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Information Transmission in the Case of Coding by Finite Automata

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Coding and decoding by finite automata non preserving the length of input words is discussed. A possible characteristic for transmission quality expressing and its estimation is presented.

1. INTRODUCTION

In this paper we shall consider a well-known transmission model: information source – coder – channel – decoder, supposing that both the coder and the decoder are finite automata. Naturally, we shall suppose, that our automata will be of the general type, which will not preserve the length of the input words. Then there are some difficulties in estimating the transmission quality which would not occur in the case of coding by n-tuples.

The first question we have to answer is thit: what changes of source properties will occur after encoding it by means of the automaton?

The secon question is connected with the circumstance, that neither in encoding nor in decoding the length of the transmissed word is necessarily preserved. How, then, can we define the error frequency? (There is no strict correspondence between time indexes of the input and output words. For instance, the word "coding" can be transmitted as "co5ng" and we have a problem whether the error frequency 1/3 or 2/3 or another value is the most suitable.)

The third and it seems the most serious question is, what the comparison of the emitted and received message can tell about an "average" error frequency in our transmitting model.

This paper will try to answer the questions stated above. From now the following designations will be used:

N – the set of all natural numbers,

I – the set of all integers,

X^n – the Cartesian product $\bigtimes_{i=1}^{n} X$ for $n \in N$ and a finite non empty set X. The set X will be called the alphabet and its elements the the letters;

X^I – the set of all sequences $(\ldots, x_{-1}, x_0, x_1, \ldots)$ where $x_i \in X$, $i \in I$. The elements of X^I will be called the messages.

$\mathbf{X}$ – σ-algebra of all subsets of X,

ξ, η, ζ – letters from Greek alphabet will designate messages containing letters which will be designated by corresponding Latin letters, e.g. $\xi = (\ldots, x_{-1}, x_0, \ldots)$, $\eta = (\ldots, y_{-1}, y_0, \ldots)$;

$*$ – end of a proof;

$\{a, b, \ldots, c\}$ – a set of elements $a, b, \ldots, c$;

$\{x \in X: V(x)\}$ – a set of such $x \in X$ that the statement $V(x)$ is valid.

A probability space $(Y^I, \mathbf{Y}^I, \mu)$ will be called a *source* (Y is assumed to be a finite and nonempty set – an alphabet of the source). $\mathbf{Y}^I$ is the σ-algebra, generated by a system of elementary cylinders on Y^I. The set $E \subset Y^I$ is called the *elementary cylinder*, if there exist integers $i_1, \ldots, i_n$ and a n-tuple $(\bar{y}_{i_1}, \ldots, \bar{y}_{i_n})$ that the following assertion is valid:

$$\eta \in E \Leftrightarrow y_{i_k} = \bar{y}_{i_k}, \quad k = 1, \ldots, n\,.$$

Then we shall say that E is determined by $(\bar{y}_{i_1}, \ldots, \bar{y}_{i_n})$. If there is a danger of misunderstanding, we add in the places $i_1, \ldots, i_n'''$. The sets from $\mathbf{Y}^I$ are called *measurable*. A finite sum of elementary cylinders is called a finite-dimensional cylinder.

The set $G(Y) = \bigcap_{n=1}^{\infty} Y^n \cup \{\Lambda\}$ is a set of words in the alphabet Y. The word Λ is called the empty word (the word without letters). If the letters are subscripted then we put $(y_k, \ldots, y_m) = \Lambda$ for $k > m$.

Let us define a *multiplication* on $G(Y)$ with following properties:

1. For every $u \in G(Y)$ $\Lambda u = u\Lambda = u$.
2. If $u = (y_1, \ldots, y_n) \in G(Y)$, $v = (\bar{y}_1, \ldots, \bar{y}_m) \in G(Y)$, then $uv = (y_1, \ldots, y_n, \bar{y}_1, \ldots, \bar{y}_m)$.

Evidently, the mulitiplication is associative.

If $u \in Y^n$, then we shall put $h(u) = n$ and call this value the *length* of the word u. For Λ we shall put $h(\Lambda) = 0$. If there is no danger of misunderstanding, we shall use the symbol h also to denote the number of elements in a finite set.

Let $a_i = (y_1^{(i)}, \ldots, y_{n_i}^{(i)}) \in G(Y)$, $n_i \geqq 0$, $i \in I$ and let $k < 0$, $l \geqq 0$ $\bar{k} \in I$, $\bar{l} \in I$. For $\eta \in Y^I$ we shall write

1. $\prod_{i=0}^{l} a_i = (y_0, \ldots, y_l)$ if and only if

$(y_0, \ldots, y_{\bar{l}}) = (y_1^{(0)}, \ldots, y_{n_0}^{(0)}, \ldots, y_1^{(l)}, \ldots, y_{n_l}^{(l)})$

2. $\prod_{i=k}^{-1} a_i = (y_{\bar{k}}, \ldots, y_{-1})$ if and only if

$(y_{\bar{k}}, \ldots, y_{-1}) = (y_1^{(k)}, \ldots, y_{n_k}^{(k)}, \ldots, y_1^{(-1)}, \ldots, y_{n_{-1}}^{(-1)})$

3. $\prod_{i=k}^{l} a_i = (y_{\bar{k}}, \ldots, y_{\bar{l}})$ if and only if

$(y_{\bar{k}}, \ldots, y_{-1}) = \prod_{i=k}^{-1} a_i, (y_0, \ldots, y_{\bar{l}}) = \prod_{i=0}^{l} a_i$

The symbol $\prod_{i \in I} a_i$ will denote such a subset of Y^I, for which following conditions are valid:

$\eta \in \prod_{i \in I} a_i$ if and only if for every $k \in I$, $l \in I$, $k \leqq -1$, $l \geqq 0$ there exist $\bar{k} \in I$, $\bar{l} \in I$ such that

$$(y_{\bar{k}}, \ldots, y_{-1}) = \prod_{i=k}^{-1} a_i , \quad (y_0, \ldots, y_{\bar{l}}) = \prod_{i=0}^{l} a_i .$$

It is evident, that if $\sum_{i=0}^{\infty} h(a_i) = \infty$, $\sum_{i=1}^{\infty} h(a_{-i}) = \infty$, then $\prod_{i \in I} a_i$ is a one-point set. If both series have a finite sum, then $\prod_{i \in I} a_i$ is an elementary cylinder.

Hence we shall use the symbol Π only in relations between sequences of words infinite in both directions and messages, resp. their finite parts. The symbol Π implies not only the identity of letters, but also a synchronism between indexes.

Let Y be an alphabet and let us define a coordinate-shift transformation T on Y^I:

If $\eta \in Y^I$, then $T\eta = \bar{\eta}$ if and only if $\bar{y}_i = y_{i+1}$ for all $i \in I$.

The inverse transformation to T will be denoted T^{-1}. If $\xi \in X^I$, $\eta \in Y^I$, then we put $T(\xi, \eta) = (T\xi, T\eta)$.

If $E \subset Y^I$, then we write

$$TE = \{\eta \in Y^I \text{: there exist } \bar{\eta} \in E,\ T\eta = \eta\} .$$

If $TE = E$ we say that E is an *invariant* set. The properties of the transformation T are described e.g. in [1], chap. X.

If the source $(Y^I, \mathbf{Y}^I, \mu)$ fulfils the condition that for every $E \in \mathbf{Y}^I$ $\mu(E) = \mu(TE)$, we call it *stationary*. Then we also say that the measure μ is stationary.

If $\mu(E)$ is either 0 or 1 for every measurable invariant set E, then we call the source (the measure) *indecomposable*. If the source (the measure) is stationary and indecomposable, then we say that it is *ergodic*.

The triple $(\mathbf{Y}^I, \nu, \mathbf{Z}^I)$ will be called the channel, if Y, Z are finite non-empty sets (input and output alphabets) and $\mathbf{Y}^I$, $\mathbf{Z}^I$ will be, as usual, the σ-algebras generated by the systems of the elementary cylinders in Y^I, resp. Z^I. Further, if ν is a function defined on $Y^I \times \mathbf{Z}^I$ (the value of this function will be denoted $\nu(E/\eta)$ for every $E \in \mathbf{Z}^I$, $\eta \in Y^I$) and, finally, if

1. for every $E \in \mathbf{Z}^I$, $\nu(E/.)$ is a measurable function on Y^I,
2. for every $\eta \in Y^I$, $\nu(./\eta)$ is a probability measure on $\mathbf{Z}^I$,
3. for every finite-dimensional cylinder $E \in \mathbf{Z}^I$ determined by the conditions for i-th, ..., j-th coordinate, the value $\nu(E/\eta)$ does not depend on $y_{j+1}, y_{j+2}, \ldots$ for every $\eta \in Y^I$.

The channel $(\mathbf{Y}^I, \nu, \mathbf{Z}^I)$ is said to be stationary, if $\nu(E/\eta) = \nu(TE/T\eta)$ for every $E \in \mathbf{Z}^I$, $\eta \in Y_I$.

The channel $(\mathbf{Y}^I, \nu, Z^I)$ is a channel with finite memory, if there exist $m \in N$ such that:

1. for every finite-dimensional cylinder $E \in \mathbf{Z}^I$, determined by the conditions for the i-th, ..., j-th coordinate and for every $\eta \in Y^I$ the value $\nu(E/\eta)$ depends only on $y_{i-m}, y_{i-m+1}, \ldots, y_j$;
2. for every finite-dimensional cylinders $E, F \in \mathbf{Z}^I$ determined by the conditions for the i-th, ..., j-th, resp. k-th, ..., l-th coordinate, where $j + m < k$ and for every $\eta \in Y^I$

$$\nu(E \cap F/\eta) = \nu(E/\eta)\left(\nu(F/\eta)\right).$$

The least integer m, which fulfils the conditions 1 and 2 will be called *the memory of the channel.*

2. FINITE AUTOMATA

Definition 2.1. The quintuple $\mathcal{T} = (X, Y, Z, g, f)$ is called a finite automaton, if X, Y, Z are non-empty finite sets (the set of *states*, of *input signals* and of *output signals*), g is a mapping from $X \times Y$ into X and f is a mapping from $X \times Y$ into the set $G(Y)$.

From now we shall write

$$g(x, \Lambda) = x\,, \quad g(X, \Lambda) = X\,,$$
$$g(x, y_1, \ldots, y_k) = g(\ldots(g(x, y_1), y_2), \ldots, y_k)\,,$$
$$g(X, y_1, \ldots, y_k) = \{x \in X\colon \text{there exist } x_0 \in X,\ g(x_0, y_1, \ldots, y_k) = x\}\,.$$

Definition 2.2. Let $\mathcal{T} = (X, Y, Z, g, f)$ be a finite automaton. Let $(y_k, \ldots, y_l) \in Y^{l-k+1}$. Then a word $(x_k, .., x_l) \in X^{l-k+1}$ is called *corresponding* to $(y_k, \ldots, y_1)$ and we write $(y_k, \ldots, y_l) \sim (x_k, \ldots, x_l)$ if $g(x_i, y_i) = x_{i+1}$ for every $i = k, k + 1, \ldots, l$.

In the same way we introduce

$$(y_k, y_{k+1}, \ldots)) \sim (x_k, x_{k+1}, \ldots) \Leftrightarrow g(x_i, y_i) = x_{i+1}, \quad i \geqq k,$$
$$(\ldots, y_{-1}, y_0, y_1, \ldots) \sim (\ldots, x_{-1}, x_0, x_1, \ldots) \Leftrightarrow g(x_i, y_i) = x_{i+1}, \quad i \in I.$$

Let $\eta \in Y^I$, $\xi \in X^I, \zeta \in Z^I, k \leqq l$, $(y_k, \ldots, y_l) \sim (x_k, \ldots, x_l)$. Then we say that $(z_{\bar{k}}, \ldots, z_{\bar{l}})$ corresponding to $[(y_k, \ldots, y_l), (x_k, \ldots, x_l)]$ and we write $(y_k, \ldots, y_l) \sim$ $\sim (x_k, \ldots, x_l) \sim (z_{\bar{k}}, \ldots, z_{\bar{l}})$

(i) in the case of $k \leqq 0$, $l \geqq -1$ if $(z_{\bar{k}}, \ldots, z_{\bar{l}}) = \prod_{i=k}^{l} f(x_i, y_i)$,

(ii) in the case of $k > 0$ if $z_0, \ldots, z_{\bar{l}}) = \prod_{l=0}^{l} f(x_i, y_i)$,

$$(z_0, \ldots, z_{\bar{k}-1}) = \prod_{i+0}^{k-1} f(x_i, y_i),$$

(iii) in the case of $l < -1$ if $(z_{\bar{k}}, \ldots, z_{-1}) = \prod_{i=k}^{-1} f(x_i, y_i)$,

$$(z_{\bar{l}+1}, \ldots, z_{-1}) = \prod_{i=l+1}^{-1} f(x_i, y_i).$$

Always we shall write $\Lambda \sim \Lambda \sim \Lambda$.

Let $\eta \in Y^I$, $\xi \in X^I, \zeta \in Z^I, \eta \sim \xi$. Then we say that ζ is *corresponding* to (ξ, η) and we write $\eta \sim \xi \sim \zeta$ or $\zeta = \varphi(\xi, \eta)$, iff $\{\zeta\} = \prod_{i \in I} f(x_i, y_i)$.

In the following lemmas we suppose that a finite automaton $\mathscr{T} = (X, Y, Z, g, f)$, which has m elements in a set X, is given.

Lemma 2.1. *For every* $(y_k, \ldots, y_l) \in Y^{l-k+1}$, *resp.* $(y_k, y_{k+1}, \ldots) \in Y^N$ *there exist exactly m different* $(x_k, \ldots, x_l)$, *resp.* $(x_k, x_{k+1}, \ldots)$, $x_i \in X$, *such that* $(y_k, \ldots, y_l) \sim$ $\sim (x_k, \ldots, x_1)$, *resp.* $(y_k, y_{k+1}, \ldots) \sim (x_k, x_{k+1}, \ldots)$.

Proof. The assertion that there exist at least m different $(x_k, \ldots)$ is evidently valid, because we can consider x_k to be an arbitrary element of X. It is also clear that the number cannot be greater, because $(y_k, \ldots)$ and x_k uniquely determine $(x_k, \ldots)$.

Lemma 2.2. *To every* $\eta \in Y^I$ *there exist at least one and at most m different* $\xi \in X^I$, *such that* $\eta \sim \xi$.

Proof. 1. We shall show the existence of one ξ. Let us suppose, on the contrary, that for a certain $\eta \in Y^I$ there does not exist $\xi \in X^I$, $\eta \sim \xi$. Let us denote the elements of the set X $x_0^{(1)}, \ldots, x_0^{(m)}$ and let them be placed in the 0-th places of m different sequences $(x_0^{(i)}, x_1^{(i)}, \ldots)$ $i = 1, \ldots, m$. Let, moreover,

$$(y_0, y_1, \ldots) \sim (x_0^{(i)}, x_1^{(i)}, \ldots), \quad i = 1, \ldots, m.$$

Now, let $k < 0$, $0 < i \leqq m$. The all $(x_k, x_{k+1}, \ldots)$ fulfilling the conditions $x_j \in X$ for every $j \geqq k$, $x_j = x_j^{(i)}$ for every $j = 0, 1, \ldots$ and $(y_k, y_{k+1}, \ldots) \sim (x_k, x_{k+1}, \ldots)$, we shall call the extension of $(x_0^{(i)}, \ldots)$ to the left until k. All the extensions are finite. According to our supposition no infinite extension of $(x_0^{(i)}, \ldots)$ exists.

There are two possibilities:

a) For every i, the extensions are bounded. Let us denote k_i the subscript until which the longest extension of $(x_0^{(i)}, \ldots)$ can be made and $k_0 = \min\limits_{i=1,\ldots,m} k_i$. Let $x \in X$ be arbitrary. The sequence $\{g(x, y_{k_0-2}, \ldots, y_l)\}_{l=k_0-2}^{\infty}$ is evidently the extension of some $(x_0^{(i)}, \ldots)$ till $k_0 - 1$, which is in contradiction with the definition of k_0.

b) Let $(x_0^{(i)}, \ldots)$ possess unbounded extensions to the left. Let us consider all extensions of the sequence until -1. One of them must also posses unbounded extensions to the left. Let us denote the extension $(x_{-1}^{(i)}, x_0^{(i)}, \ldots)$. By repeating these considerations we find $(x_{-2}^{(i)}, x_{-1}^{(i)}, x_0^{(i)}, \ldots)$ and by the induction we can define the extension $(\ldots, x_{-1}^{(i)}, x_0^{(i)}, \ldots)$, the existence of which is in contradiction with our former supposition. This concludes the proof of existence at least one $\xi \in X^I$, $\eta \sim \xi$.

2. Now we shall show that their number cannot extend m. Let us suppose, on the cotrary, that $\eta \sim \xi^{(i)}$, $i = 1, \ldots, m + 1$, $\zeta^{(i)} \neq \xi^{(k)}$, $i \neq k$. Let us denote M_{ik} the set of the all indexes j, such that $x_j^{(i)} \neq x_j^{(k)}$ $(i \neq k,\ i \leqq m + 1,\ k \leqq m + 1)$. The number of the sets M_{ik} is finite and all of them are non-empty. Therefore, some $l \in I$ must exist, such that $M_{ik} \cap \{l, l + 1, \ldots\} \neq 0$ for every $i \neq k$, $i \leqq m + 1$, $k \leqq m + 1$.

But that implies that $(x_l^{(i)}, x_{l+1}^{(i)}, \ldots)$, $i = 1, \ldots, m + 1$ are mutually different, what is in the contradiction with the assertion of the lemma 2.1.*

Lemma 2.3. *Let Y_i be the set of such $\eta \in Y^I$ that there exist exactly i different $\xi \in X^I$, corresponding to it. Then $\{Y_i\}_{i=1}^m$ is the partition of the set Y^I into invariant components.*

Proof. According to lemma 2.2 the first part of the assertion is evident and we need to prove only that *every Y_i is invariant.* Let $\xi^{(j)}$, $j = 1, \ldots, i$ be mutually different mesages from X^I: It is clear that

$$\eta \sim \xi^{(j)} \Leftrightarrow T\eta \sim T\xi^{(j)}$$

and, therefore $T\eta \in Y_i \Leftrightarrow \eta \in Y_i$.*

Lemma 2.4. *Let Y_i, $i = 1, \ldots, m$ be the sets defined by the lemma 2.3. Then $Y_i \in \mathbf{Y}^I$ for every $i = 1, \ldots, m$.*

Proof. Let $k \in I$, $l = I$, $k < l$, $i = 1, \ldots, m$. Let us denote

$$M_{kl}^{(i)} = \{\eta \in Y^I\colon g(X, y_k, \ldots, y_l) \text{ contains at least } i \text{ elements}\}\,,$$

$$M_l^{(i)} = \bigcap_{k=-\infty}^{l-1} M_{kl}^{(i)}\,,$$

$$M^{(i)} = \begin{cases} \bigcup_{l=-\infty}^{\infty} M_l^{(i)} - \bigcup_{l=-\infty}^{\infty} M_l^{(i+1)} & \text{for} \quad i = 1, \ldots, m-1\,, \\ \bigcup_{l=-\infty}^{\infty} M_l^{(i)} & \text{for} \quad i = m\,. \end{cases}$$

$M_{kl}^{(i)}$ is obviously a finite-dimensional cylinder and therefore $M^{(i)}$ is a measurable set for every $i \in I$. Our purpose is to show that $M^{(i)} = Y_i$, $i = 1, \ldots, m$.

1. Let $\eta \in M^{(i)}$, $i < m$ (in the case of $i = m$ we can use an obvious modification of our proof). Then there exist $\bar{l} \in I$, that $\eta \in M_{\bar{l}}^{(i)}$, but $\eta \notin M_l^{(i+1)}$ is valid for every $l \in I$. It is obvious that then $\eta \in M_{\bar{l}-n}^{(i)}$, $n \in N$.

From the relations stated above it follows easily that the set

$$G_{j+1} = \bigcap_{k=-\infty}^{j-1} g(X, y_k, \ldots, y_j)$$

defined for every $j \leqq l$, contains exactly i elements and that

$$g(G_j, y_j) = G_{j+1}\,. \tag{1}$$

Let us designate $x_{l+1}^{(1)}, \ldots, x_{l+1}^{(i)}$, the elements of the set G_{l+1}. We can define the sequences $(x_{l+1}^{(\alpha)}, x_{l+2}^{(\alpha)}, \ldots)$, $\alpha = 1, \ldots, i$, which are mutually different and corresponding to $(y_{l+1}, y_{l+2}, \ldots)$.

By induction we can extend the sequences uniquely to the left: If the sequence $(x_{j+1}^{(\alpha)}, x_{j+2}^{(\alpha)}, \ldots)$ for some $j \leqq l$ corresponds to $(y_{j+1}, y_{j+2}, \ldots)$ and if it is an extension of the α-th sequence $(x_{l+1}^{(\alpha)}, \ldots)$, then, according to (1), we can find the uniquely determined $x_j^{(\alpha)} \in G_j$, such that $g(x_j^{(\alpha)}, y_j) = x_{j+1}^{(\alpha)}$. Therefore there are exactly i mutually different $\xi^{(\alpha)} \in X^I$ with the property $\eta \sim \xi^{(\alpha)}$, which implies that $\eta \in Y_i$ and $M^{(i)} \subset Y_i$.

2. Let $\eta \in Y_i$. Then it is obvious that $\eta \in \bigcup_{l=-\infty}^{\infty} M_l^{(i)}$. In order to show that $\eta \in M^{(i)}$, the validity of the assertion $\eta \notin \bigcup_{i=-\infty}^{\infty} M_l^{(i+1)}$ must be investigated. We shall prove it indirectly: Let $l \in I$ have the property that $\eta \in M_l^{(i+1)}$. Then there exists an integer $j \geqq i + 1$ such that $\eta \in M^{(j)}$, $\eta \in M^{(j+1)}$ (naturally, we can omit the trivial case of $i = m$) which implies according to the first part of our proof, that $\eta \in Y_j$ and we have obtained a contradiction with the assumption $\eta \in Y_i$.

Hence we can say that $Y_i = M^{(i)} \in \mathbf{Y}^I$ for every $i = 1, \ldots, m$.*

Lemma 2.5. *Let $A \subset X^I$ be a finite-dimensional cylinder and let*

$$B_A = \{\eta \in Y^I\text{: there exists } \xi \in A,\, \eta \sim \xi\}$$

Then B_A is a measurable set, i.e. $B_A \in \mathbf{Y}^I$.

Proof. Let A be determined by the conditions for the k-th, ..., l-th coordinate and let $k \leqq l$.

Let us denote for every integer $j \leqq k$

$$B_{Aj} = \{\eta \in Y^I: \text{there exists } \xi \in A\ (y_j, y_{j+1}, \ldots) \sim (x_j, x_{j+1}, \ldots)\}.$$

Obviously, B_{Aj} are the finite-dimensional cylinders determined by conditions for the coordinates $j, \ldots, l-1$.

It is easy to see that $\eta \in B_A$ if and only if $\eta \in B_{Aj}$ for every $j \leqq k$. In other words, $B_A = \bigcap_{j=-\infty}^{k} B_{Aj} \in \mathbf{Y}^I$.*

Lemma 2.6. *Let Y_1 be defined in the same way as in the lemma 2.3. and let $A \in \mathbf{X}^I$. Then*

$$C_A = \{\eta \in Y_1: \text{there exists } \xi \in A, \eta \sim \xi\} \in \mathbf{Y}^I.$$

Proof. Let us denote $\mathbf{A} = \{A \in \mathbf{X}^I: C_A \in \mathbf{Y}^I\}$. According to lemmas 2.4 and 2.5 every finite-dimensional cylinder in $\mathbf{X}^I$ belongs to $\mathbf{A}$. Further, if $A_i \in \mathbf{A}$, $i \in N$ and $A = \bigcup_{i=1}^{\infty} A_i$, then

$$C_A = \bigcup_{i=1}^{\infty} C_{A_i} \in \mathbf{Y}^I \Rightarrow A \in \mathbf{A}.$$

If $A \in \mathbf{A}$ then $C_{(X^I - A)} = Y_1 - C_A \in \mathbf{Y}^I$ and therefore $X^I - A \in \mathbf{A}$. Hence $\mathbf{A}$ is the σ-algebra over the system of the finite-dimensional cylinders, from which it follows that $\mathbf{X}^I \subset \mathbf{A}$.*

Lemma 2.7. *Let Y_1 be defined in the same way as in the lemma 2.3. Let $H \in X^I \times \mathbf{Y}^I$. Then*

$$D_H = \{\eta \in Y_1: \text{ there exists } \xi \in X^I \text{ that } (\xi, \eta) \in H,\ \eta \sim \xi\} \in \mathbf{Y}^I.$$

Proof. Let us denote $\mathbf{H} = \{H \in \mathbf{X}^I \times \mathbf{Y}^I : D_H \in \mathbf{Y}^I\}$. If $H = E \times F$, $E \in \mathbf{X}^I$, $F \in \mathbf{Y}^I$, then $D_H = C_E \cap F \in \mathbf{Y}^I$ and therefore $H \in \mathbf{H}$.

If $H_i \in \mathbf{H}$, $i \in N$, $H = \bigcup_{i=1}^{\infty} H_i$, then $D_H = \bigcup_{i=1}^{\infty} D_{H_i} \in \mathbf{Y}^I$ and $H \in \mathbf{H}$.

If $H \in \mathbf{H}$ and $H' = X^I \times Y_I - H$, then

$$D_{H'} = Y_1 - D_H \in \mathbf{Y}^I \Rightarrow H' \in \mathbf{H}$$

what means nothing else than that $\mathbf{H}$ is the σ-algebra over the set of all measurable rectangles in $X^I \times Y^I$, and therefore $\mathbf{X}^I \times \mathbf{Y}^I \subset \mathbf{H}$.*

Definition 2.3. An automaton $\mathcal{T} = (X, Y, Z, g, f)$ is called *directable*, if there exist $n \in N$, $(y_1, \ldots, y_n) \in Y^n$ such that the set $g(X, y_1, \ldots, y_n)$ contains only one element. The word $(y_1, \ldots, y_n)$ is called the directing word of the atutomaton $\mathcal{T}$.

Lemma 2.8. *Let $\mathcal{T} = (X, Y, Z, g, f)$ and let Y_1 is defined as in the lemma 2.3. Then $\eta \in Y_1$ if and only if for every $l \in I$ there exists such $k \in I$ that $(y_k, \ldots, y_l)$ is the directing word for the automaton $\mathcal{T}$.*

Corollary. $Y_1 \neq 0$ if and only if $\mathscr{T}$ is directable.

Proof of the lemma. According to the proof of the lemma 2.4 $Y_1 = M^{(1)}$.

1. Let $\eta \in M^{(1)}$, $l \in I$. Then $\eta \notin M_l^{(2)} = \bigcap_{k=-\infty}^{l-1} M_{kl}^{(2)}$ and there exists $k < 1$ that $\eta \notin M_{kl}^{(2)}$, hence $g(X, y_k, \ldots, y_l)$ contains less than two i.e. exactly one element and therefore $(y_k, \ldots, y_l)$ is the directing word of $\mathscr{T}$.

2. Let η have a property of our lemma. Then for every $l \in I$ there exist $M_{kl}^{(2)} : \eta \notin$ $\notin M_{kl}^{(2)} \Rightarrow \eta \notin M_l^{(2)} \Rightarrow \eta \in M^{(1)} = Y_1$.*

Theorem 2.1. *Let $\mathscr{T} = (X, Y, Z, g, f)$ be a given automaton. Let $(Y^I, \mathbf{Y}^I, \mu)$ be an indecomposable source and let $\mu(Y_1) = 1$ (Y_1 is defined like in the lemma 2.3). Suppose that for every $G \in \mathbf{X}^I \times \mathbf{Y}^I$ we have a set D_G (defined in the lemma 2.7) and let $\nu(G) = \mu(D_G)$.*

Then ν is an indecomposable measure on $(X^I \times Y^I, \mathbf{X}^I \times \mathbf{Y}^I)$.

Proof. If follows from the lemma 2.7 that it is possible to define the function ν as we did.

Obviously $\nu \geqq 0$ and $\nu(\emptyset) = 0$.

Let $\{G_i\}$ be a system of mutually disjoint sets from $\mathbf{X}^I \times \mathbf{Y}^I$ and let $G = \bigcup_{i=1}^{\infty} G_i$. Then, evidently, $\{D_{G_i}\}$ is a system of mutually disjoint measurable subsets of Y_1 and $\nu(G) = \mu(D_G) = \sum_{i=1}^{\infty} \mu(D_{G_i}) = \sum_{i=1}^{\infty} \nu(G_i)$ (see the proof of the lemma 2.7).

Therefore ν is a measure and obviously a probability measure.

Now let $G \in \mathbf{X}^I \times \mathbf{Y}^I$ be an invariant set. Suppose that $\eta \in D_G$, i.e. there exists $\xi \in X^I$ with a property $(\xi, \eta) \in G$, $\eta \sim \xi$. Since G is an invariant set, for $\alpha = 1, -1$ the statement $(T^\alpha\xi, T^\alpha\eta) \in G$ is valid. Further, evidently, $T^\alpha\eta \sim T^\alpha\xi$, what means nothing else than $T^\alpha\eta \in D_G$. Hence, F_G is a measurable invariant set and $\nu(G) =$ $= \mu(D_G) \in \{0, 1\}$.*

The theorem 2.1 will be mainly utilized in the considerations concerning the "upper frequency of errors", which we shall define in § 4. But that is not the only use of the theorem.

Let us consider an ergodic source $(Y^I, \mathbf{Y}^I, \mu)$ with the property $\mu(Y_1) = 1$ and the automaton $\mathscr{T} = (X, Y, Z, g, f)$ which encodes the messages emitted by the source. We may want to know the average relative elongation of messages after encoding.

For a given $\eta \in Y_1$, $\xi \in X^I$, $\eta \sim \xi$, we can consider a limit

$$\lim_{n \to \infty} \frac{h(f(x_0, y_0)) + \ldots + h(f(x_n, y_n))}{n + 1}$$

which, if it exists, can express the relative elongation of the message η from the time $t = 0$ to infinity.

On the other hand, if we define $F_1(x) \in X^I$, $F_2(y) \in Y^I$ as the elementary cylinders determined by the conditions $x_0 = x$, resp. $y_0 = y$ for every $x \in X$, $y \in Y$ and $E(x, y) = F_1(x) \times F_2(y)$, $p(x, y) = \nu(E(x, y))$, then the value

$$s_0 = \sum_{\substack{x \in X \\ y \in Y}} h(f(x, y)) \, . \, p(x, y)$$

means the average elongation of the source by one letter at the time $t = 0$. The source $(Y^I, \mathbf{Y}^I, \mu)$ has been ergodic, i.e. stationary and indecomposable. According to the theorem 2.1 the double source $(X^I \times Y^I, \mathbf{X}^I \times \mathbf{Y}^I, \nu)$ is also indecomposable. It is evidently stationary and, therefore, ergodic. Then $\{h(f(x_n, y_n))\}_{-\infty}^{\infty}$ is an ergodic stochastic process and that's why, for the almost every $(\xi, \eta) \in X^I \times Y^I$, the assertion

$$\lim_{n \to \infty} \frac{h(f(x_0, y_0)) + \ldots + h(f(x_n, y_n))}{n + 1} = s_0$$

is valid. Hence the limit will exist and will be equal for almost all messages. Since the source was stationary, the choice of time origin $t = 0$ has no influence to the limit and therefore we can consider the value s_0 to be an average relative elongation of the source $(Y^I, \mathbf{Y}^I, \mu)$ after encoding by the automaton $\mathscr{T}$.

In order to utilize the results of the teorem 2.1 in our transmission model, let us divide the coding into two parts:

1. $$\eta \to (\xi, \eta)\,, \quad \eta \sim \xi\,,$$
2. $$(\xi, \eta) \to \zeta = \varphi(\xi, \eta)\,.$$

The theorem 2.1 tells us that after having performed the first part, the indecompossibility is preserved.

Now, let us consider the part 2. Let us suppose again that an automaton $\mathscr{T} = (X, Y, Z, g, f)$ is given. In the beginning the § 2 we showed the method of the construction of ζ for given ξ, η, such that $\eta \sim \xi$, $\zeta = \varphi(\xi, \eta)$.

Lemma 2.9. *The set* $M = \{(\xi, \eta) \in X^I \times Y^I : \eta \sim \xi\}$ *is measurable.*

Proof. Let $n \in N$ and let us denote

$$M_n = \{(\xi, \eta) \in X^I \times Y_I : (y_{-n}, \ldots, y_n) \sim (x_{-n}, \ldots, x_n)\}\,.$$

Obviously M_n is a finite-dimensional cylinder in $X^I \times Y^I$. We shall show that $M = \bigcap_{n=1}^{\infty} M_n$. The inclusion $\subset$ is evident, hence we have only to prove $\supset$. Let $(\xi, \eta) \in \bigcap_{n=1}^{\infty} M_n$ and let $k \in I$ be arbitrary. Then $(\xi, \eta) \in M_{|k|+1}$ and, therefore, $g(x_k, y_k) = x_{k+1}$ which implies that $\eta \sim \xi$.*

Lemma 2.10. *Let* $A \in \mathbf{Z}^I$. *Then*

$$E_A = \{(\xi, \eta) \in X^I \times Y^I : \text{there exists such } \zeta \in A, \text{ that } \eta \sim \xi \sim \zeta\} \in \mathbf{X}^I \times \mathbf{Y}^I .$$

Proof. Let A be an elementary cylinder, determined by $(z_k, \ldots, z_l)$ on the coordinates $k, \ldots, l$. It is sufficient to suppose that $k \leqq 0$, $l \geqq -1$, because the other cases can be reduced to this one (e.g. if $k > 0$, then A is the sum of elementary cylinders, determined by $(z_0, \ldots, z_{k-1}, z_k, \ldots, z_l)$, where $(z_0, \ldots, z_{k-1})$ are arbitrary). For $(z_k, \ldots, z_l)$ there exists at most a countable set of $(x_{\bar{k}}, \ldots, x_{\bar{l}})$, $(y_{\bar{k}}, \ldots, y_{\bar{l}})$ to which $(z_{\bar{k}}, \ldots, z_{\bar{l}})$, $\bar{k} \leqq k$, $\bar{l} \geqq l$, corresponds. Let $\mathbf{E}$ be a system of all elementary cylinders, determined by $(x_{\bar{k}}, \ldots, x_{\bar{l}})$, $(y_{\bar{k}}, \ldots, y_{\bar{l}})$ in $X^I \times Y^I$. Let $n \in N$ and let us denote

$$W_n = \{(\xi, \eta) \in X^I \times Y^I : -i \in N,\ -i \geqq n \Rightarrow f(x_i, y_i) = \Lambda\} ,$$
$$\overline{W}_n = \{(\xi, \eta) \in X^I \times Y^I : \ i \in N,\ i \geqq n \ \Rightarrow f(x_i, y_i) = \Lambda\} .$$

Obviously

$$X^I \times Y^I - \left[\bigcup_{n=1}^{\infty} W_n \cup \bigcup_{n=1}^{\infty} \overline{W}_n\right] = W \in \mathbf{X}^I \times \mathbf{Y}^I .$$

If M is the set, defined in the lemma 2.9, then

$$E_A = M \cap W \cap \bigcap_{E \in \mathbf{E}} E .$$

New, let us denote $\mathbf{A}$ the system of the sets, which fulfil the assertion of the our lemma. Wealready know, that $\mathbf{A}$ contains all elementary cylinders, determined by $(z_k, \ldots, z_l)$, $k \leqq 0$, $l \geqq -1$.

Let $A_i \in \mathbf{A}$, $i \in N$, $A = \bigcup_{i=1}^{\infty} A_i$. It is evident, that $E_A = \bigcup_{i=1}^{\infty} E_{A_i} \in \mathbf{X}^I \times \mathbf{Y}^I$ and $A \in \mathbf{A}$. That means, that $\mathbf{A}$ is closed with respect to countable disjunctions and that it contains all finite-dimensional cylinders. Let $A \in \mathbf{A}$, $A' = Z^I - A$. Then $E_{A'} = (M \cap W) - E_A \in \mathbf{X}^I \times \mathbf{Y}^I \Rightarrow A' \in \mathbf{A}$.

Hence $\mathbf{A}$ is the σ-algebra over the class of all finite-dimensional cylinders and that's why $\mathbf{Z}^I \subset \mathbf{A}$.*

Lemma 2.11. *Let* $A \subset Z^I$ *be an invariant set. Then* E_A *is also invariant.* (E_A *is defined like in the lemma* 2.10.)

Proof. *Let* $(\xi, \eta) \in E_A$. Then there exists such $\zeta \in A$, that $\eta \sim \xi \sim \zeta$ and for $\alpha = 1, -1$ such $\beta \in I$ exists, that $T^\alpha \eta \sim T^\alpha \xi \sim T^\beta \zeta$. Since $T^\beta \in A$ (A is invariant), E_A is an invariant set.*

From now we shall denote $Y_{11} = D_{\mathbf{E}_Z I}$. Obviously $Y_{11} \in \mathbf{Y}^I$.

Theorem 2.2. *Let* $(Y^I, \mathbf{Y}^I, \mu)$ *be an indecomposable source and let* $\mu(Y_{11}) = 1$. *Let* $(X^I \times Y^I, \mathbf{X}^I \times \mathbf{Y}^I, \nu)$ *be a double source as defined in the theorem* 2.1. *Let* $(Z^I, \mathbf{Z}^I, \pi)$ *be a source defined as follows:*

$$A \in \mathbf{Z}^I \Rightarrow \pi(A) = \nu(E_A)$$

(E_A *is introduced in lemma* 2.10). *Then* $(Z^I, \mathbf{Z}^I, \pi)$ *is indecomposable.*

Proof. π is evidently a probability measure. Let $A \in \boldsymbol{Z}^I$, $TA = A$. According to lemma 2.11 E_A is also invariant. Since all conditions of the theorem 2.1 are fulfilled, the double source $(X^I \times Y^I, \boldsymbol{X}^I \times \boldsymbol{Y}^I, \nu)$ is indecomposable and $\pi(A) = \nu(E_A) \in \in \{0; 1\}$.*

3. THE CHANNEL

The purpose of the present part is to show the role played by the channel in our transmission problem. We shall suppose that we have a channel with finite memory.

Theorem 3.1. *Let $(Y^I, \boldsymbol{Y}^I, \mu)$ be an indecomposable source and let there exist a stationary measure $\bar{\mu}$ equal to μ on the class of the all measurable invariant sets. Let $(\boldsymbol{Y}^I, \nu, \boldsymbol{Z}^I)$ be a stationary channel with a finite memory.*

Then the double source $(Z^I \times Y^I, \boldsymbol{Z}^I \times \boldsymbol{Y}^I, \omega)$, where

$$\omega(A) = \int \nu(A_\eta/\eta) \, \mathrm{d}\mu \text{ for every } A \in \boldsymbol{Z}^I \times \boldsymbol{Y}^I ,$$

is indecomposable (A_η is a section of A determined by η).

Proof. According to a well known theorem (see e.g. [2]) the measure $\bar{\omega}$, defined by the formula $\bar{\omega}(A) = \int \nu(A_\eta/\eta) \, \mathrm{d}\bar{\mu}$ for every $A \in \boldsymbol{Z}^I \times \boldsymbol{Y}^I$, is indecomposable, because the measure $\bar{\mu}$ has been ergodic.

Now, let $A \in \boldsymbol{Z}^I \times \boldsymbol{Y}^I$ be an invariant set. Then

$$(\zeta, \eta) \in A \Leftrightarrow (T\zeta, T\eta) \in A \Rightarrow A_{T\eta} = TA_\eta$$

and for every $\eta \in Y^I$

$$\nu(A_{T\eta}/T\eta) = \nu(TA_\eta/T\eta) = \nu(A_\eta/\eta) .$$

Let $f(\eta) = \nu(A_\eta/\eta)$ for every $\eta \in Y^I$. Then the function f is evidently invariant. From the definition of the Lebesgue integral we known that

$$\int \nu(A_\eta/\eta) \, \mathrm{d}\mu = \lim_{n \to \infty} \sum_{k=1}^{n+1} \frac{k-1}{n} \, \mu \left\{ \eta : \frac{k-1}{n} \leqq \nu(A_\eta/\eta) < \frac{k}{n} \right\} .$$

The sets on the right side are obviously invariant, therefore we can replace μ by $\bar{\mu}$ and $\omega(A) = \bar{\omega}(A) \in \{0; 1\}$. *

In the § 2 we found out that the encoding by finite automaton preserves the property of indecomposability, if some conditions are satisfied. However, no analoguous statement is possible about stationarity. We shall try to prove that for the source which arises after encoding the ergodic source by a finite automaton, there exists a stationary source equal to the previous source on the class of all measurable invariant sets. In that case we could use the theorem 3.1 which tells us that after transmission through a channel we get also an indecomposable source.

Lemma 3.1. *Let* $(Y^I, \mathbf{Y}^I, \mu)$ *be a given ergodic source and let* $\mu(Y_{11}) = 1$. *Let* $(Z^I, \mathbf{Z}^I, \pi)$ *be defined as in the theorem* 2.2.

Then π *is an indecomposable measure, to which such a stationary measure* $\bar{\pi}$ *exists that* $\pi = \bar{\pi}$ *on the class of the all measurable invariant sets.*

Proof. According to theorem 2.2, π is an indecomposable measure. Let m, $\bar{m}$, n be non-negative integers and let

$$[(x_{-\bar{m}}, \ldots, x_{\bar{m}}), (y_{-\bar{m}}, \ldots, y_m)] \in X^{m+\bar{m}+1} \times Y^{m+\bar{m}+1}, (z_{-n}, \ldots, z_n) \in G(Z) .$$

We shall write $[(x_{-\bar{m}}, \ldots, x_m), (y_{-\bar{m}}, \ldots, y_m)] \to (z_{-n}, \ldots, z_n)$ if and only if there exist integers $k \geqq 0$, $l \geqq 0$, $q \geqq 0$, $p < 0$ and such $z_{-n-k}, \ldots, z_{-n-1}, z_{n+1}; \ldots, z_{n+1}$ such that

(i) $$f(x_{-\bar{m}}, y_{-\bar{m}}) \ldots f(x_m, y_m) = (z_{-n-k}, \ldots, z_{n+l}) ,$$

(ii) $$f(x_0, y_0) = (z_{p+1}, \ldots, z_{q-1}) ,$$

(iii) $$h(f(x_{-\bar{m}}, y_{-\bar{m}})) > k , \quad h(f(x_m, y_m)) > l .$$

Let $A \in \mathbf{Z}^I$ be an elementary cylinder determined by the word $(z_{-n}, \ldots, z_n)$. Let us denote

$$F_A = \{(\xi, \eta) \in X^I \times Y^I : \text{exist } m, \bar{m}, [(x_{-\bar{m}}, \ldots, x_m), (y_{-\bar{m}}, \ldots, y_m)] \to$$
$$\to (z_{-n}, \ldots, z_n)\} .$$

For given $(z_{-n}, \ldots, z_n)$ we have only an at most countable set of $(x_{-\bar{m}}, \ldots, x_m)$, $(y_{-\bar{m}}, \ldots, y_m)$ with the property

$$[(x_{-\bar{m}}, \ldots, x_m), (y_{-\bar{m}}, \ldots, y_m)] \to (z_{-n}, \ldots, z_n)$$

which implies that F_A is the sum of the countable class of sets $E \cap W$, where W is introduced in the proof of the lemma 2.10 and E is an elementary cylinder. Hence $F_A \in \mathbf{X}^I \times \mathbf{Y}^I$.

Let us denote

$$h = \max_{\substack{x \in X \\ y \in Y}} h(f(x, y)) .$$

Let $(\xi, \eta) \in F_A$. Then $h(f(x_0, y_0)) \leqq h$ and then such integers $j \geqq 0$, $j \leqq h$ exists, that $(\xi, \eta) \in \varphi^{-1}(T^{-j}A)$. Therefore $F_A \subset \bigcup_{j=0}^{h} \varphi^{-1}(T^{-j}A)$.

Now let $r \in N$ be arbitrary and fixed. Let us define the class of the sets $\{F_i\}_{i=0}^{\infty}$, $F_i \subset F_A$:

$$(\xi, \eta) \in F_0, \text{ if } (\xi, \eta) \in F_A, \ (\xi, \eta) \in \varphi^{-1}(T^{-r}A), \text{ or if no } i \in N \cup \{0\}$$
$$\text{has a property } T^{-i}(\xi, \eta) \in \varphi^{-r}A) ,$$
$$(\xi, \eta) \in F_i, \ i \in N, \text{ if } (\xi, \eta) \in F_A, T^{-i}(\xi, \eta) \in \varphi^{-1}(T^{-r}A) \text{ and}$$
$$T^{-j}(\xi, \eta) \notin \varphi^{-1}(T^{-r}A) \text{ for } 0 \leqq j < i .$$

The sets F_i, $i \in N$ are measurable because they are countable sums of the intersections W with the elementary cylinders. It is evident, that $F_A = \bigcup_{i=0}^{\infty} F_i$ and that F_0 is measurable (F_i are mutually disjoint).

Now we shall prove, that $\varphi^{-1}(T^{-r}A) \subset \bigcup_{i=0}^{\infty} T^{-i}F_i$.

Let $(\xi, \eta) \in \varphi^{-1}(T^{-r}A)$. Then such integers $i, j \geqq 0$, $k \geqq 0$, $l \geqq 0$ exist that $f(x_i, y_i) \dots f(x_j, y_j) = z_{-n-k}, \dots, z_{n+l}$ i.e. there exists $\bar{n} \in N \cup \{0\}$, that

$$(\xi, \eta) \in T^{-\bar{n}}F_A \quad \text{and} \quad (\xi, \eta) \in \bigcup_{i=0}^{\infty} T^{-i}F_i .$$

Since ν is a stationary measure (ν defined as in the theorem 2.1), it follows

$$\pi(T^{-r}A) \leqq \nu(\bigcup_{i=0}^{\infty} T^{-i}F_i) \leqq \sum_{i=0}^{\infty} \nu(T^{-i}F_i) \leqq \sum_{i=0}^{\infty} \nu(F_i) = \nu(F_A) \leqq$$
$$\leqq \nu(\bigcup_{j=0}^{h} \varphi^{-1}(T^{-j}A)) \leqq \sum_{j=0}^{h} \pi(T^{-j}A) .$$

This inequality we have proved for all elementary cylinders determined by $(z_{-n}, \dots, z_n)$ (the minimum σ-algebra over them is already $\boldsymbol{Z}^I$).

Let us consider the class $\boldsymbol{A}$ of such sets $A \in \boldsymbol{Z}^I$, which satisfy our last inequalities. Obviously $\boldsymbol{A}$ contains all finite-dimensional cylinders, especially $Z^I \in \boldsymbol{A}$. Further, $\boldsymbol{A}$ is a monotone class over the algebra of finite-dimensional cylinders and that's why $\boldsymbol{A}$ is a σ-algebra and $\boldsymbol{Z}^I \subset \boldsymbol{A}$.

Now, let $A_k \in \boldsymbol{Z}^I$, $A_k \downarrow \emptyset$. Then

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \pi(T^{-i}A_k) \leqq \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \sum_{j=0}^{h} \pi(T^{-j}A_k) = \sum_{j=0}^{h} \pi(T^{-j}A_k)$$

and therefore

$$\lim_{k\to\infty} \left(\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \pi(T^{-i}A_k) \right) = 0 .$$

According to [4], § 31, lemma b, on $\boldsymbol{Z}^I$ there exists a stationary measure $\bar{\pi}$, equal to π on the class of all measurable invariant sets.*

The theorem 3.1 and lemma 3.1 show us that if in our fundamental transmission model the source is ergodic and has a property $\mu(Y_{11}) = 1$, then on the output of the channel we shall get an indecomposable source. Let us designate it $(\tilde{Y}^I, \tilde{\boldsymbol{Y}}^I, \tilde{\mu})$. If the decoding finite automaton $\tilde{\mathscr{F}} = (\tilde{Y}, \tilde{X}, \tilde{Z}, \tilde{g}, \tilde{f})$ has the properties $\tilde{Z} = Y$, $\tilde{\mu}(\tilde{Y}_{11}) = 1$ then on the output of the decoding automaton we shall have an indecomposable source $(Y^I, \boldsymbol{Y}^I, \tilde{\pi})$.

The validity of the condition $\tilde{\mu}(\tilde{Y}_{11}) = 1$ will depend on the properties of the channel and the coding and decoding automata. In the following lemmas we shall show a sufficient condition in the case of the channel with the same input and output alphabets.

Lemma 3.2. *Let $(\mathbf{Z}^I, \varrho, \mathbf{Z}^I)$ be a stationary channel with the finite memory. Let $n \in N$, $A \subset Z^n$ and let $F = F(A)$ be a set defined by a following conditions: $\zeta \in F(A)$ if and only if to every $k \in N$ there exist such integers $l_1 \leqq -k$, $l_2 \geqq k$, that $(z_{l_1-n+1}, \ldots, z_{l_1}) \in A$, $(z_{l_2}, \ldots, z_{l_2+n-1}) \in A$.*

Let $(Z^I, \mathbf{Z}^I, \pi)$ be an indecomposable source with the property $\pi(F) = 1$, and let such a stationary measure $\bar{\pi}$ exist that $\pi = \bar{\pi}$ on the class of the measurable invariant sets. Let the channel have a property $\mathbf{V}_A$: For every $\zeta \in F$ and such $k \in N$, that $(z_k, \ldots, z_{k+n-1}) \in A$, the relation $\varrho(B/\zeta) \neq 0$ is valid, where B is an elementary cylinder, determined by $(z_k, \ldots, z_{k+n-1})$.

Let us define $\tilde{\mu}(E) = \int \varrho(E/\zeta)\, d\pi$ for every $E \in \mathbf{Z}^I$.

Then $\tilde{\mu}(F) = 1$.

Proof. For every $E \in \mathbf{Z}^I \times \mathbf{Z}^I$ we designate $\omega(E) = \int \varrho(E_\zeta/\zeta)\, d\pi$. According to the theorem 3.1 the double source $(Z^I \times Z^I, \mathbf{Z}^I \times \mathbf{Z}^I, \omega)$ is indecomposable. F is obviously an invariant measurable set in $\mathbf{Z}^I$ and therefore $F \times Z^I$ is measurable and invariant in $\mathbf{Z}^I \times \mathbf{Z}^I$. Then $\tilde{\mu}(F) = \omega(F \times Z^I) \in \{0; 1\}$.

Now, we have to show that $\tilde{\mu}(F) = 1$. Indirectly: Let us suppose $\tilde{\mu}(F) = 0$. Let $k \in N$ and let us denote

$$F_k = \{\zeta \in Z^I : \text{exist } l_1 \leqq -k,\ l_2 \geqq k : (z_{l_1-n+1}, \ldots, z_{l_1}) \in A\,, \\ (z_{l_2}, \ldots, z_{l_2+n-1}) \in A\}\,.$$

Obviously $F_k \downarrow F$ and therefore $\tilde{\mu}(F_k) \downarrow 0$. Let us denote $\Psi_k(\zeta) = \varrho(F_k/\zeta)$ for every $\zeta \in Z^I$. Ψ_k converges in the mean to $\Psi(.) = \varrho(F/.)$. It is evident that $\Psi = 0$ almost everywhere.

Since we hawe a channel with a finite memory and since A contains only a finite number of elements, $\varrho(B/\gamma)$ for a given B, defined as above, can acquire on F only a finite number of values (different from 0).

Let us denote $p = \min_{\zeta \in F} \varrho(B/\zeta)$. Obviously $p > 0$.

From the definition of the functions Ψ_k and Ψ one can deduce that $\{\Psi_k\}$ converges to zero almost everywhere. Let $\zeta \in F$ has such property that $\Psi_k(\zeta) \to 0$. Then for $p^2 > 0$ there exists such k_0 that $\Psi_k(\zeta) < p^2$ for $k \geqq k_0$. We can suppose that $2k_0 > m$, where m is the memory of the channel.

Since $\zeta \in F$, for a number k_0 there exist such integers $l_1 \leqq -k_0$ $l_2 \geqq k_0$, that $(z_{l_1-n+1}, \ldots, z_{l_1}) \in A$, $(z_{l_2}, \ldots, z_{l_2+n-1}) \in A$. Let $\bar{B}$ be an elementary cylinder determined by $z_{l_1-n+1}, \ldots, z_{l_1}, z_{l_2}, \ldots, z_{l_2+n-1}$. It is evident that $\bar{B} \subset F_{k_0}$ and therefore $\varrho(\bar{B}/\zeta) < p^2$ which contradicts with the definition of p.

Thus the only possibility is $\tilde{\mu}(F) = \int \varrho(F/\zeta)\, d\pi = 1.*$

Lemma 3.3. *Let $\tilde{\mathcal{T}} = (\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{g}, \tilde{f})$. Let $n \in N$ and let A be a set of some directing words of the length n for $\tilde{\mathcal{T}}$, with this property:*

If $(\bar{y}_1, \ldots, \bar{y}_n) \in A$, $(\bar{y}_1, \ldots, \bar{y}_n) \sim (\tilde{x}_1, \ldots, \tilde{x}_n)$, *then such* $i \in N$, $i \leqq m$ *exists, that* $\tilde{f}(\tilde{x}_i, \bar{y}_i) \neq \Lambda$.

Let $F(A)$ *be defined like in the previous lemma* ($Z = \tilde{Y}$).

Then $F(A) \subset \tilde{Y}_{11}$.

Proof. Let $\bar{\eta} \in F(A)$. According to the lemma 2.8, $\bar{\eta}$ belongs to $\tilde{Y}_1$. Further, for every $k \in N$ there exist such integers l_1, l_2, that $l_1 \leqq -k$, $l_2 \geqq k$ and $(\bar{y}_{l_1 - n + 1}, \ldots \ldots, \bar{y}_{l_1}) \in A$, $(\bar{y}_{l_2}, \ldots, \bar{y}_{l_2 + n - 1}) \in A$. That implies, that such $i_1 \leqq l_1 \leqq -k$, $i_2 \geqq \geqq l_2 \geqq k$, that $f(\tilde{x}_{i_1}, y_{i_1}) \neq \Lambda$, $\tilde{f}(\tilde{x}_{i_2}, \bar{y}_{i_2}) \neq \Lambda$. Therefore $\bar{\eta} \in \tilde{Y}_{11}$.*

The lemmas 3.2 and 3.3 show us that in some cases we could achieve $\pi(F(A)) = 1$ (by taking a sufficiently large n) and then $\tilde{\mu}(FA)) = \tilde{\mu}(Y_{11}) = 1$.

4. THE FREQUENCY OF ERRORS

One can assume that in our transmission model we shall define the "frequency of errors" similarly as in coding and decoding by n-tuples: on the space $Y^I \times \tilde{Z}^I$ (Y^I is a space of messages emited by a source and $\tilde{Z}^I$ is a space of messages received on the output of the decoding automaton) we shall define a double source and the ingegration of some weight function will be performed over the space.

In our model, in general, it need not be sufficient to consider "finite-dimensional" weight functions; we shall have to introduce a limit weight function depending on all coordinates, beginning with some k. Since $\tilde{Z} = Y$, we shall consider the space $Y^I \times Y^I$.

Definition 4.1. Let $(\eta, \bar{\eta}) \in Y^I \times Y^I$, $n \in N$ and let $L_n \subset \{1, \ldots, n\} \times \{1, \ldots, n\}$ satisfy the conditions:

1. $$(i, j) \in L_n\,, \quad (k, l) \in L_n \Rightarrow \text{ either } \; i = k\,, \; j = l\,, \\ \text{or } \; i < k\,, \; j < l\,, \; \text{ or } \; i > k\,, \; j > l\,,$$

2. $$(i, j) \in L_n \Rightarrow y_i = \bar{y}_j\,.$$

Let $a_n(\eta, \bar{\eta}, L_n) = n - h(L_n)$ (here h denotes a number of elements in L_n) and let

$$w_n(\eta, \bar{\eta}) = \frac{1}{n} \min_{L_n} a_n(\eta, \bar{\eta}, L_n)\,.$$

The function w_n defined on $Y^I \times Y^I$ will be called the *n-dimensional weight function.*

Lemma 4.1. w_n *is a measurable function on* $(Y^I \times Y^I, \mathbf{Y}^I \times \mathbf{Y}^I)$.

Proof. w_n is evidently constant on every elementary cylinder, determined by a conditions on 1-st $-$ k-th coordinates ($k \geqq n$).*

Definition 4.2. Let us define for every $(\eta, \bar{\eta}) \in Y^I \times Y^I$

$$w(\eta, \bar{\eta}) = \limsup_{n \to \infty} w_n(\eta, \bar{\eta})$$

then we shall call w the *upper frequency weight function* on $Y^I \times Y^I$.

Definition 4.3. Let $n \in N$ and w_n be n-dimensional weight function on $Y^I \times Y^I$. Let $(Y^I \times Y^I, \mathbf{Y}^I \times \mathbf{Y}^I, \omega)$ be a double source. Then we shall call

$$\Phi(\omega) = \limsup_{n \to \infty} \int w_n(\eta, \bar{\eta}) \, d\omega$$

the upper frequency of errors of the double source.

Lemma 4.2. *Let w be an upper frequency weight function on $Y^I \times Y^I$. Then w is measurable and for every $(\eta, \bar{\eta}) \in Y^I \times Y^I$ the following statement is valid:*

$$w(\eta, \bar{\eta}) = w(T\eta, \bar{\eta}) = w(\eta, T\bar{\eta}) = w(T^{-1}\eta, \bar{\eta}) = w(\eta, T^{-1}\bar{\eta}) .$$

Proof. The measurability follows from lemma 4.1 and from the definition of w. Let $(\eta, \bar{\eta}) \in Y^I \times Y^I$. Then

$$n \, w_n(\eta, \bar{\eta}) - 1 \leqq n \, w_n(T\eta, \bar{\eta}) \leqq n \, w_n(\eta, \bar{\eta}) + 1 ,$$

$$w_n(\eta, \bar{\eta}) - \frac{1}{n} \leqq w_n(T\eta, \bar{\eta}) \leqq w_n(\eta, \bar{\eta}) + \frac{1}{n}$$

which implies that $w(\eta, \bar{\eta}) = w(T\eta, \bar{\eta})$. The other equalities can be proved in a similar way.*

Lemma 4.3. *Let B be an arbitrary Borel set. Then every section*

$$w^{-1}(B)_\eta = \{\bar{\eta} \in Y_I : (\eta, \bar{\eta}) \in w^{-1}(B)\} , \quad w^{-1}(B)_{\bar{\eta}} = \{\eta \in Y_I : (\eta, \bar{\eta}) \in w^{-1}(B)\}$$

is an invariant set.

Proof. The validity of the statement follows directly from lemma 4.2.*

Theorem 4.1. *Let $(Y^I \times Y^I, \mathbf{Y}^I \times \mathbf{Y}^I, \omega)$ be a double source and let $\omega_1(E) = \omega(E \times Y^I), \omega_2(F) = \omega(Y^I \times F)$ for every $E \in \mathbf{Y}^I$, $F \in \mathbf{Y}^I$. Let ω_1 and ω_2 be indecomposable.*

Then $\Phi(\omega) \leqq w(\eta, \bar{\eta})$ for almost all $(\eta, \bar{\eta})$ with respect to ω.

Proof. Let B be a Borel set and let $\bar{\eta} \in Y_I$. Then, according to the lemma 4.3, $\omega_1(w^{-1}(B)_{\bar{\eta}}) \in \{0; 1\}$.

Let us denote $W = \{\bar{\eta} \in Y^I : \omega_1(w^{-1}(B)_{\bar{\eta}} = 1\}$. From the lemma 4.2 it follows that W is an invariant set and $\omega_2(W) \in \{0; 1\}$. Therefore

$$\omega(w^{-1}(B)) = \int \omega_1(w^{-1}(B)_{\bar{\eta}} \, d\omega_2 \in \{0; 1\} .$$

It is now evident, that a real number $c \geqq 0$ exists such that $w(\eta, \bar{\eta}) = c$ for almost all $(\eta, \bar{\eta})$. Therefore

$$\Phi(\omega) = \limsup_{n \to \infty} \int w_n(\eta, \bar{\eta})\, d\omega \leqq \int w(\eta, \bar{\eta})\, d\omega = c = w(\eta, \bar{\eta})$$

for almost every $(\eta, \bar{\eta})$.*

The next theorem gives the sufficient conditions for the possibility of the estimation of the mean frequency of errors represented by the upper frequency of errors $\Phi(\omega)$ by comparing emitted and received messages.

Theorem 4.2. *Let $(Y^I, \mathbf{Y}^I, \mu)$ be an ergodic source. Let $\mathcal{T} = (X, Y, Z, g, f)$ and $\tilde{\mathcal{T}} = (\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{g}, \tilde{f})$ be the coding, resp. the decoding automata.* Let $\tilde{Y} = Z$, $Y = \tilde{Z}$. *Let the designations Y_{11}, $\tilde{Y}_{11}$ have the meaning introduced in § 2 for $\mathcal{T}$, resp. $\tilde{\mathcal{T}}$. Let $\mu(Y_{11}) = 1$. Let for every $\eta \in Y_{11}$, $\eta \sim \xi \sim \zeta$ be denoted $\alpha(\eta) = \zeta$; let $\tilde{\alpha}$ be defined on Y_{11} in a similar way and let α, $\tilde{\alpha}$ be defined on $Y^I - Y_{11}$ resp. $\tilde{Y}^D - \tilde{Y}_{11}$ as an arbitrary constant. Let A, $F(A)$ be defined as in the lemma 3.3 and let $\mu(\alpha^{-1}(F(A))) = 1$. Let $(\mathbf{Z}^I, \varrho, \mathbf{Z}^I)$ be a stationary channel with a finite memory and with the property $\mathbf{V}_A$ (lemma 3.2). Let us denote $G_\eta = \{\eta^{(2)} \in Y^I : : \eta, \eta^{(2)}) \in G\}$ for every $\eta \in Y^I$, $G \in \mathbf{Y}^I \times \mathbf{Y}^I$ and $\omega(G) = \int \varrho(\alpha^{-1}(G_\eta)/\alpha(\eta))\, d\mu$.*

Then for almost every $(\eta, \bar{\eta}) \in Y^I \times Y^I$ (with respect to ω)

$$\Phi(\omega) \leqq w(\eta, \bar{\eta}) .$$

Proof. Let us consider a source $(\tilde{Y}^I, \mathbf{Y}^I, \tilde{\mu})$ defined by the relation $\tilde{\mu}(E) = \int \varrho(E/\alpha(\eta))\, d\mu$. This source is, according to the lemma 3.1 and the theorem 3.1, indecomposable (see the note after the lemma 3.1). From the conditions of the present theorem it follows that on the input of the chanell $\pi(F(A)) = 1$ and then according to lemma 3.2, $\tilde{\mu}(F(A)) = 1$ on its output, which implies (lemma 3.3) $\tilde{\mu}(Y_{11}) = 1$. The source on the input of the decoding automaton fulfils hence the conditions of the theorem 2.2 and therefore the measure $\omega_2(.) = \omega(Y^I \times .)$ is indecomposable. Since $\omega_1 = \mu$ is also indecomposable, according to the theorem 4.1, $\Phi(\omega) \leqq w(\eta, \bar{\eta})$ is valid for almost every $(\eta, \bar{\eta})$ (with respect to ω).*

§ 5. CONCLUSION

The author hopes that the results of this paper, mainly the theorem 4.2, possess some technical applications e.g. in considerations about tape teleprinters and the quality of their transmission. In fact the coder and the decoder can be considered like finite automata with two states (the digit state and the letter state) and it seems that all conditions from the Theorem 4.2 are practically always fulfilled. However, applications of the above results may present some practical difficulties connected with the absence of answers to following two questions:

1. Can be the limsup in the theorem 4.2 replaced by a limit?
2. What is the estimation of the difference $w_n(\eta, \bar{\eta}) - \Phi(\omega)$ depending on n?

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VÝTAH

Prenos informácií pri kódovaní a dekódovaní pomocou konečných automatov

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V práci sa uvažuje prenos zpráv, vysielaných ergodickým zdrojom s konečnou abecedou, cez stacionárny kanál s konečnou pamäťou, pričom kódovanie a dekódovanie sa vykonáva pomocou konečných automatov. Tieto automaty sú zobecnením Mealyho automatov v tom, že každej dvojici, tvorenej vstupným písmenom a vnútorným stavom automatu, odpovedá niektoré výstupné slovo (prípadne i prázdne). Pretože sa jedná o kódovanie, nezachovávajúce dĺžku, predpokladá sa súlad medzi vstupnou a výstupnou zprávou pri prechode medzi indexami -1 a 0. Z týchto dôvodov je treba zmeniť i doteraz obvyklú definíciu rizika (frekvencie chýb).

V práci sa študujú otázky, ako vplýva takéto kódovanie na pravdepodobnostné vlastnosti zdroja na výstupe kódovacieho automatu a aké vlastnosti majú mať prvky sponímaného prenosového modelu, aby bolo možné z porovnania vyslanej a prijatej zprávy vysloviť nejaké tvrdenie o strednej frekvencii chýb prenosu.

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