > z_j$ for some $j \in I$, as $x \neq z$, Let us consider a coalition $K \subset I$ such that $j \in K$, where $x_j > z_j$. As $z \in V^*(K)$, and $x_j < z_j$, $x_i \geqq z_i$ for all $i \in K$, then $x \notin V(K)$ for any $K \subset I$ for which $j \in K$. It means that $x \notin V(\mathscr{K})$ for any coalition structure $\mathscr{K}$, and then x cannot be strongly stable. Hence, there is no strongly stable imputation but z in the considered game. □

2. COOPERATIVE MARKET

After introducing the game theoretical concepts, the market model can be presented in this section.

Let us suppose that i is a finite and non-empty set of *agents*, and that there exist m kinds of *goods*. By R^m_+ we denote the set of all real-valued m-dimensional vectors with non-negative components. Vectors from R^m_+ denote the quantities of goods. Let us suppose that there exists a vector

$$(2.1) \qquad \xi = (\xi_i)_{i \in I}, \quad \xi_i \in R^m_+, \quad \xi_i = (\xi_i^1, \xi_i^2. \ldots, \xi_i^m),$$

describing the *initial quantities* of goods owned by agents at the beginning of the considered situation. If $\alpha \in R^m_+$ and $\beta \in R^m_+$ then we write $\alpha > \beta$ iff $\alpha^j \geqq \beta^j$ for all $j = 1, \ldots, m$ and $\alpha^k > \beta^k$ for some $k \in \{1, \ldots, m\}$. Let there exist continuous and increasing mappings $u_i : R^m_+ \to R$, $i \in I$, where

$$(2.2) \qquad \textit{if} \quad \eta_i \in R^m_+, \quad \zeta_i \in R^m_+, \quad \eta_i > \zeta_i \quad \textit{then} \quad u_i(\eta_i) > u_i(\zeta_i), \quad i \in I.$$

Then the mappings u_i are called *utility functions* of agents. Let us suppose, finally, that $p = (p_j)_{j=1,\ldots,m}$, $p_j \in R$, $p_j > 0$, $j = 1, \ldots, m$, is a real-valued vector of *prices* of the respective goods, and that the scalar product $p \cdot \eta$ for $\eta \in R^m_+$ has sense. Then the quintuple

$$\mathfrak{m} = (I, R^m_+, (\xi_i)_{i \in I}, (u_i)_{i \in I}, p)$$

is called a market.

It is useful to define the following sets for any coalition of agents $K \subset I$.

$$(2.3) \qquad \mathbb{A}(K) = \{\alpha_K = (\alpha_i)_{i \in K} : \alpha_i \in R^m_+, \sum_{i \in K} \alpha_i \leqq \sum_{i \in K} \xi_i\},$$

$$(2.4) \qquad \mathbb{B}(K) = \{\beta_K = (\beta_i)_{i \in K} : \beta_i \in R^m_+, \sum_{i \in K} p \cdot \beta_i \leqq \sum_{i \in K} p \cdot \xi_i\},$$

called the *aggregation set* and the *budget set* of the coalition K, respectively. The following statements hold for those sets.

Lemma 2. The aggregation sets $\mathbb{A}(K)$ are non-empty, bounded, closed and convex for all $K \subset I$, $K \neq \emptyset$.

Proof. The sets $\mathbb{A}(K)$ are non-empty, as follows from (2.3) immediately. As the vectors α_i for $i \in K$ are re-distributions of the initial quantities of goods ξ_i, the sets $\mathbb{A}(K)$ are closed and bounded. Let us consider $\alpha_K \in \mathbb{A}(K)$, $\beta_K \in \mathbb{A}(K)$, $\lambda \in R$, $0 \leqq \lambda \leqq 1$. Then for

$$\gamma_K = \lambda \alpha_K + (1 - \lambda) \beta_K$$

is

$$\lambda \sum_{i \in K} \alpha_i + (1 - \lambda) \sum_{i \in K} \beta_i \leqq \lambda \sum_{i \in K} \xi_i + (1 - \lambda) \sum_{i \in K} \xi_i = \sum_{i \in K} \xi_i .$$

Hence, $\gamma_K \in \mathbb{A}(K)$, and the convexity of $\mathbb{A}(K)$ is proved. □

Lemma 3. The budget sets $\mathbb{B}(K)$ are non-empty, bounded, closed and convex for all coalitions $K \subset I$, $K \neq \emptyset$.

Proof. If $K \subset I$, $K \neq \emptyset$, and $\xi_K = (\xi_i)_{1 \in K}$ then obviously $\xi_K \in \mathbb{B}(K) \neq \emptyset$. As $p_j > 0$ for all $j = 1, \ldots, m$, the sets $\mathbb{B}(K)$ are necessarily bounded and (2.4) implies thet they are closed. Let $\alpha_K \in \mathbb{B}(K)$, $\beta_K \in \mathbb{B}(K)$, $\lambda \in R$, $0 \leqq \lambda \leqq 1$, $\gamma_K = \lambda \alpha_K + (1 - \lambda) \beta_K$. Then

$$\sum_{i \in K} \boldsymbol{p} \cdot \gamma_i = \lambda \sum_{i \in K} \boldsymbol{p} \cdot \alpha_i + (1 - \lambda) \sum_{i \in K} \boldsymbol{p} \cdot \beta_i \leqq \lambda \sum_{i \in K} \boldsymbol{p} \cdot \xi_i + (1 - \lambda) \sum_{i \in K} \boldsymbol{p} \cdot \xi_i = \sum_{i \in K} \boldsymbol{p} \cdot \xi_i ,$$

and the convexity of the sets $\mathbb{B}(K)$ is proved. □

If $\eta = (\eta_i)_{i \in I}$, $\eta_i \in R^m_+$, then the pair $(\eta, \boldsymbol{p})$ is called a *state of* the considered *market* $\mathfrak{m}$. The states of market represent the results of activities realized in the market, namely the exchange done under the existing prices, and other manipulations with goods.

If $(\eta, \boldsymbol{p})$ is a state of market such that

(2.5) $$\eta \in \mathbb{A}(I) ,$$

(2.6) $$\eta_i \in \mathbb{B}(\{i\}) \quad \textit{for all} \quad i \in I ,$$

(2.7) $$u_i(\eta_i) = \max \{u_i(\beta_i) : \beta_i \in \mathbb{B}(\{i\})\} \quad \textit{for all} \quad i \in I ,$$

then it is called a *general equilibrium* of the considered market $\mathfrak{m}$.

There exists a relation between market model introduced in this section and some general coalition-games. The relation is investigated in the following section.

3. MARKET GAMES

The mutual connections between markets and some types of coalition-games are investigated in the literature, e.g. in [1] and [8]. The types of games investigated there are the coalition-games with side-payments (cf. [2], [8], [9]). There exists a difference between the game theoretical model of cooperation and that one assumed in market. The general equilibrium is a set of individual achievements satisfying particular agents, and the cooperation is reduced to the exchange of goods. It is achievable only under the assumption of analogical exchange possibilities and suitable demans of all agents. On the other hand, the coalition-games with side-payments represent higher degree of cooperation among players forming coalitions and maximizing the guaranteed profit of those coalitions.

It can be useful to apply the game theoretical concept of cooperation also to the market models, and to consider coalitions of agents operating with common goods and maximizing the common profit. If this model of cooperation is applied, then the coalition-games with side-payments are not an adequate tool for its description. Namely, there appears a necessity to assume the existence of some representation of utility which should be transferable among players, linear and identical for all of them. This assumption is rather strong and not very realistic.

The general coalition-games theory offers game theoretical tools which are more adequate to the cooperative market models. They are not limited by some assumptions typical of coalition-games with side-payments, and consequently, the transition of utility among agents can be considered without the strong assumption about its representative.

In this section, a game theoretical analogy of the cooperative market based on the general coalition-games concept is suggested.

Wishing to derive a game or more games representing the considered market, it is useful to introduce the following notations. If $K \subset I$ and $\mathbb{A}(K)$, $\mathbb{B}(K)$ are the aggregation and budget sets by (2.3) and (2.4), respectively, then we denote

$$V_a(K) = \{x \in R^I : \exists \alpha_K \in \mathbb{A}(K), \forall i \in K, x_i \leqq u_i(\alpha_i)\} , \tag{3.1}$$

$$V_b(K) = \{x \in R^I : \exists \beta_K \in \mathbb{B}(K), \forall i \in K, x_i \leqq u_i(\beta_i)\} . \tag{3.2}$$

Theorem 4. The pairs (I, V_a) and (I, V_b) are general coalition-games with general characteristic functions V_a and V_b.

Proof. By Lemma 2 and Lemma 3, the sets $\mathbb{A}(K)$ and $\mathbb{B}(K)$ are closed in $(R^m_+)^K$. If $\alpha_K \in \mathbb{A}(K)$, $\beta_K \in \mathbb{B}(K)$, and if $\hat{\alpha}_K \in (R^m_+)^K$, $\hat{\beta}_K \in (R^m_+)^K$, where

$$\hat{\alpha}^j_i \leqq \alpha^j_i, \hat{\beta}^j_i \leqq \beta^j_i \quad \text{for all} \quad i \in K , \quad j = 1, \ldots, m ,$$

then also $\hat{\alpha}_K \in \mathbb{A}(K)$ and $\hat{\beta}_K \in \mathbb{B}(K)$, as follows from (2.3) and (2.4). As the utility functions u_i, $i \in I$, are continuous and increasing, formulas (3.1) and (3.2) imply the validity of (1.1), (1.2), (1.3) and (1.4) for V_a and V_b. □

The games (I, V_a) and (I, V_b) characterize the cooperation possibilities in the considered market $\mathfrak{m}$.

Lemma 4. For every coalition $K \subset I$ is $\mathbb{A}(K) \subset \mathbb{B}(K)$ and $V_a(K) \subset V_b(K)$.

Proof. Both inclusions follow immediately from (2.3), (2.4) and (3.1), (3.2), respectively. □

Lemma 5. If V_b^* is the superoptimum in the game (I, V_b) then for every pair of disjoint coalitions $K, L \subset I, K \cap L = \emptyset$, is

$$V_b^*(K \cup L) \supset V_b^*(K) \cap V_b^*(L) .$$

Proof. Let $y \in R^I$, $y \notin V_b^*(K \cup L)$. Then $y \in V_b(K \cup L) - V_b^*(K \cup L)$ and there exists $x \in V_b(K \cup L)$ such that

(3.3) $$x_i \geqq y_i \quad \text{for all} \quad i \in K \cup L, \quad x_i > y_i \quad \text{for some} \quad i \in K \cup L.$$

As $x \in V_b(K \cup L)$, there exists $\beta_{K \cup L} \in \mathbb{B}(K \cup L)$ such that $x_i \leqq u_i(\beta_i)$ for all $i \in K \cup L$, and

(3.4) $$\sum_{i \in K \cup L} p \cdot \beta_i \leqq \sum_{i \in K \cup L} p \cdot \xi_i .$$

Then also

$$\sum_{i \in K} p \cdot \beta_i \leqq \sum_{i \in K} p \cdot \xi_i \quad \text{or} \quad \sum_{i \in L} p \cdot \beta_i \leqq \sum_{i \in L} p \cdot \xi_i .$$

Let us suppose that the first one of the two inequalities is fulfilled, i.e.

(3.5) $$\beta_K = (\beta_i)_{i \in K} \in \mathbb{B}(K) .$$

If $x_i > y_i$ for some $i \in K$ then $y \notin V_b^*(K)$, as $x \in V_b(K)$ by (3.4) and (3.5). If

(3.6) $$x_i = y_i \quad \text{for all} \quad i \in K$$

then there are two possibilities. Either

$$x_j < u_j(\beta_j) \quad \text{for some} \quad j \in K$$

and then there exists $z \in R^I$ such that

$$z \in V_b(K) , \quad z_i = u_i(\beta_i) \quad \text{for all} \quad i \in K ,$$
$$z_j > y_j , \quad z_i \geqq y_i \quad \text{for all} \quad i \in K .$$

Hence, $y \notin V_b^*(K)$. Or

$$x_i = u_i(\beta_i) \quad \text{for all} \quad i \in K .$$

Then either

(3.7) $$\sum_{i \in K} p \cdot \beta_i < \sum_{i \in K} p \cdot \xi_i$$

or

(3.8) $$\sum_{i \in K} \boldsymbol{p} \cdot \beta_i = \sum_{i \in K} \boldsymbol{p} \cdot \xi_i .$$

If (3.7) holds then there exists $\delta_K \in \mathbb{B}(K)$ such that $\delta_i > \beta_i$ for all $i \in K$. Then the strict monotonicity of u_i implies that there exists $z = (z_i)_{i \in I} \in V_b(K)$ such that

$$z_i = u_i(\delta_i) > u_i(\beta_i) = x_i = y_i$$

for all $i \in K$, and then $y \notin V_b^*(K)$. If (3.8) holds then necessarily

$$\sum_{i \in L} \boldsymbol{p} \cdot \beta_i < \sum_{i \in L} \boldsymbol{p} \cdot \xi_i$$

as follows from (3.4). Then $x \in V_b(L)$, and $x_i > y_i$ for some $i \in L$, as follows from (3.3) and (3.6). Then $y \notin V_b^*(L)$. It means that if $y \notin V_b^*(K \cup L)$ then either $y \notin V_b^*(K)$ or $y \notin V_b^*(L)$, and the statement is proved. □

Lemma 6. For any $K, L \subset I$, $K \cap L = \emptyset$ is

$$V_b(K \cup L) \supset V_b(K) \cap V_b(L) .$$

Proof. If $x \in V_b(K)$, $y \in V_b(L)$ then there exist $\beta'_K \in \mathbb{B}(K)$ and $\beta''_i \in \mathbb{B}(L)$ such that

$$x_i \leqq u_i(\beta'_i) \quad \text{for all} \quad i \in K , \quad y_i \leqq u_i(\beta''_i) \quad \text{for all} \quad i \in L .$$

Let us construct

$$\beta_{K \cup L} = (\beta_i)_{i \in K \cup L} , \quad \beta_i = \beta'_i \quad \text{for all} \quad i \in K ,$$
$$\beta_i = \beta''_i \quad \text{for all} \quad i \in L .$$

Then

$$\sum_{i \in K \cup L} \boldsymbol{p} \cdot \beta_i = \sum_{i \in K} \boldsymbol{p} \cdot \beta' + \sum_{i \in L} \boldsymbol{p} \cdot \beta'' \leqq \sum_{i \in K \cup L} \boldsymbol{p} \cdot \xi_i ,$$

and $\beta_{K \cup L} \in \mathbb{B}(K \cup L)$. An imputation $z \in R^I$ such that

$$z_i = x_i \quad \text{for all} \quad i \in K , \quad z_i = y_i \quad \text{for all} \quad i \in L$$

belongs to $V_b(K \cup L)$, and the statement holds. □

Theorem 5. General coalition-game (I, V_b) is additive.

Proof. The statement follows immediately from Lemma 5 and Lemma 6. □

Theorem 6. General coalition-game (I, V_a) is superadditive.

Proof. Let us consider $K, L \subset I$, $K \cap L = \emptyset$, and $x \in V_a(K)$, $y \in V_a(L)$. Then there exist $\alpha'_K \in \mathbb{A}(K)$, $\alpha''_L \in \mathbb{A}(L)$ such that

$$x_i \leqq u_i(\alpha'_i) \quad \text{for all} \quad i \in K ,$$
$$y_i \leqq u_i(\alpha''_i) \quad \text{for all} \quad i \in L .$$

If $\alpha_{K\cup L} = (\alpha_i)_{i\in K\cup L}$, $\alpha_i \in R^m_+$, $\alpha_i = \alpha'_i$ for all $i \in K$, $\alpha_i = \alpha''_i$ for all $i \in L$, then $\alpha_{K\cup L} \in \mathbb{A}(K \cup I)$, and an imputation $z \in R^I$, $z_i = x_i$ for $i \in K$, $z_i = y_i$ for $i \in L$, belongs to $V_a(K \cup L)$. □

Lemma 7. If $i \in I$, $\{i\} \in 2^I$ is one-player coalition containing exactly i, and if $y \in R^I$ then

$$y \in V_b(\{i\}) \cap V_b(\{i\}) \Leftrightarrow y_i = \max \{u_i(\beta_i) : \beta_i \in \mathbb{B}(\{i\})\} .$$

Proof. If $y \in V_b(\{i\})$, then $y \in V^*_b(\{i\})$ if and only if $y_i \geqq x_i$ for all $x \in V_b(\{i\})$, i.e. if and only if

$$y_i = \max \{u_i(\beta_i) : \beta_i \in \mathbb{B}(\{i\})\} ,$$

as follows from (2.4) and (3.2). □

Theorem 7. Any state (η, p) of the market $\mathbf{m}$ is a general equilibrium if and only if $\eta \in \mathbb{A}(I)$ and for all $i \in I$ is $\eta_i \in \mathbb{B}(\{i\})$ and $(u_k(\eta_k))_{k\in I} \in V^*_b(\{i\})$.

Proof. The statement follows immediately from Lemma 7 and from the definition of general equilibrium. □

Lemma 8. If (η, p) is a general equilibrium and if $z \in R^I$ is the imputation for which $z_i = u_i(\eta_i)$ for all $i \in I$, then $z \in V_b(K) \cap V^*_b(K)$ for all $K \subset I$.

Proof. By Theorem 7 and by the definition of general equilibrium is $z \in V^*_b(\{i\})$ for all $i \in I$. The additivity of (I, V_b) implies that $z \in V^*_b(K)$ for all $K \subset I$. Lemma 7 and the definition of general equilibrium imply that $z \in V_b(\{i\})$ for all $i \in I$. As (I, V_b) is additive, $z \in V_b(K)$ for all $K \subset I$. □

Theorem 8. If (η, p) is a state of market $\mathbf{m}$, and if $z \in R^I$ is the imputation for which $z_i = u_i(\eta_i)$ for all $i \in I$, then (η, p) is general equilibrium if and only if $z \in V_a(I)$ and z is a strongly stable imputation in (I, V_b).

Proof. The statement follows from Lemma 8 and from the definition of general equilibrium immediately. □

Theorem 9. If there exists a general equilibrium (η, p) in a market $\mathbf{m}$ then for arbitrary coalition structure $\mathcal{K}$ is

$$V_a(I) \cap V^*_b(\mathcal{K}) \neq \emptyset .$$

Proof. If (η, p) is a general equilibrium and $x = (x_i)_{i\in I} \in R^I$ is the imputation for which $x_i = u_i(\eta_i)$ for all $i \in I$ then the definition of general equilibrium and Lemma 8 imply

$$x \in V_a(I) \quad \text{and} \quad x \in V^*_b(K) \quad \text{for all} \quad K \subset I . \qquad \square$$

Theorem 10. If $(\eta, \boldsymbol{p})$ is a general equilibrium in market $\mathbf{m}$ and if $\boldsymbol{y} \in R^I$ is the imputation for which $y_i = u_i(\eta_i)$ for all $i \in I$ then $\boldsymbol{y}$ is strongly stable in general coalition-game $(I, \boldsymbol{V}_a)$.

Proof. If $(\eta, \boldsymbol{p})$ is a general equilibrium then, by Theorem 7, $\eta \in \mathbb{A}(I)$ and for all $i \in I$ is

$$\boldsymbol{y} \in \boldsymbol{V}_b(\{i\}) \cap \boldsymbol{V}_b^*(\{i\}) .$$

It means that $\boldsymbol{y} \in \boldsymbol{V}_a(I)$ and $\boldsymbol{y} \in \boldsymbol{V}_b^*(K)$ for all coalitions $K \subset I$, as follows from Lemma 5. Moreover, by Lemma 4 is

$$\boldsymbol{V}_b^*(K) \subset \boldsymbol{V}_a^*(K)$$

for all $K \subset I$, as

$$\boldsymbol{V}_a(K) \cup \boldsymbol{V}_a^*(K) = R^I = \boldsymbol{V}_b(K) \cup \boldsymbol{V}_b^*(K) .$$

Hence, $\boldsymbol{y} \in \boldsymbol{V}_a^*(K)$ for all $K \subset I$, and the statement holds. □

It was shown in this section that there exist strong relations between general coalition-games and the considered type of markets. Some of the results presented here are rather similar to those known from the literature, e.g. from [1] or [8]. It concerns especially the results on mutual relations between the strongly stable imputations in games and general equilibrium in markets. However, the analogy between markets and coalition-games with side-payments shown in the literature is not as strong as the one between markets and general coalition-games. For example, the general equilibrium implies the existence of core in a coalition-game with side-payments (cf. [8]) but it need not be equivalent with it except of the asymptotic case (c.f. [1]). It is also obvious from the comparison of this paper with the classical literature (cf. [1], [8], [10]) that the proofs based on the general coalition-games are more simple and more lucid.

The general coalition-games can be applied also for modelling some weaker solutions of cooperation in the considered markets in which the exiestence of general equilibrium is not guaranted. Such model is suggested in the next section.

4. WEAK FORM OF GENERAL EQUILIBRIUM

The general equilibrium concept described in Section 2 and investigated also in Section 3 represents a stable and commonly acceptable solution of the market cooperation. Nevertheless, it does not exist in many real markets, and it is useful to formulate some weaker analogy of equilibrium defining possible rational behaviour of agents if the general equilibrium is not achievable.

A realistic way is to suppose stronger cooperation among agents who cannot reach the individualistic general equilibrium. We suppose that agents form coalitions in which they gather their goods together (form the aggregation sets), dispose with it in order to optimize the final set of goods, and then re-distribute the obtained goods in certain sense proportionally to the input utility and to the preferences of the coalition members. This kind of cooperation cannot usually offer the maximal utility theoretically covered by their budget set, but it brings more utility than the original distribution of goods. It is possible to show that such cooperating coalitions have chance to satisfy their requirements and to be realized if the general equilibrium does not exist.

This model accents the importance of stronger cooperation aiming to the optimization of common profit. Moreover, it presents a weaker solution reducible to the general equilibrium if it exists.

The theoretical model assumed here allows to take arbitrary classes of coalitions into consideration. However, it is obvious that the typical classes of such realized coalitions should form coalition structures or their unions.

Let us consider a class of coalitions $\mathcal{M} \subset 2^I$, and a state $(\eta, \boldsymbol{p})$ of market $\mathbf{m}$. Then $(\eta, \boldsymbol{p})$ is called an *$\mathcal{M}$-equilibrium* iff $\eta \in \mathbb{A}(I)$ and for any $K \in \mathcal{M}$ is

$$\eta_K = (\eta_i)_{i \in K} \in \mathbb{B}(K),$$
$$(u_i(\eta_i))_{i \in I} \in \boldsymbol{V}_b^*(K).$$

Remark 3. If $\mathcal{M} \subset \mathcal{N} \subset 2^I$ and if $(\eta, \boldsymbol{p})$ is an $\mathcal{N}$-equilibrium then it is also an $\mathcal{M}$-equilibrium.

Remark 4. If $\mathcal{M} = (\{i\})_{i \in I}$, i.e. $\mathcal{M}$ is the class of all one-agent coalitions, then the $\mathcal{M}$-equilibrium is identical with the general equilibrium of the considered market, as follows from the definitions of $\mathcal{M}$-equilibrium and general equilibrium and from Theorem 7.

Remark 5. If $\mathcal{M} \subset 2^I$, $\mathcal{N} \subset 2^I$ and if $(\eta, \boldsymbol{p})$ is an $\mathcal{M}$-equilibrium and an $\mathcal{N}$-equilibrium simultaneously then $(\eta, \boldsymbol{p})$ is also an $(\mathcal{M} \cup \mathcal{N})$-equilibrium as follows from the definition immediately.

Theorem 11. Let $\mathcal{M} \subset \mathcal{N} \subset 2^I$, let $L_1, \ldots, L_n \in \mathcal{M}$, $L_i \cap L_j = \emptyset$ for all $i, j \in \{1, \ldots, n\}$, $i \neq j$, let $L = L_1 \cup \ldots \cup L_n$, and let $\mathcal{N} = \mathcal{M} \cup \{L\}$. Then any state of market $(\eta, \boldsymbol{p})$ is an $\mathcal{M}$-equilibrium if and only if it is an $\mathcal{N}$-equilibrium.

Proof. Let us suppose that $(\eta, \boldsymbol{p})$ is an $\mathcal{M}$-equilibrium. Then $\eta \in \mathbb{A}(I)$, $\eta_K \in \mathbb{B}(K)$ for all $K \in \mathcal{M}$, and if $\boldsymbol{x} \in \boldsymbol{R}^I$, $x_i = u_i(\eta_i)$ for all $i \in I$ then $\boldsymbol{x} \in \boldsymbol{V}_b^*(K)$ for all $K \in \mathcal{M}$. Let us consider $L_1, \ldots, L_k \in \mathcal{M}$ fulfilling the assumptions of this theorem, and $L = L_1 \cup \ldots \cup L_n$. Then $\eta_L = (\eta_i)_{i \in L} \in \mathbb{B}(L)$ as follows from (2.4) immediately, and $\boldsymbol{x} \in V_b^*(L)$ as follows from Lemma 5. It means that $(\eta, \boldsymbol{p})$ is also an $\mathcal{N}$-equilibrium. The opposite implication follows from the definition (cf. Remark 3) immediately. □

Theorem 12. If $(\eta, \boldsymbol{p})$ is a general equilibrium in the market $\mathbf{m}$ then $(\eta, \boldsymbol{p})$ is also an $\mathcal{M}$-equilibrium for any $\mathcal{M} \subset 2^I$.

Proof. Repetitive application of Theorem 11 implies that $(\eta, \boldsymbol{p})$ is an $\mathscr{M}$-equilibrium for any $\mathscr{M} \subset 2^I$ containing all one-agent coalitions. It means by Remark 3 that it is an $\mathscr{M}$-equilibrium for all $\mathscr{M} \subset 2^I$. □

Remark 6. Theorem 11 and Remark 3 imply analogously to Theorem 12 that if $\mathscr{K}$ is a coalition structure, $\mathscr{M} \subset 2^I$, $\mathscr{K} \subset \mathscr{M}$, and if there exists an $\mathscr{M}$-equilibrium then there exists an $\{I\}$-equilibrium where $\{I\}$ is the class of coalitions containing exactly one coalition of all agents.

Theorem 13. If $\mathscr{M} = \{I\}$ contains exactly the coalition of all agents then there exists an $\mathscr{M}$-equilibrium in the considered market if and only if $V_a(I) \cap V_b^*(I) \neq \emptyset$.

Proof. If $(\eta, \boldsymbol{p})$ is a state of the market and $\boldsymbol{x} \in R^I$, $x_i = u_i(\eta_i)$ for all $i \in I$ then $(\eta, \boldsymbol{p})$ is an $\{I\}$-equilibrium iff $\eta \in \mathbb{A}(I) \cap \mathbb{B}(I)$ and $\boldsymbol{x} \in V_b^*(I)$. As $\mathbb{A}(I) \subset \mathbb{B}(I)$ by (2.3) and (2.4), $(\eta, \boldsymbol{p})$ is an $\{I\}$-equilibrium iff $\boldsymbol{x} \in V_a(I) \cap V_b^*(I)$, where (3.1) was used.

Theorem 14. Let $\mathscr{M} \subset 2^I$, $(\eta, \boldsymbol{p})$ be a state of market $\mathbf{m}$, and let $\boldsymbol{x} \in R^I$, $x_i = u_i(\eta_i)$ for all $i \in I$. If $(\eta, \boldsymbol{p})$ is an $\mathscr{M}$-equilibrium then $\boldsymbol{x} \in V_a^*(K)$ for all $K \in \mathscr{M}$.

Proof. By definition of $\mathscr{M}$-equilibrium, $\boldsymbol{x} \in V_b^*(K)$ for all $K \in \mathscr{M}$. As $V_a(K) \subset \subset V_b(K)$ for all $K \subset I$ and

$$V_a(K) \cup V_a^*(K) = R^I = V_b(K) \cup V_b^*(K).$$

the statement holds. □

5. DEVELOPMENT OF MARKET

The market model investigated in the preceding sections was static, describing one isolated exchange process with fixed quantities of goods. Such situation can be considered also as a repetition of similar exchanges of goods realized by the same agents but with rather different structure of goods. The variability of the structure of goods can be caused by different initial distributions or by production and consumption influencing the quantity of existing goods.

It is possible to suppose that there exist certain general characteristics of the market which are preserved. The aim of this section is to study the existence of general equilibrium and $\mathscr{M}$-equilibrium in such changing market.

Let us consider a market $\mathbf{m} = (I, R_+^m, (\xi_i)_{i \in I}, (u_i)_{i \in I}, \boldsymbol{p})$ satisfying the assumptions introduced in Section 2. Let us denote

$$\mathbb{H} = \{\eta = (\eta_i)_{i \in I} : \eta_i \in R_+^m\},$$

and consider a transformation S of the set $\mathbb{H}$ into itself. For arbitrary $\eta \in \mathbb{H}$

$$S\eta = (S\eta_i)_{j \in I}, \quad S\eta_i \in R_+^m.$$

Then the transformation of market $\mathbf{m}$ into another one can be described as a transformation of the vector $\xi = (\xi_i)_{i\in I}$ into another vector of initial quantities of goods $S\xi = (S\xi_i)_{i\in I}$, where the set of agents keeps unchanged but utilities and prices can be generally changed. This type of transformations of the market is considered in this section.

If the original market was denoted

(5.1) $$\mathfrak{m} = (I, R^m_+, (\xi_i)_{i\in I}, (u_i)_{i\in I}, \boldsymbol{p})$$

then the transformed market is

(5.2) $$S\mathfrak{m} = (I, R^m_+, (S\xi_i)_{i\in I}, (v_i)_{i\in I}, \boldsymbol{q}),$$

and for all coalitions $K \subset I$ we have

$$S\,\mathbb{A}(K) = \{\alpha_K = (\alpha_i)_{i\in I} : \alpha_i \in R^m_+, \sum_{i\in K} \alpha_i \leqq \sum_{i\in K} S\xi_i\},$$

$$S\,\mathbb{B}(K) = \{\beta_K = (\beta_i)_{i\in K} : \beta_i \in R^m_+, \sum_{i\in K} \boldsymbol{q} \cdot \beta_i \leqq \sum_{i\in K} \boldsymbol{q} \cdot S\xi_i\}.$$

Symbols $SV_a(K)$ and $SV_b(K)$ for $K \subset I$ denote the sets of imputations derived from $S\mathbb{A}(K)$ and $S\mathbb{B}(K)$ analogously to formulas (3.1) and (3.2),

$$SV_a(K) = \{x \in R^I : \exists\alpha_K \in S\,\mathbb{A}(K), \forall i \in K, x_i \leqq v_i(\alpha_i)\},$$

$$SV_b(K) = \{x \in R^I : \exists\beta_K \in S\,\mathbb{B}(K), \forall i \in K, x_i \leqq v_i(\beta_i)\}.$$

Then the pairs (I, SV_a) and (I, SV_l) are general coalition-games as follows from Theorem 4.

Here we consider transformations of markets preserving some basic characteristics of the market structure.

We say that the transformation S of market $\mathfrak{m}$ into market $S\mathfrak{m}$ *preserves quantities* iff for any $\gamma \in \mathbb{H}$, $\delta \in \mathbb{H}$

$$\sum_{i\in I} \gamma_i = \sum_{i\in I} \delta_i \Leftrightarrow \sum_{i\in I} S\gamma_i = \sum_{i\in I} S\delta_i.$$

We say that S *preserves price relations* iff for any $\gamma \in \mathbb{H}$, $\delta \in \mathbb{H}$ the following relation holds

$$\boldsymbol{p} \cdot \gamma_i \geqq \boldsymbol{p}\delta_i \Leftrightarrow \boldsymbol{q} \cdot S\gamma_i \geqq \boldsymbol{q} \cdot S\delta_i \quad \text{for all} \quad i \in I.$$

Similarly we say that S *preserves utility relations* iff for any $\gamma \in \mathbb{H}$, $\delta \in \mathbb{H}$

$$u_i(\gamma_i) \geqq u_i(\delta_i) \Leftrightarrow v_i(S\gamma_i) \geqq v_i(S\delta_i) \quad \text{for all} \quad i \in I.$$

Lemma 9. If the transformation S preserves quantities then for any $\alpha \in \mathbb{H}$

$$\alpha \in \mathbb{A}(I) \Leftrightarrow S\alpha \in S\,\mathbb{A}(I).$$

Proof. If S preserves quantities then

$$\sum_{i\in I} \alpha_i \leqq \sum_{i\in I} \xi_i \Leftrightarrow \sum_{i\in I} S\alpha_i \leqq \sum_{i\in I} S\xi_i. \qquad \square$$

Lemma 10. If S preserves price relations then for any $K \subset I$ and any $\beta \in \mathbb{H}$

$$\beta \in \mathbb{B}(K) \Leftrightarrow S\beta \in S\,\mathbb{B}(K)\,.$$

Proof. If S preserves price relations then for any $K \subset I$

$$\sum_{i \in K} \boldsymbol{p} \cdot \beta_i \leqq \sum_{i \in K} \boldsymbol{p} \cdot \xi_i \Leftrightarrow \sum_{i \in K} \boldsymbol{q} \cdot S\beta_i \leqq \sum_{i \in K} \boldsymbol{q} \cdot S\xi_i\,. \qquad \square$$

Theorem 15. Let the transformation S of market $\mathfrak{m}$ into market $S\mathfrak{m}$ preserve quantities, price relations and utility relations, and let $(\eta, \boldsymbol{p})$ and $(S\eta, \boldsymbol{q})$ be states of markets $\mathfrak{m}$ and $S\mathfrak{m}$, respectively. Then $(\eta, \boldsymbol{p})$ is a general equlibrium in $\mathfrak{m}$ if and only if $(S\eta. \boldsymbol{q})$ is a general equilibrium in $S\mathfrak{m}$.

Proof. Lemma 9 implies that $\eta \in \mathbb{A}(I)$ iff $S\eta \in S\,\mathbb{A}(I)$ and by Lemma 10 is $\eta \in \mathbb{B}(\{i\})$ iff $S\eta \in S\,\mathbb{B}(\{i\})$ for all $i \in I$. If S preserves utility relations then, using Lemma 10,

$$u_i(\eta_i) = \max\,\{u_i(\beta_i) : \beta_i \in \mathbb{B}(\{i\})\}$$

iff

$$v_i(S\eta_i) = \max\,\{v_i(\beta_i) : \beta_i \in S\,\mathbb{B}(\{i\})\}$$

for all $i \in I$. $\square$

Theorem 16. Let the transformation S of market $\mathfrak{m}$ into market $S\mathfrak{m}$ preserve quantities, price relations and utility relations, let $(\eta, \boldsymbol{p})$ and $(S\eta, \boldsymbol{q})$ be states of markets $\mathfrak{m}$ and $S\mathfrak{m}$, respectively, and let $\mathscr{M} \subset 2^I$ be a class of coalitions. Then $(\eta, \boldsymbol{p})$ is an $\mathscr{M}$-equilibrium in $\mathfrak{m}$ if and only if $(S\eta, \boldsymbol{q})$ is an $\mathscr{M}$-equilibrium in $S\mathfrak{m}$.

Proof. Lemma 9 and 10 imply that $\eta \in \mathbb{A}(I)$ iff $S\eta \in S\,\mathbb{A}(I)$ and for any $K \subset I$ is $\eta \in \mathbb{B}(K)$ iff $S\eta \in S\,\mathbb{B}(K)$. For any $K \subset I$

$$(u_i(\eta_i))_{i \in I} \in V_b^*(K)$$

iff there does not exist $\boldsymbol{y} \in V_b(K)$ such that $y_i \geqq u_i(\eta_i)$ for all $i \in K$ and $y_j > u_j(\eta_j)$ for some $j \in K$. It is true iff there does not exist $\beta_K = (\beta_i)_{i \in K} \in \mathbb{B}(K)$ such that

$$u_i(\beta_i) \geqq u_i(\eta_i) \quad \text{for all} \quad i \in K\,,$$

$$u_j(\beta_j) > u_j(\eta_j) \quad \text{for some} \quad j \in K\,.$$

If S preserves utility relations, the preceding condition is fulfilled iff there does not exist any $\gamma_K = (\gamma_i)_{i \in K} \in S\,\mathbb{B}(K)$ such that

$$v_i(\gamma_i) \geqq v_i(S\eta_i) \quad \text{for all} \quad i \in K\,,$$

$$v_j(\gamma_j) > v_j(S\eta_j) \quad \text{for some} \quad j \in K\,.$$

But this is true iff

$$(v_i(S\eta_i))_{i \in I} \in S\,V_b^*(K)\,. \qquad \square$$

Theorem 17. Let the transformation S of market $\mathfrak{m}$ into market $S\mathfrak{m}$ preserve price relations and utility relations. Then there exists a strongly stable imputation in the game (I, V_b) if and only if there exists a strongly stable imputation in the game (I, SV_b), and a coalition structure $\mathscr{K}$ is strongly stable in (I, V_b) if and only if it is strongly stable in (I, SV_b).

Proof. Let us suppose that there exists a strongly stable imputation $\boldsymbol{x} = (x_i)_{i \in I} \in R^I$ in (I, V_b). It means that there exists a coalition structure $\mathscr{K}$ such that $\boldsymbol{x} \in V_b(\mathscr{K})$, and $\boldsymbol{x} \in V_b^*(L)$ for all $L \subset I$. As $\boldsymbol{x} \in V_b(\mathscr{K})$ and $\boldsymbol{x} \in V_b^*(\mathscr{K})$, there exists $\beta_K = (\beta_i)_{i \in K} \in \mathbb{B}(K)$ for any $K \in \mathscr{K}$, such that $x_i = u_i(\beta_i)$ for all $i \in K$. If this statement is not true for some $K \in \mathscr{K}$, then there certainly exists $\beta_K \in \mathbb{B}(K)$ for which $x_i \leqq u_i(\beta_i)$ for all $i \in K$ and $x_j < u_j(\beta_j)$ for some $j \in K$. But the vector $\boldsymbol{y} \in R^I$ such that $y_i = u_i(\beta_i)$ for all $i \in K$ also belongs to $V_b(K)$. Then we obtain the relation $\boldsymbol{x} \notin V_b^*(K)$ contradicting the assumption of strong stability of $\boldsymbol{x}$. Hence, for any $K \in \mathscr{K}$ there exists a vector $\beta_K \in \mathbb{B}(K)$ such that $x_i = u_i(\beta_i)$ for all $i \in K$. Let us denote $\beta = (\beta_i)_{i \in I}$, where β_i are components of vectors β_k for respective disjoint coalitions from $\mathscr{K}$. If $S\beta = (S\beta_i)_{i \in I}$ is the transformation of β then we denote

$$S\boldsymbol{x} = (Sx_i)_{i \in I} \in R^I ,$$

the imputation for which

$$Sx_i = v_i(S\beta_i) \quad \text{for all} \quad i \in I .$$

Lemma 10 implies that $S\beta \in S\mathbb{B}(K)$ for all $K \in \mathscr{K}$, and then $S\boldsymbol{x} \in SV_b$ for all $K \in \mathscr{K}$. We know that $\boldsymbol{x} \in V_b^*(L)$ for all $L \subset I$. Let us suppose that $S\boldsymbol{x} \notin SV_b^*(L)$ for some $L \subset I$, where the set of imputations $SV_b^*(L)$ is derived from $SV_b(L)$ by means of (1.5). Then there exists $\boldsymbol{y} \in SV_b(L)$ such that

$$y_i \geqq Sx_i \quad \text{for all} \quad i \in L,$$

$$y_j > Sx_j \quad \text{for some} \quad j \in L.$$

As $\boldsymbol{y} \in S\,V_b(L)$, there necessarily exists $\gamma \in \mathbb{B}(L)$ and $\delta = S\gamma \in S\,\mathbb{B}(L)$ such that $v_i(\delta_i) = y_i$ for all $i \in L$. By Lemma 10

$$v_i(\delta_i) \geqq v_i(S\beta_i) \quad \text{for all} \quad i \in L,$$

$$v_j(\delta_j) > v_j(S\beta_j) \quad \text{for some} \quad j \in L,$$

iff

$$u_i(\gamma_i) \geqq u_i(\beta_i) = x_i \quad \text{for all} \quad i \in L,$$

$$u_j(\gamma_j) > u_j(\beta_j) = x_j \quad \text{for some} \quad j \in L.$$

This contradicts the assumption of strong stability of $\boldsymbol{x}$ in the game (I, V_b), namely the relation $\boldsymbol{x} \in V_b^*(L)$ for all $L \subset I$. It means that the imputations $\boldsymbol{x}$ and $S\boldsymbol{x}$ are strongly stable in (I, V_b) and $(I, S\,V_b)$, respectively. Moreover, the coalition structure $\mathscr{K}$ is strongly stable in both games. □

6. CONCLUSIVE REMARKS

It has been shown in the foregoing sections that the concept of general coalition-games is suitable fot modelling certain type of markets and their equilibria. They are adequate to some specific market properties, especially to different forms of super-additivity subadditivity and additivity. Further, they can be used for modelling some modified forms of equilibrium, the $\mathcal{M}$-equilibrium, representing an advanced degree of cooperation in the considered market. This application of general coalition-games is possible without strong assumptions about transferability of profit which are necessary if some other game models are used for similar purpose.

There has been also suggested a method of modelling the dynamics of market including potential development of more components like initial distributions of goods, utility functions and prices. There were formulated sufficient conditions under which the solutions of market and corresponding market games can be preserved during the market transformation. It seems that some further results concerning the parallel development of market and market games can be derived by using the game theoretical tools in future.

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