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APPROXIMATIONS OF BELIEVEABILITY FUNCTIONS UNDER INCOMPLETE IDENTIFICATION OF SETS OF COMPATIBLE STATES

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The believeability function has been introduced and investigated in the Dempster–Shafer theory as a numerical characteristic of uncertainty ascribing to each set of possible answers to a question, or set of possible states of an investigated system, under another interpretation, the probability with which the obtained random empirical data (observations) are such that the true answer, or the actual state of the system, can be proved to belong to the set in question. In other words, this value is defined by the probability that the set of all answers or states compatible with the at random obtained data is a subset of the set the believeability of which is to be defined. In this paper we shall investigate the case when the set of compatible states cannot be completely defined so that we have at hand just a class of sets of answers or states containing the set of compatible states. Using this class of sets, we shall define and compute an approximation of the desired value of the believeability function, which can be useful in some decision-making problems when not the value of this function itself, but rather the fact whether this value exceeds some threshold value or not is important.

1. INTRODUCTION

Dempster–Shafer (D.-S., abbreviately) approach to uncertainty quantification and processing or, as it is often called, D.-S. theory, has been developed since the last more than twenty years (the basic Dempster's paper [1] originates from 1967) and represents, in our days, an interesting mathematical model which can be seen, from the purely formal point of view, as a non-traditional application of probability theory, but which offers interpretations going out of the frameworks of the usual interpretations of the probability theory. The aim of this paper is to modify the mathematical apparatus of the D.-S. theory in order to be able to express formally some ideas generalizing and weakening the assumptions of the original D.-S. way of reasoning. So, it would be quite sufficient to restrict ourselves to the very simple abstract presentation of the D.-S. theory as will be introduced in Chapter 2 below, and to kindly invite the reader to consult very numerous and easily accessible references as far as the motivation, philosophical and methodological background, interpretations and applications of the D.-S. theory are concerned. Nevertheless, aiming to present the intended generalization of the D.-S. theory not only as a matter of purely mathematical speculation but also as an effort motivated by certain rather deeply going philosophical and methodological reasons, we take as useful to introduce here, very briefly, a way of reasoning leading to the D.-S. theory and to an intuitive interpretation of this theory.

Consider a system the nature of which (technical, biological, ecological,...) is irrelevant for our purposes and which is situated in just one internal state s_0, called the actual state of the system and belonging to a nonempty set S of potential internal states of the system. The system works or developes in an environment and this environment is situated in just one state ω_0, called the actual state of the environment, or the actual elementary state, belonging to a nonempty set Ω of possible states of the environment (elementary states). Hence, the assumption of closed world is accepted as far as the system in question as well as the environment are concerned. The relation between the environment and the system is deterministic in the sense that s_0 is strictly determined by ω_0, in symbols, there exists a function (total) σ^* defined on Ω and taking its values in S ($\sigma^* : \Omega \to S$, abbreviately) such that $\sigma^*(\omega)$ is the actual state of the system supposing that ω is the actual elementary state, in particular, $\sigma^*(\omega_0) = s_0$ by definition.

The aim of the user (observer, investigator) of the system in question is to identify the actual state s_0 of the system or at least to answer, correctly, the question whether the relation $s_0 \in T$ holds or does not hold for a nonempty proper subset T of S, T being, as a rule, the set of states of the investigated system possessing a property important or interesting because of some reasons connected with an intended application of the system. However, neither s_0 nor ω_0 are directly observable, so that the only possibility is to obtain some partial information about them indirectly. Namely, the user of the system can obtain some empirical data concerning the system and the environment (observations, measurements, results of experiments), formally expressed by a value X from the space E of possible empirical values. This space E may be also a vector space to cover the possibility of more observations, measurements, etc. taken altogether. So, if $E = \mathbb{X}_{i=1}^{n} E_i$, then, e.g. some E_i may be two-element (say, zero-one) spaces to express the results of qualitative observations and experiments (yes – no, holds – does not hold), other E_i's may be real lines or appropriate intervals of real numbers to express the results of quantitative measurements, and so on. In every case, also the values of X are supposed to be unambiguously determined by the actual elementary state of the environment, so that X will be taken as a total function $X : \Omega \to E$.

In the most general case, what the user knows is not the precise value of $X = X(\omega_0)$, but only the fact that $X(\omega) \in F$ holds for some nonempty proper subset F of E. The basic question now reads: what can be told about the validity of the inclusion $s_0 = \sigma^*(\omega_0) \in T$ for a given $\emptyset \neq T \subset S$, $T \neq S$ supposing that we know that $X(\omega_0) \in F \subset E$? From the Platonist point of view and at the level of a three-valued logic the answer is almost trivial:

(i) If $\{\omega : \omega \in \Omega, \sigma^*(\omega) \in T\} \supset \{\omega : \omega \in \Omega : X(\omega) \in F\}$, then $\sigma^*(\omega_0) \in T$ certainly holds.

(ii) If $\{\omega : \omega \in \Omega, \sigma^*(\omega) \in S - T\} \supset \{\omega : \omega \in \Omega, X(\omega) \in F\}$, then $\sigma^*(\omega_0) \in T$ certainly does not hold, so that $\sigma(\omega_0) \in S - T$ certainly holds.

(iii) If neither (i) nor (ii) is the case, i.e., if there are $\omega_1, \omega_2 \in \Omega$ such that $X(\omega_1) \in F$, $X(\omega_2) \in F$, $\sigma^*(\omega_1) \in T$, but $\sigma^*(\omega_2) \notin T$, then no sure answer to the question whether $\sigma^*(\omega_0) \in T$ can be given on the ground of knowing that $X(\omega_0) \in F$ holds, in other words, each such answer or decision will be charged by an uncertainty and possible risk resulting when applying such a decision in practice. Here we are just at the point where we must choose, in order to quantify and process this uncertainty and/or risk, between the way offered by the classical probability theory and by the statistical decision making theory based on these grounds, and between the D.-S. theory. Here we explain only the last case, referring to [3] or to some classical textbooks of statistical decision making theory like [10] or [4] as far as the former approach is concerned.

Very roughly speaking, D.-S. theory is based on ascribing probability values, summing to one, to the three disjoint cases (i) – (iii) listed below instead of the statistical decision theory approach, when the case (iii) is distributed among (i) and (ii), so that the decision is always "yes" or "no", but it is connected with a probability of error, or with a numerically quantifiable risk, when applied; the way how to distribute (iii) among (i) and (ii) is proposed with the aim to minimize this probability of error (risk). To be able to define the three probabilities mentioned above correctly and consistently, some formal structures over the sets Ω, S and E should be defined.

First, we must be able to ascribe numerical probability values to at least some subsets of Ω, S, and E. For the sake of technical convenience we shall suppose that the systems of these subsets are σ-fields, hence, we shall suppose that three nonempty σ-fields $\tilde{\mathcal{A}} \subset \mathcal{P}(\Omega)$, $\mathcal{S} \subset \mathcal{P}(S)$, and $\mathcal{E} \subset \mathcal{P}(E)$ are given, here $\mathcal{P}(A)$ denotes the power-set over A, i.e., the system of all subsets of a set A. Throughout all this paper we shall suppose, to simplify our formal model, that the set S of all potential internal states of the system in question is finite and that $\mathcal{S} = \mathcal{P}(S)$. The mappings $\sigma^* : \Omega \rightarrow S$, $X : \Omega \rightarrow E$ are supposed to be measurable in the sense that the inclusions $\{\{\omega : \omega \in \Omega, \sigma^*(\omega) \in T\} : T \subset S\} \subset \tilde{\mathcal{A}}$ and $\{\{\omega : \omega \in \Omega, X(\omega) \in F\} : F \in \mathcal{E}\} \subset \tilde{\mathcal{A}}$ hold. Finally, a probability measure $P : \tilde{\mathcal{A}} \rightarrow \langle 0, 1\rangle$ is defined on the sets from $\tilde{\mathcal{A}}$, so that σ (X, resp.) is a random variable defined on the probability space $\langle \Omega, \tilde{\mathcal{A}}, P\rangle$ and taking its values in the measurable space $\langle S, \mathcal{P}(S)\rangle$ ($\langle E, \mathcal{E}\rangle$, resp.).

An internal state $s \in S$ is called *compatible* with an empirical value $x \in E$, if $\{\omega \in \Omega : X(\omega) = x\} \cap \{\omega \in \Omega, \sigma^*(\omega) = s\} \neq \emptyset$. More generally, s is compatible with a set $F \subset E$ of empirical states, if s is compatible with at least one empirical value $x \in F$, i.e., if $\{\omega \in \Omega : X(\omega) \in F\} \cap \{\omega \in \Omega, \sigma^*(\omega) = s\} \neq \emptyset$. Let $\mathcal{E}^* \subset \mathcal{P}(\mathcal{P}(E))$ be a nonempty σ-field of subsets of $\mathcal{E}$ and let $\mathcal{F} : \Omega \rightarrow \mathcal{E}$ be a measurable mapping, i.e., $\{\{\omega \in \Omega : \mathcal{F}(\omega) \in \mathcal{G}\} : \mathcal{G} \in \mathcal{E}^*\} \subset \tilde{\mathcal{A}}$ holds. So, the result of an experiment, measurement, observation, etc., is expressed not by an exact empirical value $X(\omega)$, but rather by a set (an interval, e.g.) of such values supposed to cover the exact value of the observed or measured quantity. The former case when these values are directly accessible is then defined when taking simply $\mathcal{F}(\omega) = \{X(\omega)\}$, where $\{\ \}$ denotes the singleton.

Let $U(\omega) = U(\mathcal{F}(\omega)) \subset S$ denote the set of internal states compatible with the set of empirical values $\mathcal{F}(\omega) \subset E$ with respect to X in symbols,

$$\begin{aligned} & U(\omega) = U(\mathcal{F}(\omega)) = \\ = \; & \{s \in S : \{\eta \in \Omega : \sigma^*(\eta) = s\} \cap \{\eta \in \Omega : X(\eta) \in \mathcal{F}(\omega)\} \neq \emptyset\}. \end{aligned} \tag{1.1}$$

The value (supposing that $\{\omega \in \Omega : U(\omega) \subset T\} \in \tilde{\mathcal{A}}$ holds for each $T \subset S$)

$$\mathrm{Bel}(T) = P\left(\{\omega : \omega \in \Omega,\, U(\omega) \subset T\}\right) \tag{1.2}$$

is then defined for each $T \subset S$ and is called the *believeability* of the subset $T \subset S$. Under our interpretation the actual empirical value $X(\omega)$, compatible with the actual internal state $s_0(\omega)$ of the investigated system, is in $\mathcal{F}(\omega)$, so that $s_0(\omega) \in U(\omega)$ holds and $U(\omega) \subset T$ immediately implies that $s_0(\omega) \in T$. Consequently, the value $\mathrm{Bel}(T)$ defines the probability with which the case (i) above occurs, i.e. the case when we can be sure that $s_0(\omega) \in T$ holds. Analogously, the probability of the case (ii) is defined by $\mathrm{Bel}(S - T)$ and the probability of the case (iii) by $1 - \mathrm{Bel}(T) - \mathrm{Bel}(S - T)$. The value $1 - \mathrm{Bel}(S - T)$ is denoted by $\mathrm{Pl}(T)$ and is called the *plausibility* of T; it defines the probability with which the possibility that $s_0(\omega) \in T$ holds cannot be ultimately avoided. The values $\mathrm{Bel}(T)$ and $\mathrm{Pl}(T)$ are the basic numerical characteristics (quantifications, degrees) of uncertainty ascribed to subsets of S and processed in the D.-S. theory.

The most usual way of introducing the D.-S. theory differs from our presentation in two aspects. First, only the case when empirical values are directly observable is considered, so that $U(\omega)$ is defined by $U(\{X(\omega)\})$. Second, not only results of observations, measurements, etc., are taken into consideration, but also other statements concerning the environment and the system and not necessarily supported by empirical data, so that the data taken altogether may be confusing or logically inconsistent. Consequently, the relation $s_0(\omega) \in U(\omega)$ does not necessarily hold, and the set $U(\omega)$ can be even empty (the data are inconsistent), cf. [3, 8] for a more detailed discussion. This problem is solved, within the framework of the D.-S. theory, by avoiding the case of inconsistent data from consideration and by renormalizing the probabilities of the three cases (i) – (iii) with respect to the case when the data are logically consistent, i.e., when the set $U(\omega)$ of compatible states is nonempty. Hence, (1.2) is replaced by the conditional probability

$$\mathrm{Bel}(T) = P\left(\{\omega : \omega \in \Omega,\, U(\omega) \in T\} \,/\, \{\omega : \omega \in \Omega,\, U(\omega) \neq \emptyset\}\right). \tag{1.3}$$

Very often, the D.-S. theory is presented in such a way that the sets $U(\omega)$ of states compatible with random data are taken as the primary point of further considerations, so that all the way leading to this sets and briefly outlined above is neglected. As the set S is taken as finite, all what is then needed is a probability distribution over the (finite) power-set $\mathcal{P}(S)$, i.e., a mapping $m : \mathcal{P}(S) \to \langle 0, 1 \rangle$ such that $m(\emptyset) < 1$ and $\sum_{A \subset S} m(A) = 1$; it is the so called *basic probability assignment.* Given $T \subset S$, $\mathrm{Bel}(T)$ is then simply defined by $(1 - m(\emptyset))^{-1} \sum_{\emptyset \neq A \subset T} m(A)$.

An important feature of the model of D.-S. reasoning explained above, and this holds true for all the models presented till now, consists in the fact that the set $U(X(\omega))$ or $U(\mathcal{F}(\omega))$ of the states compatible with at random sampled empirical value $X(\omega)$, or set $\mathcal{F}(\omega)$ of such values, is always supposed to be at our disposal as a crisp set containing just all the states in question. Only under this condition we are able to decide, theoretically and practically, given $T \subset S$, whether the inclusion $U(X(\omega)) \subset T$ holds or not, when defining and computing the value $\mathrm{Bel}(T)$. However, if the set S is large enough (or infinite, but this case will be mostly avoided from our considerations below, because of technical difficulties), and if $U(\omega)$ $(= U(X(\omega)) = U(\mathcal{F}(\omega))$ must be constructed by a sequential verifying of the compatibility of every particular state from S with the given or obtained empirical data, the assumption that we are able to process actually the set $U(\omega)$ as a whole, unique, single and terminated object may be rather strong and hard to satisfy. It is why we propose, in the next chapter, a generalized formal model for the D.-S.-theory, which replaces this assumption by a weaker one: that we are able to decide for *some* but, in general, *not for all* systems of $\mathcal{A}$ of subsests of S, whether $\mathcal{A}$ contains $U(\omega)$ or not, leaving this question perhaps undecided for some other systems $\mathcal{A} \subset \mathcal{P}(S)$. In the intensional setting, *we are able to recognize some properties of the set* $U(\omega)$, but not to identify it completely in order to be able to separate this set from no matter which other set(s). Obviously, if the power-sets $\mathcal{P}(T) \subset \mathcal{P}(S)$ are among the decidable systems for all $T \subset S$, the model explained above immediately proves to be a special case of the generalized one.

2. DEMPSTER–SHAFER MODEL OF UNCERTAINTY PROCESSING WITH INCOMPLETE IDENTIFICATION

Like as in the model investigated above, our reasoning begins with a nonempty (and usually finite) set S of possible internal states of an investigated system and with a nonempty set E (perhaps a many-dimensional vector space) of empirical values, sampled at random and used in order to take some decision or to answer some question concerning the directly unobservable and inaccessible actual internal state of the system in question. Instead of supposing that both the actual state of the system and the empirical data are determined by elementary states of the environment in which the system as well as its observer are situated, we shall suppose that we have immediately at our disposal an *incomplete compatibility relation over the spaces* S *and* E.

Definition 2.1. Incomplete compatibility relation over nonempty spaces S and E is a mapping ρ defined on the Cartesian product $S \times E$ and taking its values in the set $\{1, *, 0\}$ or in any set containing just three different elements of no matter which nature.

The interpretation behind reads as follows. Given $s \in S$ and $x \in E$, the equality $\rho(s, x) = 1$ means that the state s is compatible with the observation x in the sense that if the observed empirical value is x, it is possible that the system is in the internal state s according to the laws and dependences governing the behaviour of the system and of the procedures generating the observed values and binding them together, moreover, the subject (i. e., the observer, investigator, user) knows this fact. If $\rho(s, x) = 0$, then s is incompatible with x, i.e., the possibility that the system is in the state s can be logically excluded using the mentioned laws and dependences, and the subject knows this fact. Finally, $\rho(s, x) = *$ means that the subject does not know whether s is compatible with x or not, perhaps because of lack of knowledge about the system and observations, because of lack of time to deduce the necessary conclusions, etc.

Setting $\rho(T, F) = \inf_{s \in T} \sup_{x \in F} \rho(s, x)$ for each $T \subset S$ and $F \subset E$, where sup and inf are taken with respect to the linear ordering $0 \prec * \prec 1$ of the set $\{1, *, 0\}$, the incomplete compatibility relation ρ can be extended to the Cartesian product $\mathcal{P}(S) \times \mathcal{P}(E)$. The extension is conservative in the sense that $\rho(\{s\}, \{x\}) = \rho(s, x)$ for each $s \in S$ and $x \in E$, so that we can also define $\rho(T, x)$ by $\rho(T, \{x\})$ and $\rho(s, F)$ by $\rho(\{s\}, F)$. The intuition behind is obvious: $\rho(T, F) = 1$ iff each state $s \in T$ is compatible with at least one value $x \in F$, hence, no $s \in T$ can be eliminated from consideration if the only fact the subject knows is that the observed empirical value is in F. Dually, if $\rho(T, F) = 0$, then there exists a state $s_0 \in T$ incompatible with every value $x \in F$, so that s_0 must be avoided from further considerations as far as the actual state of the system is concerned, if the subject knows that the observed empirical value is in F.

Each incomplete compatibility relation ρ over $S \times E$ defines two subsets of states compatible with an empirical value $x \in E$, namely $U^0(x) = \{s \in S : \rho(s, x) = 1\}$, and $U(x) = \{s \in S : \rho(s, x) \neq 0\}$. It is just the case of $U(x)$ which reflects more tightly the basic idea of D.-S. theory: a state s is compatible with an empirical value x, if s cannot be avoided from consideration as a possible candidate to the actual state of the system, no matter whether the reasons are given by the objective nature of the system and its environment, or by reasons of subjective nature limiting the observer's decision making abilities.

The limited subject's abilities as far as the identification of subsets of the set S in general and of the sets of states compatible with given empirical data in particular are concerned are formally defined by an *incomplete identification relation over the set S*.

Definition 2.2. Incomplete identification relation over a nonempty set S is a mapping σ defined on the Cartesian product $\mathcal{P}(S) \times \mathcal{P}(\mathcal{P}(S))$, taking its values in the set $\{1, *, 0\}$ and such that, for each $A \subset S$ and each $\mathcal{A} \subset \mathcal{P}(S)$, $\sigma(A, \mathcal{A}) = 1$ implies that $A \in \mathcal{A}$ and $\sigma(A, \mathcal{A}) = 0$ implies that $A \in \mathcal{P}(S) - \mathcal{A}$. If $\sigma(A, \mathcal{A}) \in \{1, 0\}$ for each $A \subset S$ and each $\mathcal{A} \subset \mathcal{P}(S)$, the adjective "incomplete" is omitted and the relation σ is called *trivial*. A set $A \subset S$ (a system $\mathcal{A} \subset \mathcal{P}(S)$, resp.) is called *decidable* with respect to σ, if $\sigma(A, \mathcal{B}) \in \{0, 1\}$ for all $\mathcal{B} \subset \mathcal{P}(S)$ (if $\sigma(B, \mathcal{A}) \in \{1, 0\}$ for all $B \subset S$, resp.). A set $A \subset S$ (a system $\mathcal{A} \subset \mathcal{P}(S)$, resp.) is called *vague* with respect to σ, if $\sigma(A, \mathcal{B}) = *$ for all $\mathcal{B} \subset \mathcal{P}(S)$ (if $\sigma(B, \mathcal{A}) = *$ for all $B \subset S$, resp.). The relation σ is called *simple* with respect to σ, if every $\mathcal{A} \subset \mathcal{P}(S)$ is either decidable or vague. The relation σ is called *inclusively closed* if the system $\mathcal{P}(T)$ is decidable for each $T \subset S$.

Let $\langle \Omega, \tilde{\mathcal{A}}, P \rangle$ be an abstract probability space, let $\mathcal{E}$ be a nonempty σ-field of subsets of E, let X be a random variable, i.e., a measurable mapping defined on the probability space $\langle \Omega, \tilde{\mathcal{A}}, P \rangle$ and taking its values in the measurable space $\langle E, \mathcal{E} \rangle$. If ρ is a complete compatibility relation over $S \times E$, i.e., if $\rho(s,x) \in \{1,0\}$ holds for each $s \in S$ and $x \in E$, and if σ is an inclusively closed (incomplete) identification relation such that the class $\mathcal{P}(S) - \{\emptyset\}$ of subsets of S is decidable, the usual definition (1.3) of the believeability function can be rewritten as follows.

$$\begin{aligned} \mathrm{Bel}(T) = \; & P(\{\omega : \omega \in \Omega,\, \sigma(\{s \in S : \rho(s, X(\omega)) = 1\}, \mathcal{P}(T)) = 1\} / \\ & \{\omega : \omega \in \Omega,\, \sigma(\{s \in S : \rho(s, X(\omega)) = 1\}, \mathcal{P}(S) - \{\emptyset\}) = 1\}), \end{aligned} \tag{2.1}$$

where $\{s \in S : \rho(s, X(\omega)) = 1\}$ stands for $U(\omega)$, $\sigma(U(\omega), \mathcal{P}(T)) = 1$ stands for $U(\omega) \in \mathcal{P}(T)$, i.e., for $U(\omega) \subset T$, and $\sigma(U(\omega), \mathcal{P}(S) - \{\emptyset\}) = 1$ stands for $U(\omega) \in \mathcal{P}(S) - \{\emptyset\}$, i.e. for $U(\omega) \neq \emptyset$. A reasonable (as will be argued below) generalization of (1.3) and (2.1) to the case of incomplete compatibility relation and identification relation reads as follows.

Definition 2.3. Let S and E be nonempty sets, let S be finite, let $\mathcal{E}$ be a nonempty σ-field of subsets of E, let X be a random variable defined on the probability space $\langle \Omega, \tilde{\mathcal{A}}, P \rangle$ and taking its values in the measurable space $\langle E, \mathcal{E} \rangle$, let ρ be an incomplete compatibility relation on $S \times E$ and let σ be an incomplete identification relation on S such that $\{\omega \in \Omega : \sigma(\{s \in S : \rho(s, X(\omega)) = \alpha\}, \mathcal{A}) = \beta\} \in \tilde{\mathcal{A}}$ holds for each α, $\beta \in \{1, 0, *\}$ and each $\mathcal{A}$ belonging to the minimal σ-field generated in $\mathcal{P}(\mathcal{P}(S))$ by the set $\{\mathcal{P}(T) : T \subset S\}$ of systems of subsets of S. *Generalized believeability function* Bel^* ($\mathrm{Bel}^*(X, \rho, \sigma)$, in more detail), generated on $\mathcal{P}(S)$ by X, ρ, and σ, is then defined, for each $T \subset S$, by the conditional probability

$$\begin{aligned} \mathrm{Bel}(T) = \; & P(\{\omega : \omega \in \Omega,\, \sigma(\{s \in S : \rho(s, X(\omega)) \neq 0\}, \mathcal{P}(T)) = 1\} / \\ & \{\omega : \omega \in \Omega,\, \sigma(\{s \in S : \rho(s, X(\omega)) = 1\}, \mathcal{P}(S) - \{\emptyset\}) = 1\}), \end{aligned} \tag{2.2}$$

if $P(\{\omega \in \Omega : \sigma(\{s \in S : \rho(s, X(\omega)) = 1\}, \mathcal{P}(S) - \{\emptyset\}) = 1\}) > 0$ holds, or by

$$\begin{aligned} \mathrm{Bel}^*(T) = \; & P(\{\omega : \omega \in \Omega,\, \sigma(U(X(\omega)), \mathcal{P}(T)) = 1\} / \\ & \{\omega : \omega \in \Omega,\, \sigma(U^0(X(\omega)), \mathcal{P}(S) - \{\emptyset\}) = 1\}), \end{aligned} \tag{2.3}$$

when using the definitions of $U(x)$ and $U^0(x)$ introduced above.

At the first sight, we could consider also other variants of the original function Bel, e.g., replacing U by U^0 in the conditioned random event in (2.3) and/or replacing U^0 by U in the conditioning event, however, only the variant introduced in (2.3) conserves the main idea of the D.-S. reasoning. Or, (2.3) defines the probability of occurrence of such empirical data that

(1) we are sure that the data are consistent as we are sure that there exists at least one state of the investigated system which is compatible with these data by virtue of the objective properties of the system and environment, not only by virtue of our ignorance and limited abilities, and

(2) we are sure that every state compatible with the data in this objective sense, no matter whether we are able to verify this compatibility for every such state in

particular, must lie in the critical set T the degree of believeability of which is to be numerically quantified. In other words,

(3) $\mathrm{Bel}^*(T)$ defines the probability of occurrence of such empirical data which enable to conclude, supposing that these data holds true in the actual state of the system and of the environment, that the actual state of the system must be in the critical set T, and this deduction is not charged by any kind and degree of uncertainty. An easy re-consideration of other possible variants of the generalized believeability function shows that no of them conserves this basic property of the D.-S. reasoning.

3. INCOMPLETE IDENTIFICATION RELATIONS RELATED TO EQUIVALENCE RELATIONS OF INDISTINGUISHABILITY

In this chapter we shall investigate the special case of the model described above arising when we are not able to distinguish the set of states compatible with the obtained empirical values from some other subsets of S, so that we are able to decide that $U(X) \subset T$ ($U^0(x) \subset T$, resp.) holds only if this inclusion holds for every $A \subset S$ indistinguishable from $U(X)$ ($U^0(x)$, resp.). In order to simplify the situation we shall suppose, throughout this chapter, that

(a) the set S is finite,

(b) the compatibility relation ρ on $S \times E$ is complete, so that $\rho(s,x) = 1$ or 0 for each $s \in S$, $x \in E$ and, consequently, $U^0(x) = U(x)$ for each $x \in E$,

(c) the indistinguishability relation on $\mathcal{P}(S) \times \mathcal{P}(S)$ is defined by an equivalence relation $\sim$ on $\mathcal{P}(S) \times \mathcal{P}(S)$, so that two subsets A, B of S are indistinguishable iff $A \sim B$ holds,

(d) the incomplete identification relation σ on $\mathcal{P}(S) \times \mathcal{P}(\mathcal{P}(S))$ is such that, for each A, $T \subset S$,

$$\sigma(A, \mathcal{P}(T)) = 1 \text{ iff } B \subset T \text{ (i.e., } B \in \mathcal{P}(T)), \text{ holds for each } B \sim A,$$
$$\sigma(A, \mathcal{P}(S) - \mathcal{P}(\emptyset)) = 1 \text{ iff } B \neq \emptyset \text{ (i.e., } B \in \mathcal{P}(S) - \mathcal{P}(\emptyset)), \text{ holds for each } B \sim A,$$

(e) the measurable space $\langle E, \mathcal{E} \rangle$, in which the random variable X defined on the probability space $\langle \Omega, \tilde{\mathcal{A}}, P \rangle$ takes its values, is rich enough so that the set $\bigcup_{B \sim A} \{x \in E : U(x) = B\}$ is in $\mathcal{E}$ for each $A \subset S$.

Denoting, for each $A \subset S$, by $[A] = \{B : B \subset S,\ B \sim A\}$ the equivalence class in the factor-space $\mathcal{P}(S)/\sim$, to which A belongs, we can rewrite the definition of Bel^* as follows. Let $T \subset S$, then

$$\begin{aligned} \mathrm{Bel}^*(T) &= P(\{\omega \in \Omega : \sigma(U(X(\omega)), \mathcal{P}(T)) = 1\} / \\ &\quad \{\omega \in \Omega : \sigma(U(X(\omega)), \mathcal{P}(S) - \mathcal{P}(\emptyset)) = 1\}) = \\ &= P(\{\omega \in \Omega : B \subset T \text{ for all } B \sim U(X(\omega))\} / \\ &\quad \{\omega \in \Omega : B \neq \emptyset \text{ for all } B \sim U(X(\omega))\}) = \\ &= P(\{\omega \in \Omega : B \in \mathcal{P}(T) \text{ for all } B \sim U(X(\omega))\} / \end{aligned} \tag{3.1}$$

$$\{\omega \in \Omega : \emptyset \notin [U(X(\omega))]\}) =$$
$$= P\left(\{\omega \in \Omega : [U(X(\omega))] \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : [U(X(\omega))] \neq [\emptyset]\}\right) =$$
$$= P\Big(\bigcup_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \subset \mathcal{P}(T)} \{\omega \in \Omega : [U(X(\omega))] = \mathcal{A}\} \,/$$
$$\bigcup_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset]} \{\omega \in \Omega : [U(X(\omega))] = \mathcal{A}\}\Big)$$

For each A, $B \subset S$, $B \in [A]$ holds iff $[B] = [A]$, so that, for each $\mathcal{A} \in \mathcal{P}(S)/\sim$, i.e., for each $\mathcal{A} = [A]$ for some $A \subset S$, an easy calculation yields that

$$\{\omega \in \Omega : [U(X(\omega))] = \mathcal{A}\} = \{\omega \in \Omega : [U(X(\omega))] = [A]\} = \tag{3.2}$$
$$= \{\omega \in \Omega : U(X(\omega)) \in [A]\} = \{\omega \in \Omega : X(\omega) \in \{x \in E : U(X) \in [A]\}\} =$$
$$= \Big\{\omega \in \Omega : X(\omega) \in \bigcup_{B \in [A]} \{x \in E : U(x) = B\}\Big\} =$$
$$= \{\omega \in \Omega : X(\omega) \in \bigcup_{B \sim A} \{x \in E : U(x) = B\}\}.$$

The condition (e) above yields that $\bigcup_{B \sim A}\{x \in E : U(x) = B\} \in \mathcal{E}$ holds for each $A \subset S$ and X is a random variable defined on $\langle \Omega, \tilde{\mathcal{A}}, P\rangle$ and taking its values in $\langle E, \mathcal{E}\rangle$, hence, the subset of Ω defined in (3.2) is in $\tilde{\mathcal{A}}$, consequently, $P(\{\omega \in \Omega : [U(X(\omega))] = \mathcal{A}\})$ is defined for each $\mathcal{A} \in \mathcal{P}(S)/\sim$. Combining (3.1) and (3.2) we obtain that

$$\mathrm{Bel}^*(T) = \frac{\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset],\, \mathcal{A} \subset \mathcal{P}(T)} P(\{\omega \in \Omega : [U(X(\omega))] = \mathcal{A}\})}{\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset]} P(\{\omega \in \Omega : [U(X(\omega))] = \mathcal{A}\})}. \tag{3.3}$$

This expression can be easily written in the form using the basic probability assignment and yielding the usual believeability function, but this time with $\mathcal{P}(S)$ playing the role of the basic space S. Let $m : \mathcal{P}(\mathcal{P}(S)) \to \langle 0, 1\rangle$ be defined, for each $\mathcal{A} \in \mathcal{P}(S)$, by

$$m(\mathcal{A}) = P(\{\omega \in \Omega : [U(X(\omega))] = \mathcal{A}\}) \tag{3.4}$$

if $\mathcal{A} \in \mathcal{P}(S)/\sim$, i.e., $\mathcal{A} = [A]$ for some $A \subset S$, $m(\mathcal{A}) = 0$ otherwise. Then, obviously, $\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim} m(\mathcal{A}) = 1$ and

$$\mathrm{Bel}^*(T) = \frac{\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset],\, \mathcal{A} \subset \mathcal{P}(T)} m(\mathcal{A})}{\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset]} m(\mathcal{A})} = \frac{\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset],\, \mathcal{A} \subset \mathcal{P}(T)} m(\mathcal{A})}{1 - m([\emptyset])}, \tag{3.5}$$

supposing that $m([\emptyset]) < 1$, otherwise $\mathrm{Bel}^*(T)$ is not defined.

The interpretation behind may be such that a more complicated system with several possible compatibility functions is considered. The states of this new system are sets of states of the original system and a state of this new system is compatible with some data, if it is indistinguishable, with respect to the equivalence relation $\sim$, from the actual state of the new system by which the data in question have been generated.

It is perhaps worth investigating, in particular, the case when the equivalence relation $\sim$ coincides with the identity relation Id on $\mathcal{P}(S) \times \mathcal{P}(S)$. Then, evidently,

$[A] = \{A\}$ for each $A \subset S$ and $\mathcal{P}(S)/\mathrm{Id} = \{\{A\} : A \subset S\}$, so that (3.1) implies

$$\begin{aligned}\mathrm{Bel}^*(T) &= P\left(\{\omega \in \Omega : \{U(X(\omega))\} \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \{U(X(\omega))\} \neq \{\emptyset\}\}\right) \quad (3.6)\\ &= P\left(\{\omega \in \Omega : U(X(\omega)) \subset T\} \,/\, \{\omega \in \Omega : U(X(\omega)) \neq \emptyset\}\right),\end{aligned}$$

hence, $\mathrm{Bel}^*(T)$ agrees with the usual believeability function $\mathrm{Bel}(T)$ generated on $\mathcal{P}(S)$ by X and ρ, as could be expected. The condition (e) yields, in this case, that the probability

$$\begin{aligned} & P\left(\{\omega \in \Omega : U(X(\omega)) = A\}\right) = \quad (3.7)\\ &= P\left(\{\omega \in \Omega : X(\omega) \in \{x \in E : U(x) \in \{A\}\}\}\right)\\ &= P\left(\{\omega \in \Omega : X(\omega) \in \{x \in E : U(x) = A\}\}\right)\end{aligned}$$

is defined for each $A \subset S$. Denoting this value by $m^0(A)$ we easily obtain that m^0 is a basic probability assignment of $\mathcal{P}(S)$ and, by (3.4),

$$\begin{aligned} & m([A]) = m(\{A\}) = \quad (3.8)\\ &= P\left(\{\omega \in \Omega : \{U(X(\omega))\} = \{A\}\}\right) = P\left(\{\omega \in \Omega : U(X(\omega)) = A\}\right) = m^0(A),\end{aligned}$$

so that

$$\mathrm{Bel}^*(T) = \frac{\sum_{\{\mathcal{A}\} \in \mathcal{P}(S)/\mathrm{Id},\, \{\mathcal{A}\} \neq \{\emptyset\},\, \{\mathcal{A}\} \subset \mathcal{P}(T)} m(\mathcal{A})}{\sum_{\{\mathcal{A}\} \in \mathcal{P}(S)/\mathrm{Id},\, \{\mathcal{A}\} \neq \{\emptyset\}} m(\mathcal{A})} = \frac{\sum_{A \neq \emptyset,\, A \subset T} m^0(A)}{\sum_{A \neq \emptyset} m^0(A)}, \quad (3.9)$$

and this agrees with the definition of usual believeability function over $\mathcal{P}(S)$ through a basic probability assignment.

Let us turn to the case with a general, not necessary identity, equivalence relation $\sim$ on $\mathcal{P}(S) \times \mathcal{P}(S)$, but let us suppose that the condition (e) holds for the identity relation Id (its validity for each equivalence relation on $\mathcal{P}(S) \times \mathcal{P}(S)$ then easily follows from the finiteness of the basic space S). Hence, the values $m^0(A)$ are defined, by (3.7), for each $A \subset S$. (3.2) and (3.4) then yield that, for each $\mathcal{A} = [A] \in \mathcal{P}(S)/\sim$,

$$\begin{aligned} m(\mathcal{A}) &= m([A]) = P\left(\{\omega \in \Omega : [U(X(\omega))] = [A]\}\right) = \quad (3.10)\\ &= P\left(\{\omega \in \Omega : U(X(\omega)) \in [A]\}\right) =\\ &= \textstyle\sum_{B,\, B \sim A} P\left(\{\omega \in \Omega : U(X(\omega)) = B\}\right) = \sum_{B \in \mathcal{A}} m^0(B),\end{aligned}$$

so that the expression (3.5) for $\mathrm{Bel}^*(T)$ can be expressed directly by the values of the basic probability assignment m^0, namely,

$$\mathrm{Bel}^*(T) = \frac{\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset],\, \mathcal{A} \subset \mathcal{P}(T)} \sum_{A \in \mathcal{A}} m^0(A)}{\sum_{\mathcal{A} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \neq [\emptyset]} \sum_{A \in \mathcal{A}} m^0(A)}. \quad (3.11)$$

Obviously, $\mathcal{A} = [A]$, $\mathcal{A} \subset \mathcal{P}(T)$ implies that $A \subset T$ but, in general, $A \subset T$ does not imply that $[A] \subset \mathcal{P}(T)$, so that, if $m([\emptyset]) = 0$ (consequently, $m^0(\emptyset) = 0$), we obtain that $\mathrm{Bel}^*(T) \leq \mathrm{Bel}(T)$ holds for each $T \subset S$. Again, this inequality could be intuitively expected when taking into consideration the interpretation of the values in question. As can be almost immediately seen, everything what has been proved above for the case of identity equivalence relation Id remains valid for any equivalence relation $\approx$ on $\mathcal{P}(S) \times \mathcal{P}(S)$ refining the relation $\sim$, i.e., such that $A \approx B$ implies $A \sim B$ for each $A, B \subset S$. The mapping m^0 will then be a basic probability assignment on the equivalence classes generated in $\mathcal{P}(S)$ by the finer equivalence relation $\approx$, i.e., on $\mathcal{P}(S)/\approx$.

It may be perhaps worth presenting the obtained results once more, in the form of an assertion.

Theorem 3.1. Let $\langle \Omega, \tilde{\mathcal{A}}, P\rangle$ be an abstract probability space, $\langle E, \mathcal{E}\rangle$ a measurable space of empirical values, X a random variable taking $\langle \Omega, \tilde{\mathcal{A}}, P\rangle$ into $\langle E, \mathcal{E}\rangle$, S a finite set, ρ a complete compatibility relation over $S \times E$, and σ an incomplete identification relation on $\mathcal{P}(S) \times \mathcal{P}(\mathcal{P}(S))$ such that the conditions (a)–(e) hold. Let $m(\mathcal{A})$ be defined, for each $\mathcal{A} \subset \mathcal{P}(S)$, by (3.4). Then, for each $T \subset S$, (3.5) holds. If the condition (e) is valid for the identity relation on $\mathcal{P}(S) \times \mathcal{P}(S)$ and if $m^0(A)$ is defined for each $A \subset S$ by (3.7), then (3.11) holds for each $T \subset S$, If $m([\emptyset]) = 0$, if the condition (e) holds with respect to an equivalence relation $\approx$ refining the original relation $\sim$, and if $\text{Bel}^{**}(T)$ is defined with respect to $\approx$, then $\text{Bel}^*(T) \leq \text{Bel}^{**}(T)$ holds for each $T \subset S$. In particular, if $\approx = \text{Id}$, then $\text{Bel}^*(T) \leq \text{Bel}(T)$ holds for each $T \subset S$.

4. DEMPSTER COMBINATION RULE FOR GENERALIZED BELIEVEABILITY FUNCTIONS

In the introductory part of this paper we describe the way of reasoning paradigmatic for the D.-S. theory and yielding the believeability function as the principal numerical characteristic of uncertainty introduced and investigated within the framework of this theory. Under the presented interpretation, a subject (observer, user,...) obtains some empirical data of random nature concerning the investigated system and the environment in which this system works, and combining these data with her/his a priori knowledge, she/he arrives at the set $U(X(\omega))$ of all possible internal states of the system compatible with the empirical data $X(\omega)$. Given a subset T of the set S of all possible internal states of the system in question, the believeability $\text{Bel}_U(T)$ ascribed to this subset is then defined by the probability with which every state compatible with $X(\omega)$ is in T, hence, as the probability with which the inclusion $U(X(\omega)) \subset T$ holds.

Consider, now, the case when two subjects observe the same system, the first one obtains empirical data $X_1(\omega)$ and the other one empirical data $X_2(\omega)$, and they combine these data separately and independently of each other, with their individual a priori knowledge into sets $U(X_1(\omega))$ and $U(X_2(\omega))$, where $U(X_i(\omega))$ denotes the set of states compatible with the data obtained by the i-th subject ($i = 1, 2$). A third subject, to whom the results of both the former subjects represented by the set $U(X_1(\omega))$ and $U(X_2(\omega))$ are accessible, wants to combine them in a way improving both the particular results in the sense that the believeability ascribed to a subset T of S should increase supposing that the data obtained by both the subjects are true and that the actual state of the investigated system is, in fact, in T. This is an optimistic way of combination of the particular results, a pessimistic way of combination when any new uncertain knowledge can only deteriorate the information being already at hand is also possible and worth a more detailed investigation, but we shall not take this possibility into consideration here.

The most simple optimistic combination can be easily defined as follows. The third subject takes a state of the system as compatible with the joined data $\langle X_1(\omega), X_2(\omega)\rangle$, if it is taken as compatible by both the first and the second subjects. In other words, the reasons leading at least one of the original two subjects to the statement that a state is incompatible are fully accepted by the third subject (the believing of the third subject to what the first or the second one say expresses, in a sense, an optimistic point of view of the third subject). So, the third subject defines her/his own set $U(\langle X_1(\omega), X_2(\omega)\rangle$ of states compatible with the joined data $\langle X_1(\omega), X_2(\omega)\rangle$ by the set-theoretic joint $U(X_1(\omega)) \cap U(X_2(\omega))$, i.e. by $U_1(\omega) \cap U_2(\omega)$, if abbreviating $U(X_i(\omega))$ by $U_i(\omega)$. Then the third subject defines her/his own believeability function $\mathrm{Bel}_{U_1 \cap U_2}$ in the usual way, setting $(U_1 \cap U_2)(\omega) = U_1(\omega) \cap U_2(\omega)$ for all $\omega \in \Omega$. So, the third subject obtains that

$$\mathrm{Bel}_{U_1 \cap U_2}(T) = P(\{\omega \in \Omega : (U_1 \cap U_2)(\omega) \subset T\} \,/\, \{\omega \in \Omega : (U_1 \cap U_2)(\omega) \neq \emptyset\}) \tag{4.1}$$

for each $T \subset S$. This new believeability function can be written in the form $\mathrm{Bel}_{U_1 \cap U_2}(T) = f(\mathrm{Bel}_{U_1}, \mathrm{Bel}_{U_2})(T)$ and interpreted as the result of a binary operation f applied to the original believeability functions Bel_{U_1} and Bel_{U_2}. Under some simplifying conditions, $\mathrm{Bel}_{U_1 \cap U_2}$ can be expressed in a more explicit way through Bel_{U_1} and Bel_{U_2}. Namely, if

(i) S is finite, so that $m_i(A) = P(\{\omega \in \Omega : U_i(\omega) = A\})$ is defined for each $A \subset S$ and for both $i = 1, 2$,

(ii) $\sum_{A,B \in S,\, A \cap B = \emptyset} m_1(A)\, m_2(B) < 1$ holds,

(iii) the set-valued random variables U_1, U_2, defined on the probability space $\langle \Omega, \tilde{\mathcal{A}}, P\rangle$ and taking their values in the measurable space $\langle \mathcal{P}(S), \mathcal{P}(\mathcal{P}(S))\rangle$ are statistically independent, i.e.,

$$\begin{aligned} & P(\{\omega \in \Omega : U_1(\omega) = A,\, U_2(\omega) = B\}) = \\ = \; & P(\{\omega \in \Omega : U_1(\omega) = A\}) \cdot P(\{\omega \in \Omega : U_2(\omega) = B\}) \end{aligned} \tag{4.2}$$

holds for each A, $B \subset S$, then (4.1) converts into the combinatoric expression

$$\mathrm{Bel}_{U_1 \cap U_2}(T) = \frac{\sum_{A,B \subset S,\, \emptyset \neq A \cap B \subset T} m_1(A)\, m_2(B)}{\sum_{A,B \subset S,\, \emptyset \neq A \cap B} m_1(A)\, m_2(B)}. \tag{4.3}$$

In this simplified case when (4.3) holds we usually write $\mathrm{Bel}_{U_1 \cap U_2}(T) = (\mathrm{Bel}_{U_1} \oplus \mathrm{Bel}_{U_2})(T)$, instead of a general operation f, and the operation $\oplus$ is called the *Dempster combination rule* yielding the believeability function $\mathrm{Bel}_{U_1 \cap U_2}$ as the result of combination of the original believeability functions Bel_{U_1} and Bel_{U_2}. If the state space S is supposed to be finite, (4.3) is often immediately introduced as an abstract axiomatic definition of the Dempster combination rule applied to two believeability functions defined by their basic probability assignments m_1, m_2. Obviously, $\mathrm{Bel}_{U_1 \cap U_2}$ can be also defined through a new probability assignment m_3, setting

$$m_3(C) = \textstyle\sum_{A,B \subset S,\, A \cap B = C} m_1(A)\, m_2(B) \tag{4.4}$$

for each $C \subset S$.

In order to investigate a possibility how to extend the Dempster combination rule to generalized believeability function or how to define another rule playing the same or similar role let us reconsider the relation (2.1) when believeability functions are defined through complete compatibility relations and complete identification relations taken as the primary notions. So, let $\rho_1, \rho_2 : S \times E \to \{0,1\}$ be two complete compatibility relations corresponding to the two subjects in question and inducing, consecutively, two set-valued random variables U_1, U_2 such that $U_i(\omega) = U(X_i(\omega)) = \{s \in S : \rho(s, X_i(\omega)) = 1\}$. Hence,

$$\begin{aligned} \mathrm{Bel}_{U_i}(T) \;=\; & P\left(\{\omega \in \Omega : \sigma(\{s \in S : \rho_i(s, X_i(\omega)) = 1\}, \mathcal{P}(T)) = 1\}\,/\right. \\ & \left./\,\{\omega \in \Omega : \sigma(\{s \in S : \rho_i(s, X_i(\omega)) = 1\}, \mathcal{P}(S) - \{\emptyset\}) = 1\}\right) \end{aligned} \tag{4.5}$$

for all $T \subset S$ and for both $i = 1, 2$, supposing that $\mathrm{Bel}_{U_i}(T)$ is defined. Defining a new compatibility relations ρ_3 on $S \times (E \times E)$ by

$$\rho_3(s, \langle x_1, x_2 \rangle) = \min\{\rho_1(s, x_1),\ \rho_2(s, x_2)\} \tag{4.6}$$

for all $s \in S$, x_1, $x_2 \in E$, we easily obtain that for each such x_1, x_2,

$$\begin{aligned} & U_2(\langle x_1, x_2 \rangle) = \{s \in S : \rho_3(s, \langle x_1, x_2 \rangle) = 1\} = \\ = \; & \{s \in S : \rho_1(s, x_1) = 1\} \cap \{s \in S : \rho_2(s, x_2) = 1\} = U_1(x_1) \cap U_2(x_2), \end{aligned} \tag{4.7}$$

so that $\mathrm{Bel}_{U_1 \cap U_2}(T) = \mathrm{Bel}_{U_3}(T)$ can be expressed, for each $T \subset S$, by

$$\begin{aligned} & \mathrm{Bel}_{U_1 \cap U_2}(T) = P\left(\{\omega \in \Omega : \sigma(\{s \in S : \rho_3(s, \langle X_1(\omega), X_2(\omega) \rangle) = 1\}, \mathcal{P}(T)) = 1\}\,/\right. \\ & \left./\,\{\omega \in \Omega : \sigma(\{s \in S : \rho_3(s, \langle X_1(\omega), X_2(\omega) \rangle) = 1\}, \mathcal{P}(S) - \{\emptyset\}) = 1\}\right) = \\ & = P\left(\{\omega \in \Omega : \sigma(\{s \in S : \rho_1(s, X_1(\omega)) = \rho_2(s, X_2(\omega)) = 1\}, \mathcal{P}(T)) = 1\}\,/\right. \\ & \left./\,\{\omega \in \Omega : \sigma(\{s \in S : \rho_1(s, X_1(\omega)) = \rho_2(s, X_2(\omega)) = 1\}, \mathcal{P}(S) - \{\emptyset\}) = 1\}\right), \end{aligned} \tag{4.8}$$

or, in still another notation,

$$\begin{aligned} & \mathrm{Bel}_{U_1 \cap U_2}(T) = P\left(\{\omega \in \Omega : \sigma(\{s \in S : \min\{\rho_i(s, X_i(\omega)),\ i = 1, 2\} = 1\}, \mathcal{P}(T)) = 1\}\,/\right. \\ & \left./\,\{\omega \in \Omega : \sigma(\{s \in S : \min\{\rho_i(s, X_i(\omega)),\ i = 1, 2\} = 1\}, \mathcal{P}(S) - \{\emptyset\}) = 1\}\right). \end{aligned} \tag{4.9}$$

This last expression for $\mathrm{Bel}_{U_1 \cap U_2}(T)$ seems to be the most appropriate for being immediately generalized to the case of three-valued incomplete compatibility functions. Consider, again, two independent subjects, but this time represented by incomplete compatibility functions $\rho_1, \rho_2 : S \times E \to \{1, *, 0\}$. A third subject uses the incomplete compatibility function $\rho_3 : S \times (E \times E) \to \{1, *, 0\}$ defined as follows: she/he takes a state s as surely (or provably) compatible with a pair $\langle x_1, x_2\rangle \in E \times E$ of observations, iff s is surely compatible with x_1 for the first subject and with x_2 for the second subject, hence, $\rho_3(s, \langle x_1, x_2\rangle) = 1$ iff $\rho_i(s, x_i) = 1$ for both $i = 1, 2$. Also, the third subject takes s as surely incompatible with $\langle x_1, x_2\rangle$, if s is surely incompatible with x_i with respect to ρ_i for at least one of the two original subjects, hence, $\rho_3(s, \langle x_1, x_2\rangle) = 0$ iff either $\rho_1(s, x_1) = 0$, or $\rho_2(s, x_2) = 0$. Finally, the third subject takes the compatibility of s with $\langle x_1, x_2\rangle$ as uncertain, i.e., $\rho_3(s, \langle x_1, x_2\rangle) = *$, in all the other cases. An easy reasoning proves that ρ_3 is the only three-valued extension of the two-valued case of ρ_3 investigated above which preserves its two basic principles: each subject in particular decides ultimately about the incompatibility of a state, on the other side, if s is considered as compatible with data $\langle x_1, x_2\rangle$, then s is in fact (at an objective level) compatible with x_1 as well as with x_2, consequently, if x_1 and x_2 are true (valid) in the actual state s_0 of the system, then s_0 is compatible with x_1, x_2, and $\langle x_1, x_2\rangle$, no matter whether the corresponding subjects are able to deduce this fact. E. g., when setting $\tilde{\rho}_3(s, \langle x_1, x_2\rangle) = \min\{\rho_1(s, x_1), \rho_2(s, x_2)\}$, if $\rho_i(s, x_i) \in \{1, 0\}$ for both $i = 1, 2$, and setting $\tilde{\rho}_3(s, \langle x_1, x_2\rangle) = \rho_i(s, x_i)$, if $\rho_j(s, x_j) = *$ for $i, j = 1, 2,\ i \neq j$, then $\tilde{\rho}_3$ does not possess the property mentioned above, even if it may be also taken as reasonable from a certain point of view (an uncertain answer about the compatibility of a state given by one subject is neglected supposing that the other subject offers a certain answer).

If follows immediately that the incomplete compatibility relation ρ_3 can be formally defined by

$$\rho_3(s, \langle x_1, x_2\rangle) = \mu\{\rho_1(s, x_1),\ \rho_2(s, x_2)\}, \tag{4.10}$$

where μ is the minimum operation on $\{1, *, 0\}$ defined by the linear ordering $0 \prec * \prec 1$ on this set. In order to be able to write an analogy to the Dempster combination rule for generalized believeability functions in a form as close to (4.9) as possible, let us realize that the random variable U_1 (U_2, resp.) in (4.5) is uniquely defined by the (complete) compatibility function ρ_1 (ρ_2, resp.) and by the random variable X, so that we could also write $\mathrm{Bel}_{\rho_i}(T)$ instead of $\mathrm{Bel}_{U_i}(T)$. In the case of incomplete compatibility functions, the same role is played by the pairs $\mathcal{U}_i(\omega) = \langle U_i^0(\omega),\ U_i(\omega)\rangle$, $i = 1, 2$, of random variables defined by

$$U_i^0(\omega) \;=\; \{s \in S : \rho_i(s, X(\omega)) = 1\}\,, \tag{4.11}$$

$$U_i(\omega) \;=\; \{s \in S : \rho_i(s, X(\omega)) \neq \emptyset\}\,. \tag{4.12}$$

Obviously, if $\mathcal{U}_3(\omega) = \langle U_3^0(\omega),\ U_3(\omega)\rangle$ is generated by ρ_3 and ρ_3 is defined by (4.10), then $U_3^0(\omega) = U_1^0(\omega) \cap U_2^0(\omega)$ and $U_3(\omega) = U_1(\omega) \cap U_2(\omega)$. Denoting these last two equalities together by $\mathcal{U}_3(\omega) = \mathcal{U}_1(\omega) \sqcap \mathcal{U}_2(\omega)$, or simply by $\mathcal{U}_3 = \mathcal{U}_1 \sqcap \mathcal{U}_2$, we arrive at the following Dempster combination rule $\oplus^*$ for generalized believeability functions. If $\mathrm{Bel}_{\mathcal{U}_i}(T)$, $i = 1, 2$ are defined by (2.2) with ρ replaced by ρ_1 and ρ_2 and with $\mathcal{U}_i$ generated by ρ_i according to (4.11) and (4.12), then for each $T \subset S$,

$$\begin{aligned} &\left(\mathrm{Bel}^*_{\mathcal{U}_1} \oplus^* \mathrm{Bel}^*_{\mathcal{U}_2}\right)(T) = \mathrm{Bel}^*_{\mathcal{U}_1 \sqcap \mathcal{U}_2}(T) = \\ = \;& P(\{\omega \in \Omega : \sigma(\{s \in S : \mu\{\rho_i(s, X_i(\omega)) : i = 1, 2\} \neq 0\},\ \mathcal{P}(T)) = 1\}\,/ \\ & /\, \{\omega \in \Omega : \sigma(\{s \in S : \mu\{\rho_i(s, X_i(\omega)) : i = 1, 2\} = 1\},\ \mathcal{P}(S) - \{\emptyset\}) = 1\})\,. \end{aligned} \tag{4.13}$$

In the case of generalized believeability functions with complete compatibility relations and with incomplete identification relations related to equivalence relations and investigated in Chapter 3 we have seen that these generalized believeability functions can be defined by usual believeability functions over appropriate factor-spaces generated by the equivalence relation in question in the power-set $\mathcal{P}(S)$ over the state space S. A natural question arises, whether the generalized Dempster combination rule defined by (4.13) could be expressed, in this case, through the usual Dempster combination rule applied to the corresponding usual believeability functions over the factor-spaces in question.

As will be proved below, the answer can be affirmative only under certain additional restrictive conditions imposed on the equivalence relation $\sim$ defined on $\mathcal{P}(S)$. Or, when considering the original set-valued random variables $U_1 = U(X_1(\cdot))$, $U_2 = U(X_2(\cdot))$, the random event occurring when $U_1(\omega) \cap U_2(\omega) = A$ for some $A \subset S$ can be easily defined by random events $U_1(\omega) = B$ and $U_2(\omega) = C$ for appropriate B, $C \subset S$, namely,

$$\begin{aligned} &\{\omega \in \Omega : U_1(\omega) \cap U_2(\omega) = A\} = \\ &= \bigcup_{B, C \subset S,\, B \cap C = A} \{\omega \in \Omega : U_1(\omega) = B,\, U_2(\omega) = C\}. \end{aligned} \tag{4.14}$$

This relation cannot be, in general, extended to the random variables $[U_1]=[U(X_1(\cdot))]$ and $[U_2] = [U(X_2(\cdot))]$, as the random event $\{\omega \in \Omega : [U_1(\omega) \cap U_2(\omega)] = \mathcal{A}\}$, $\mathcal{A} \in \mathcal{P}(S)/\sim$, cannot be defined by random events $\{\omega \in \Omega : [U_1(\omega)] = \mathcal{B}\}$, $\{\omega \in \Omega : [U_2(\omega)] = \mathcal{C}\}$, for some $\mathcal{B}, \mathcal{C} \in \mathcal{P}(S)/\sim$. The reason is that if $\mathcal{B} = [B]$, $\mathcal{C} = [C]$ for some B, $C \subset S$, and if $B_1 \in \mathcal{B}$, $C_1 \in \mathcal{C}$, i.e. if $B_1 \sim B$ and $C_1 \sim C$ hold, then $B_1 \cap C_1 \sim B \cap C$ need not hold. In other words, $[U_1(\omega) \cap U_2(\omega)]$ need not be definable by $[U_1(\omega)]$ and $[U_2(\omega)]$. The following definition will restrict our consideration to the particular case when such a relation can be defined.

Definition 4.1. An equivalence relation $\sim$ on $\mathcal{P}(S)$ is called *conservative with respect to a binary set operation* φ on S, if for each A, B, A_1, $B_1 \subset S$ the implication

$$(A \sim A_1)\,\&\,(B \sim B_2) \Longrightarrow \varphi(A, B) \sim \varphi(A_1, B_1) \tag{4.15}$$

holds. The relation $\sim$ is *conservative with respect to a unary set operation* χ on S, if for each A, $A_1 \subset S$ the implication

$$A \sim A_1 \Longrightarrow \chi(A) \sim \chi(A_1) \tag{4.16}$$

holds.

Let the equivalence relation $\sim$ be conservative with respect to a binary set operation φ on S. Then its *extension* to $\mathcal{P}(S)/\sim$ is the binary operation φ^0 defined, for each $[A]$, $[B] \in \mathcal{P}(S)/\sim$, by

$$\varphi^0([A], [B]) = [\varphi(A, B)]. \tag{4.17}$$

Let the equivalence relation $\sim$ be conservative with respect to a unary set operation χ on S. Then its *extension* to $\mathcal{P}(S)/\sim$ is the unary operation χ^0 defined, for each $[A] \in \mathcal{P}(S)/\sim$, by

$$\chi^0([A]) = [\chi(A)]. \tag{4.18}$$

The relations (4.15) and (4.16) yield immediately that the definitions (4.17) and (4.18) are correct in the sense that the classes $\varphi^0([A],[B])$ and $\chi^0([A])$ in $\mathcal{P}(S)/\sim$ are defined uniquely no matter how the representatives A and B of the classes $[A]$ and $[B]$ are chosen. If φ is the operation $\cup$ of set-theoretic union (operation $\cap$ of set-theoretic joint, resp.), we shall write $[A]\sqcup[B]$ and $[A]\sqcap[B]$ instead of $\bigcup^0([A],[B])$ and $\bigcap^0([A],[B])$. Let us recall that $[A]\cap[B]=[A]=[B]$, if $B\sim A$, $[A]\cap[B]=\emptyset$ otherwise.

The following trivial assertion shows that Definition 4.1 is non-trivial, i.e., non-empty.

Lemma 4.1. The identity relation $=$ on $\mathcal{P}(S)$ is conservative with respect to each extensional binary and unary set operation on S.

Proof. Let φ (χ, resp.) be a binary (unary, resp.) operation on S, let A, B, A_1, $B_1 \subset S$ be such that $A = A_1$ and $B = B_1$. The extensionality of the operations on S then yields that $\varphi(A,B)=\varphi(A_1,B_1)$ and $\chi(A)=\chi(A_1)$ hold. □

Theorem 4.1. Let $\langle\Omega,\tilde{\mathcal{A}},P\rangle$ be an abstract probability space, let $\langle E_i,\mathcal{E}_i\rangle$, $i=1,2$, be two measurable spaces of empirical values, let X_i, $i=1,2$, be a random variable taking $\langle\Omega,\tilde{\mathcal{A}},P\rangle$ into $\langle E_i,\mathcal{E}_i\rangle$. Let S be a finite set, let ρ_i, $i=1,2$, be a complete compatibility relation over $S\times E_i$, let $U_i(\omega)=U(X_i(\omega))=\{s\in S:\rho(s,X_i(\omega))=1\}$ for both $i=1,2$. Let σ be an incomplete identification relation on $\mathcal{P}(S)\times\mathcal{P}(\mathcal{P}(S))$ such that the conditions (a)–(e) introduced in Chapter 3 hold for $i=1,2$. Let $m_i(\mathcal{A})=P(\{\omega\in\Omega:[U_i(\omega)]=\mathcal{A}\})$ for each $\mathcal{A}\in\mathcal{P}(S)/\sim$ and for both $i=1,2$, let $\mathrm{Bel}^*_{\mathcal{U}_i}$ be defined by (2.2) for $i=1,2$ with ρ and X replaced by ρ_i and X_i, let $\mathrm{Bel}^*_{\mathcal{U}_1}\oplus^*\mathrm{Bel}^*_{\mathcal{U}_2}$ be defined by (4.13). Let the random variables $[U_1(\cdot)]$ and $[U_2(\cdot)]$, defined on the probability space $\langle\Omega,\tilde{\mathcal{A}},P\rangle$ and taking their values in the measurable spaces $\langle\mathcal{P}(S)/\sim,\ \mathcal{P}(\mathcal{P}(S)/\sim)\rangle$, be statistically independent, i.e., let the equality

$$\begin{aligned} & P\left(\{\omega\in\Omega:[U_1(\omega)]=\mathcal{A},\ [U_2(\omega)]=\mathcal{B}\}\right)= \\ = \; & P\left(\{\omega\in\Omega:[U_1(\omega)]=\mathcal{A}\}\right)\,P\left(\{\omega\in\Omega:[U_2(\omega)]=\mathcal{B}\}\right) \end{aligned} \tag{4.19}$$

hold for each $\mathcal{A}$, $\mathcal{B}\in\mathcal{P}(S)/\sim$. Let the equivalence relation $\sim$ on $\mathcal{P}(S)$ be conservative with respect to the set operation of joint on S. Then, for each $T\subset S$,

$$\left(\mathrm{Bel}^*_{\mathcal{U}_1}\oplus^*\mathrm{Bel}^*_{\mathcal{U}_2}\right)(T)=\frac{\sum_{\mathcal{A},\mathcal{B}\in\mathcal{P}(S)/\sim,\,\mathcal{A}\sqcap\mathcal{B}\neq[\emptyset],\,\mathcal{A}\sqcap\mathcal{B}\subset\mathcal{P}(T)}m_1(\mathcal{A})\,m_2(\mathcal{B})}{\sum_{\mathcal{A},\mathcal{B}\in\mathcal{P}(S)/\sim,\,\mathcal{A}\sqcap\mathcal{B}\neq[\emptyset]}m_1(\mathcal{A})\,m_2(\mathcal{B})} \tag{4.20}$$

supposing that $\sum_{\mathcal{A},\mathcal{B}\in\mathcal{P}(S)/\sim,\,\mathcal{A}\sqcap\mathcal{B}\neq[\emptyset]}m_1(\mathcal{A})\,m_2(\mathcal{B})>0$ holds, otherwise $(\mathrm{Bel}^*_{\mathcal{U}_1}\oplus^*\mathrm{Bel}^*_{\mathcal{U}_2})(T)$ is not defined.

Proof. As the compatibility relations ρ_1,ρ_2 are complete, $\mathcal{U}_i=\langle U_i,U_i\rangle=\langle U_i^0,U_i^0\rangle$ for both $i=1,2$. An easy calculation yields that

$$\left(\mathrm{Bel}^*_{\mathcal{U}_1}\oplus^*\mathrm{Bel}^*_{\mathcal{U}_2}\right)=\mathrm{Bel}^*_{\mathcal{U}_1\sqcap\mathcal{U}_2}(T)= \tag{4.21}$$

$$= P(\{\omega \in \Omega : \sigma(U(X_1(\omega)) \cap U(X_2(\omega)), \mathcal{P}(T)) = 1\} /$$
$$/ \{\omega \in \Omega : \sigma(U(X_1(\omega)) \cap U(X_2(\omega)), \mathcal{P}(S) - \{\emptyset\}) = 1\}) =$$
$$= P(\{\omega \in \Omega : [U_1(\omega) \cap U_2(\omega)] \subset \mathcal{P}(T)\} / \{\omega \in \Omega : [U_1(\omega) \cap U_2(\omega)] \neq [\emptyset]\}) =$$
$$= P(\{\omega \in \Omega : [U_1(\omega)] \sqcap [U_2(\omega)] \subset \mathcal{P}(T)\} / \{\omega \in \Omega : [U_1(\omega)] \sqcap [U_2(\omega)] \neq [\emptyset]\}) =$$
$$= \frac{P(\{\omega \in \Omega : [\emptyset] \neq [U_1(\omega)] \sqcap [U_2(\omega)] \subset \mathcal{P}(T)\})}{P(\{\omega \in \Omega : [\emptyset] \neq [U_1(\omega)] \sqcap [U_2(\omega)]\})} =$$
$$= \frac{\sum_{\mathcal{A},\mathcal{B} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \sqcap \mathcal{B} \neq [\emptyset],\, \mathcal{A} \sqcap \mathcal{B} \subset \mathcal{P}(T)} P(\{\omega \in \Omega : [U_1(\omega)] = \mathcal{A},\, [U_2(\omega)] = \mathcal{B}\})}{\sum_{\mathcal{A},\mathcal{B} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \sqcap \mathcal{B} \neq [\emptyset]} P(\{\omega \in \Omega : [U_1(\omega)] = \mathcal{A},\, [U_2(\omega)] = \mathcal{B}\})} =$$
$$= \frac{\sum_{\mathcal{A},\mathcal{B} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \sqcap \mathcal{B} \neq [\emptyset],\, \mathcal{A} \sqcap \mathcal{B} \subset \mathcal{P}(T)} P(\{\omega \in \Omega : [U_1(\omega)] = \mathcal{A}\}) P(\{\omega \in \Omega : [U_2(\omega)] = \mathcal{B}\})}{\sum_{\mathcal{A},\mathcal{B} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \sqcap \mathcal{B} \neq [\emptyset]} P(\{\omega \in \Omega : [U_1(\omega) = \mathcal{A}\}) P(\{\omega \in \Omega : [U_2(\omega)] = \mathcal{B}\})} =$$
$$= \frac{\sum_{\mathcal{A},\mathcal{B} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \sqcap \mathcal{B} \neq [\emptyset],\, \mathcal{A} \sqcap \mathcal{B} \subset \mathcal{P}(T)} m_1(\mathcal{A})\, m_2(\mathcal{A})}{\sum_{\mathcal{A},\mathcal{B} \in \mathcal{P}(S)/\sim,\, \mathcal{A} \sqcap \mathcal{B} \neq [\emptyset]} m_1(\mathcal{A})\, m_2(\mathcal{B})}.$$

The assertion is proved. □

When $\sim$ is the identity relation Id on $\mathcal{P}(S)$, i. e. when $[A] = \{A\}$ for each $A \subset S$ and $\mathcal{P}(S)/\mathrm{Id} = \{\{A\} : A \subset S\}$, an easy verification yields that, defining $m_i^0(A)$ by $m_i(\{A\})$ for each $A \subset S$ and for both $i = 1, 2$, we obtain

$$\left(\mathrm{Bel}^*_{\mathcal{U}_1} \oplus^* \mathrm{Bel}^*_{\mathcal{U}_2}\right)(T) = \tag{4.22}$$
$$\frac{\sum_{\{A\},\{B\} \in \mathcal{P}(S)/\mathrm{Id},\, \{A\} \sqcap \{B\} \neq \{\emptyset\},\, \{A\} \sqcap \{B\} \subset \mathcal{P}(T)} m_1(\{A\})\, m_2(\{B\})}{\sum_{\{A\},\{B\} \in \mathcal{P}(S)/\mathrm{Id},\, \{A\} \sqcap \{B\} \neq \{\emptyset\}} m_1(\{A\})\, m_2(\{B\})}$$
$$\frac{\sum_{A,B \subset S,\, A \cap B \neq \emptyset,\, A \cap B \subset T} m_1^0(A)\, m_2^0(B)}{\sum_{A,B \subset S,\, A \cap B \neq \emptyset} m_1^0(A)\, m_2^0(B)} = \left(\mathrm{Bel}_{U_1} \oplus \mathrm{Bel}_{U_2}\right)(T),$$

where Bel_{U_1} and Bel_{U_2} are usual believeability functions defined on $\mathcal{P}(S)$ by the usual basic probability assignments m_1^0 and m_2^0.

5. AN ABSTRACT MODEL FOR INCOMPLETELY IDENTIFIABLE SETS OF COMPATIBLE STATES

During our introductory explanation of basic ideas of D.-S. theory we started from an extra-mathematical interpretation and motivation for the presented ideas and notions, just at a secondary level mentioning the possibility of a purely abstract presentation of this theory. Also when investigating the abilities how to process the case with incompletely identifiable sets of compatible states we used a rather intuitive approach based, e. g., on the idea of indistinguishability of sets within the same equivalence class generated on $\mathcal{P}(S)$. An alternative, purely abstract and formalized model may be settled as follows.

For the sake of simplicity we shall suppose, again, that the basic space S is finite so that all the probabilities under consideration will be definable by appropriate basic probability assignments. Let $\nu : \mathcal{P}(S) \to \mathcal{P}(\mathcal{P}(S))$ be the operation of neighbourhood ascribing to each $A \subset S$ a system $\nu(A) \subset \mathcal{P}(S)$ of subsets of S in such a way that $A \in \nu(A)$ holds for each $A \subset S$. The interpretation behind may be that if $B \in \nu(A)$, then A and B are so close to each other that they are indistinguishable from each other in the sense that, for each $\mathcal{A} \subset \mathcal{P}(S)$, $A \in \mathcal{A}$ can be proved iff $B \in \mathcal{A}$ holds for each $B \in \nu(A)$, hence, iff $\nu(A) \subset \mathcal{A}$ holds. Consequently, $\nu(A)$ immediately generalizes the equivalence class $[A]$ introduced and investigated above. Let $X = \langle \Omega, \tilde{\mathcal{A}}, P \rangle \to \langle E, \mathcal{E} \rangle$ be a random variable the values of which are the empirical results being to the subject's disposal, let ρ be a complete compatibility relation on $S \times E$, let $U(X(\omega)) = U(\omega) = \{s \in S : \rho(X(\omega)) = 1\}$ be defined as above. Then the *ν-induced generalized believeability function* Bel_U^ν is defined, on $\mathcal{P}(S)$, by

$$\mathrm{Bel}_U^\nu(T) = P\left(\{\omega \in \Omega : \nu(U(X(\omega))) \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \nu(U(X(\omega)))\}\right) \quad (5.1)$$

for each $T \subset S$ supposing that $P(\{\omega \in \Omega : \emptyset \notin \nu(U(X(\omega)))\} > 0$ holds, Bel_U^ν being undefined otherwise. Defining $m^* : \mathcal{P}(\mathcal{P}(S)) \to \langle 0, 1 \rangle$ by

$$m^*(\mathcal{A}) = P\left(\{\omega \in \Omega : \nu(U(X(\omega))) = \mathcal{A}\}\right) \quad (5.2)$$

for each $\mathcal{A} \subset \mathcal{P}(S)$, we easily obtain that

$$\mathrm{Bel}_U^\nu(T) = \frac{\sum_{\mathcal{A} \in \mathcal{P}(\mathcal{P}(S)), \emptyset \notin \mathcal{A}, \mathcal{A} \subset \mathcal{P}(T)} m^*(\mathcal{A})}{\sum_{\mathcal{A} \in \mathcal{P}(\mathcal{P}(S)), \emptyset \notin \mathcal{A}} m^*(\mathcal{A})}. \quad (5.3)$$

Obviously, instead of the random variable $\nu(U(X(\cdot)))$, defined on $\langle \Omega, \tilde{\mathcal{A}}, P \rangle$ and taking its values in $\mathcal{P}(\mathcal{P}(S))$, we can take the basic probability assignment defined on $\mathcal{P}(\mathcal{P}(S))$ as the keystone of all further considerations and constructions.

Let us recall that if $\nu(A) = [A]$ for some equivalence relation $\sim$, then $\nu(B) = \nu(A)$ for each $B \in [A]$, i.e. for each $B \sim A$, however, in general the implication $B \in \nu(A) \Rightarrow \nu(B) = \nu(A)$ need not hold.

The following theorem proves that if the case of inconsistent data is strictly separable from all other cases, then each ν-induced generalized believeability function Bel_U^ν can be defined by the usual believeability function $\mathrm{Bel}_{\overline{\overline{U}}}$ for an appropriate set-valued random variable $\overline{\overline{U}} : \langle \Omega, \tilde{\mathcal{A}}, P \rangle \longrightarrow \langle \mathcal{P}(S), \mathcal{P}(\mathcal{P}(S)) \rangle$. Set $\overline{\mathcal{A}} = \bigcup_{B \in \mathcal{A}} B$ for each nonempty $\mathcal{A} \subset \mathcal{P}(S)$, set $\overline{\overline{A}} = \overline{\nu(A)}$ for each $A \subset S$, set $\overline{\emptyset}^* = \emptyset$ for the empty subset ϕ^* of $\mathcal{P}(S)$.

Theorem 5.1. Let S finite, $\langle \Omega, \tilde{\mathcal{A}}, P \rangle$, $\langle E, \mathcal{E} \rangle$, $X : \langle \Omega, \tilde{\mathcal{A}}, P \rangle \longrightarrow \langle E, \mathcal{E} \rangle$, and $U : E \to \mathcal{P}(S)$ be as above, let $\nu : \mathcal{P}(S) \to \mathcal{P}(\mathcal{P}(S))$ be such that $A \in \nu(A)$ holds for each $A \subset S$, $\nu(\emptyset) = \{\emptyset\}$, and $\emptyset \notin \nu(A)$ for each $A \neq \emptyset$, $A \subset S$. Then

$$\mathrm{Bel}_U^\nu(T) = \mathrm{Bel}_{\overline{\overline{U}}}(T) = P\left(\{\omega \in \Omega : \overline{\overline{U}}(\omega) \subset T\} \,/\, \{\omega \in \Omega : \overline{\overline{U}}(\omega) \neq \emptyset\}\right) \quad (5.4)$$

holds for each $T \subset S$ supposing that $P(\{\omega \in \Omega : \overline{\overline{U}}(\omega) \neq \emptyset\}) > 0$. If $m^* : \mathcal{P}(\mathcal{P}(S)) \to \langle 0, 1 \rangle$ for each $\mathcal{A} \subset \mathcal{P}(S)$, and if $m : \mathcal{P}(S) \to \langle 0, 1 \rangle$ is defined by

$$m(A) = \sum_{\mathcal{A} \subset \mathcal{P}(S), \overline{\mathcal{A}} = A} m^*(\mathcal{A}), \quad (5.5)$$

then $m : \mathcal{P}(S) \to \langle 0, 1\rangle$ is a basic probability assignment on S and

$$\mathrm{Bel}_U^{\nu}(T) = \frac{\sum_{A\subset S,\, A\neq\emptyset,\, A\subset T} m(A)}{\sum_{A\subset S,\, A\neq\emptyset} m(A)} \tag{5.6}$$

for each $T \subset S$ supposing that this value is defined.

Proof. The conditions imposed to ν yield that if $\emptyset \notin \nu(U(X(\omega))$, then $U(X(\omega)) = U(\omega) \neq \emptyset$, hence, as $U(\omega) \in \nu(U(\omega))$ and $U(\omega) \subset \overline{\overline{U}}(\omega)$, also $\overline{\overline{U}}(\omega) \neq \emptyset$. As far as the inverse implication is concerned, if $\overline{\overline{U}}(\omega) \neq \emptyset$, then there exists $\emptyset \neq A \in \nu(U(\omega))$, so that $\nu(U(\omega)) \neq \{\emptyset\}$. Consequently, $\nu(U(\omega)) \neq \nu(\emptyset)$ and $\emptyset \notin \nu(U(\omega))$ hold. Moreover, $\nu(U(\omega)) \subset \mathcal{P}(T)$ is valid iff $A \in \mathcal{P}(T)$ is valid for each $A \in \nu(U(\omega))$, but this relation holds iff $\bigcup_{A\in\nu(U(\omega))} A = \overline{\nu(U(\omega))} = \overline{\overline{U}}(\omega) \subset T$. An easy calculation then yields, using (5.1), that for each $T \subset S$,

$$\begin{aligned}\mathrm{Bel}_U^{\nu}(T) &= P\left(\{\omega\in\Omega : \nu(U(\omega)) \subset \mathcal{P}(T)\} \,/\, \{\omega\in\Omega : \emptyset \notin \nu(U(X(\omega)))\}\right) = \\ &= P\left(\{\omega\in\Omega : \overline{\overline{U}}(X(\omega)) \subset T\} \,/\, \{\omega\in\Omega : \overline{\overline{U}}(X(\omega)) \neq \emptyset\}\right) = \mathrm{Bel}_{\overline{\overline{U}}}(T)\end{aligned} \tag{5.7}$$

whenever this conditional probability is defined.

If $m^*(\mathcal{A}) = P(\{\omega\in\Omega : \nu(U(X(\omega))) = \mathcal{A}\})$ for each $\mathcal{A} \subset \mathcal{P}(S)$, then m is evidently a basic probability assignment on S. Moreover, by (5.3),

$$\begin{aligned}\mathrm{Bel}_U^{\nu}(T) &= \frac{\sum_{\mathcal{A}\in\mathcal{P}(\mathcal{P}(S)),\,\{\emptyset\}\neq\mathcal{A},\,\mathcal{A}\subset\mathcal{P}(T)} m^*(\mathcal{A})}{\sum_{\mathcal{A}\in\mathcal{P}(\mathcal{P}(S)),\,\{\emptyset\}\neq\mathcal{A}} m^*(\mathcal{A})} = \\ &= \frac{\sum_{A\in\mathcal{P}(S),\,\emptyset\notin\nu(A),\,A\subset T} \left(\sum_{\mathcal{A}\in\mathcal{P}(\mathcal{P}(S)),\overline{\mathcal{A}}=A} m^*(\mathcal{A})\right)}{\sum_{A\in\mathcal{P}(S),\,\emptyset\notin\nu(A)} \left(\sum_{\mathcal{A}\in\mathcal{P}(\mathcal{P}(S)),\overline{\mathcal{A}}=A} m^*(\mathcal{A})\right)} = \frac{\sum_{A\subset S,\,\emptyset\neq A,\,A\subset T} m(A)}{\sum_{A\subset S,\,\emptyset\neq A} m(A)},\end{aligned} \tag{5.8}$$

as $m^*(\mathcal{A}) \neq \emptyset$ may hold only if $\mathcal{A} = \nu(A)$ for some $A \subset S$, and for each $A \subset S$ the summation condition $A \subset T$ is equivalent to $\nu(A) \subset \mathcal{P}(T)$. The theorem is proved. □

An immediate corollary of Theorem 5.1 reads that under the conditions of this theorem the inequality $\mathrm{Bel}_U^{\nu}(T) \leq \mathrm{Bel}_U^{\mathrm{Id}}(T) = \mathrm{Bel}_U(T)$ holds for each T whenever the values are defined, more generally, $\mathrm{Bel}_U^{\nu_1}(T) \leq \mathrm{Bel}_U^{\nu_2}(T)$ holds for each T supposing that ν_1, ν_2 satisfy the demands of Theorem 5.1 and $\nu_1(A) \supset \nu_2(A)$ is valid for each $A \subset S$. The assertion immediately follows from the obvious fact that $U(\omega) = \overline{\overline{U}}^{\mathrm{Id}}(\omega) \subset \overline{\overline{U}}^{\nu_2}(\omega) \subset \overline{\overline{U}}^{\nu_1}(\omega)$ holds for ν_1, ν_2 in question and for all $\omega \in \Omega$, here $\nu = \mathrm{Id}$ means that $\nu(A) = \{A\}$ for each $A \subset S$. Hence, $\mathrm{Bel}_U^{\nu}(T)$ can be interpreted and used as a lower approximation of $\mathrm{Bel}_U(T)$, consequently, if we have to decide whether $\mathrm{Bel}_U(T) \geq \alpha$ holds for some threshold value α, we may arrive at the positive answer supposing that we prove the inequality $\mathrm{Bel}_U^{\nu}(T) \geq \alpha$ for some neighbouring operation ν satisfying the conditions of Theorem 5.1. For $\nu = \mathrm{Id}$ these conditions are obviously fulfilled.

It may be a matter of interest to define and briefly discuss the Dempster combination rule for the case of ν-induced generalized believeability functions $\mathrm{Bel}_{U_1}^{\nu}$, $\mathrm{Bel}_{U_2}^{\nu}$.

From an apriori point of view at least the three following combination rules $\oplus^1, \oplus^2, \oplus^3$ should be considered; we write $U_i(\omega)$ for $U(X_i(\omega))$ and for both $i = 1, 2$.

$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^1 \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \tag{5.9}$$
$$= P\left(\{\omega \in \Omega : \nu(U_1(\omega) \cap U_2(\omega)) \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \nu(U_1(\omega) \cap U_2(\omega))\}\right),$$
$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^2 \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \tag{5.10}$$
$$= P\left(\{\omega \in \Omega : \nu(U_1(\omega)) \cap \nu(U_2(\omega)) \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \nu(U_1(\omega)) \cap \nu(U_2(\omega))\}\right),$$
$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^3 \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \tag{5.11}$$
$$= P\left(\{\omega \in \Omega : \nu(U_1(\omega)) \sqcap \nu(U_2(\omega)) \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))\}\right),$$

where $\mathcal{A} \sqcap \mathcal{B} = \{A \cap B : A \in \mathcal{A}, B \in \mathcal{B}\}$ for $\mathcal{A}, \mathcal{B} \subset \mathcal{P}(S)$. Of course, all the three functions are defined only when the corresponding conditional probabilities are defined, i.e., when the apriori probabilities of the conditioning random events in question are positive. The following simple assertion shows, whether these rules are compatible with the usual Dempster combination rule in some simple particular cases; an eventual incompatibility should eliminate the corresponding combination rule from the scale of possible candidates to the role of Dempster combination rule for ν-induced generalized believeability functions.

Theorem 5.2. **(a)** Let the notations of Theorem 5.1 hold, let $A \in \nu(A)$ hold for each $A \subset S$, let the random variables X_1, X_2 be equivalent in the sense that for all $\omega \in \Omega$ the equality

$$U_1(\omega) = U(X_1(\omega)) = U_2(\omega) = U(X_2(\omega)) \tag{5.12}$$

hold. Then

$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^j \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \mathrm{Bel}^{\nu}_{U_1}(T) = \mathrm{Bel}^{\nu}_{U_2}(T) \tag{5.13}$$

holds for all $T \subset S$ (supposing that the values are defined), if $j = 1$ or $j = 2$, but not, in general, for $j = 3$.

(b) Under the same notations and for $\nu = \mathrm{Id}$, the equality

$$\left(\mathrm{Bel}^{\mathrm{Id}}_{U_1} \oplus^j \mathrm{Bel}^{\mathrm{Id}}_{U_2}\right)(T) = \left(\mathrm{Bel}_{U_1} \oplus \mathrm{Bel}_{U_2}\right)(T) \tag{5.14}$$

holds for all $T \subset S$ (supposing that the values are defined), if $j = 1$ or $j = 3$, but not, in general, for $j = 2$.

Proof. (a) If $U_1(\omega) = U_2(\omega)$ for all $\omega \in \Omega$, then $\nu(U_1(\omega)) = \nu(U_2(\omega))$, so that $U_1(\omega) \cap U_2(\omega) = U_1(\omega) = U_2(\omega)$ and $\nu(U_1(\omega) \cap U_2(\omega)) = \nu(U_1(\omega)) = \nu(U_2(\omega))$. So, the definitions (5.9) and (5.10) immediately imply that (5.13) holds for each $T \subset S$ and for both $j = 1, 2$.

For $j = 3$, we have to take into consideration the subset $\nu(U_1(\omega)) \sqcap \nu(U_2(\omega))$ of $\mathcal{P}(S)$, i.e., for $U_1(\omega) = U_2(\omega)$, the subset $\nu(U_1(\omega)) \sqcap \nu(U_1(\omega)) = \{A \cap B : A, B \in \nu(U_1(\omega))\}$ of $\mathcal{P}(S)$. Let $\nu(U_1(\omega)) \subset \mathcal{P}(T)$. Then, for each $A, B \in \nu(U_1(\omega))$, we have $A, B \in \mathcal{P}(T)$, i.e., $A, B \subset T$, so that $A \cap B \subset T$ and $A \cap B \in \mathcal{P}(T)$ also hold, consequently, $\nu(U_1(\omega)) \sqcap \nu(U_1(\omega)) \subset \mathcal{P}(T)$ is valid. Let $\nu(U_1(\omega)) \sqcap \nu(U_1(\omega)) \subset \mathcal{P}(T)$.

Then $\{A \cap B : A,\, B \in \nu(U_1(\omega))\} \subset \mathcal{P}(T)$, so that also $\{A \cap A : A \in \nu(U_1(\omega))\} = \nu(U_1(\omega)) \subset \mathcal{P}(T)$ holds. So, $\nu(U_1(\omega)) \subset \mathcal{P}(T)$ iff $\nu(U_1(\omega)) \sqcap \nu(U_1(\omega)) \subset \mathcal{P}(T)$, and analogously for U_2, however, this equivalence is not sufficient to assure the validity of (5.13) for $j = 3$, as the following counterexample proves.

Let $\nu(A) = \{A, S - A\}$ for all $A \subset S$, let $\emptyset \neq U_i(\omega) \neq S$ hold for all $\omega \in \Omega$ and for both $i = 1, 2$. Then, for both $i = 1, 2$,

$$\begin{aligned}\mathrm{Bel}^{\nu}_{U_i}(T) &= P\left(\{\omega \in \Omega : \nu(U_i(\omega)) \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \nu(U_i(\omega))\}\right) = \\ &= P\left(\{\omega \in \Omega : \{U_i(\omega),\, S - U_i(\omega)\} \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \{U_i(\omega),\, S - U_i(\omega)\}\}\right).\end{aligned} \tag{5.15}$$

Hence, $\mathrm{Bel}^{\nu}_{U_i}(T) = 1$, if $T = S$, and $\mathrm{Bel}^{\nu}_{U_i}(T) = 0$ for $T \neq S$. Combining $\mathrm{Bel}^{\nu}_{U_1}$ and $\mathrm{Bel}^{\nu}_{U_2}$ by the operation $\oplus^3$ we obtain, for $U_1(\omega) = U_2(\omega)$, that

$$\begin{aligned}&\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^3 \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \\ &= P\left(\{\omega \in \Omega : \nu(U_1(\omega)) \sqcap \nu(U_2(\omega)) \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \nu(U_1(\omega)) \sqcap \nu(U_1(\omega))\}\right).\end{aligned} \tag{5.16}$$

But,

$$\begin{aligned}&\nu(U_1(\omega)) \sqcap \nu(U_2(\omega)) = \\ &= \{A \cap B : A,\, B \in \nu(U_1(\omega))\} = \{\emptyset, U_1(\omega), S - U_1(\omega)\},\end{aligned} \tag{5.17}$$

so that

$$P\left(\{\omega \in \Omega : \emptyset \notin \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))\}\right) = 0. \tag{5.18}$$

Consequently, $(\mathrm{Bel}^{\nu}_{U_1} \oplus \mathrm{Bel}^{\nu}_{U_2})(T)$ is not defined for no matter which $T \subset S$, hence, (5.13) does not hold for $j = 3$. The assertion (a) is proved.

(b) Let $\nu = \mathrm{Id}$. Two easy calculations yield that

$$\begin{aligned}&\left(\mathrm{Bel}^{\mathrm{Id}}_{U_1} \oplus^1 \mathrm{Bel}^{\mathrm{Id}}_{U_2}\right)(T) = \\ &= P\left(\{\omega \in \Omega : \{U_1(\omega) \cap U_2(\omega)\} \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \{U_1(\omega) \cap U_2(\omega)\}\}\right) = \\ &= P\left(\{\omega \in \Omega : U_1(\omega) \cap U_2(\omega) \subset T\} \,/\, \{\omega \in \Omega : \emptyset \neq U_1(\omega) \cap U_2(\omega)\}\right) \\ &= (\mathrm{Bel}_{U_1} \oplus \mathrm{Bel}_{U_2})(T),\end{aligned} \tag{5.19}$$

$$\begin{aligned}&\left(\mathrm{Bel}^{\mathrm{Id}}_{U_1} \oplus^3 \mathrm{Bel}^{\mathrm{Id}}_{U_2}\right)(T) = \\ &= P\left(\{\omega \in \Omega : \{U_1(\omega)\} \sqcap \{U_2(\omega)\} \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \{U_1(\omega)\} \sqcap \{U_2(\omega)\}\}\right) = \\ &= P\left(\{\omega \in \Omega : \{U_1(\omega) \cap U_2(\omega)\} \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \{U_1(\omega) \cap U_2(\omega)\}\}\right) = \\ &= \left(\mathrm{Bel}^{\mathrm{Id}}_{U_1} \oplus^1 \mathrm{Bel}^{\mathrm{Id}}_{U_2}\right)(T) = (\mathrm{Bel}_{U_1} \oplus \mathrm{Bel}_{U_2})(T)\end{aligned} \tag{5.20}$$

due to (5.19), as the equality $\{A\} \sqcap \{B\} = \{A \cap B\}$ trivially holds for each $A,\, B \subset S$. So, (5.14) is proved for $j = 1$ and $j = 3$.

Let U_1, U_2 be such that $\emptyset \neq U_1(\omega) \cap U_2(\omega)$ and $U_1(\omega) \neq U_2(\omega)$ hold for each $\omega \in \Omega$, let $\emptyset^*$ denote the empty subset of $\mathcal{P}(T)$ to distinguish it from the empty subset $\emptyset$ of S. Then

$$\begin{aligned}&(\mathrm{Bel}_{U_1} \oplus \mathrm{Bel}_{U_2})(\emptyset) = \\ &= P\left(\{\omega \in \Omega : U_1(\omega) \cap U_2(\omega) \subset \emptyset\} \,/\, \{\omega \in \Omega : U_1(\omega) \cap U_2(\omega) \neq \emptyset\}\right) = 0,\end{aligned} \tag{5.21}$$

but

$$\left(\mathrm{Bel}^{\mathrm{Id}}_{U_1} \oplus^2 \mathrm{Bel}^{\mathrm{Id}}_{U_2}\right)(T) = \tag{5.22}$$
$$= P\left(\{\omega \in \Omega : \{U_1(\omega)\} \cap \{U_2(\omega)\} \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \{U_1(\omega)\} \cap \{U_2(\omega)\}\}\right) =$$
$$= P\left(\{\omega \in \Omega : \emptyset^* \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \emptyset^*\}\right) = 1,$$

as $U_1(\omega) \neq U_2(\omega)$ for all $\omega \in \Omega$ implies that $\{U_1(\omega)\} \cap \{U_2(\omega)\} = \emptyset^*$ and the relations $\emptyset^* \subset \mathcal{P}(T)$, $\emptyset \notin \emptyset^*$ are obviously valid. Hence, (5.14) does not hold for $j = 2$ and the proof of Theorem 5.2 is completed. □

A natural, interesting, and immediately arising question can be formulated as follows. Let the conditions of Theorem 5.1 hold for two random variables X_1, X_2, so that $\mathrm{Bel}^{\nu}_{U_i}(T) = \mathrm{Bel}_{\overline{\overline{U}}_i}(T)$ for both $i = 1, 2$. Applying the usual Dempster combination rule $\oplus$ to $\mathrm{Bel}_{\overline{\overline{U}}_1}$ and $\mathrm{Bel}_{\overline{\overline{U}}_2}$ we can ask, whether the identity

$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^j \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \left(\mathrm{Bel}_{\overline{\overline{U}}_1} \oplus \mathrm{Bel}_{\overline{\overline{U}}_2}\right)(T) \tag{5.23}$$

for all $T \subset S$ holds for some $j = 1, 2, 3$. In other terms, we can define a new Dempster combination rule $\oplus^4$, setting

$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^4 \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \left(\mathrm{Bel}_{\overline{\overline{U}}_1} \oplus \mathrm{Bel}_{\overline{\overline{U}}_2}\right)(T) \tag{5.24}$$

for all $T \subset S$ and supposing that the values are defined and that both the random variables X_1, X_2 and the neigbourhood mapping ν satisfy the conditions of Theorem 5.1, and we can ask whether $\oplus^4$ is identical with $\oplus^j$ for some $j = 1, 2, 3$, or, more generally, which are the relations between $\oplus^4$ and $\oplus^1$, $\oplus^2$, $\oplus^3$.

The identity $\oplus^1 = \oplus^4$, i.e.

$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^1 \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \left(\mathrm{Bel}_{\overline{\overline{U}}_1} \oplus \mathrm{Bel}_{\overline{\overline{U}}_2}\right)(T) \tag{5.25}$$

could hold only if $\overline{\overline{U_1(\omega) \cap U_2(\omega)}} = \overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega)$ held for all $\omega \in \Omega$, hence, only if $\overline{\overline{A \cap B}} = \overline{\overline{A}} \cap \overline{\overline{B}}$ held for all A, $B \subset S$. However, setting $\nu(A \cap B) = \{A \cap B, S\}$ for fixed A, $B \subset S$ such that $S \neq A \neq A \cap B \neq \emptyset$, $S \neq B \neq A \cap B$, and setting $\nu(C) = \{C\}$ for all other $C \subset S$, in particular also $\nu(A) = \{A\}$ and $\nu(B) = \{B\}$, we obtain that

$$\overline{\overline{A \cap B}} = \bigcup \{C : C \in \nu(A \cap B)\} = (A \cap B) \cup S = S \neq A \cap B = \tag{5.26}$$
$$= \left(\bigcup \{A_1 : A_1 \in \nu(A)\}\right) \cap \left(\bigcup \{B_1 : B_1 \in \nu(B)\}\right) = \overline{\overline{A}} \cap \overline{\overline{B}}.$$

Consequently, (5.25) cannot hold identically, so that $\oplus^4 \neq \oplus^1$.

A hypothetical identity $\oplus^2 = \oplus^4$ can be easily converted to the identity

$$\overline{\nu(U_1(\omega)) \cap \nu(U_2(\omega))} = \overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega). \tag{5.27}$$

However, for U_1 and U_2 such that $U_1(\omega) \neq U_2(\omega)$ and $\emptyset \neq U_1(\omega) \cap U_2(\omega)$ hold for each $\omega \in \Omega$, and for $\nu = \mathrm{Id}$ we easily obtain that $\nu(U_1(\omega)) \cap \nu(U_2(\omega)) = \{U_1(\omega)\} \cap$

$\{U_2(\omega)\} = \emptyset^*$, and $\overline{\overline{\emptyset^*}} = \emptyset$, but $\overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) = U_1(\omega) \cap U_2(\omega) \neq \emptyset$, so that (5.27) cannot hold identically, consequently, $\oplus^2 \not\equiv \oplus^4$.

The neighbourhood mapping $\nu : \mathcal{P}(S) \to \mathcal{P}(\mathcal{P}(S))$ is called *consistence preserving*, if for all A, $B \subset S$ the following implication holds: if $A \cap B = \emptyset$, then $A_1 \cap B_1 = \emptyset$ for all $A_1 \in \nu(A)$, $B_1 \in \nu(B)$. Consequently, if $A_1 \cap B_1 \neq \emptyset$ for some $A_1 \in \nu(A)$ and some $B_1 \in \nu(B)$, then $A \cap B \neq \emptyset$. If $\nu = \mathrm{Id}$, then ν is obviously consistence preserving.

Theorem 5.3. Let the notations and conditions of Theorem 5.1 hold, let ν be consistence preserving. Then $\oplus^3$ and $\oplus^4$ are identical, so that

$$\left(\mathrm{Bel}^{\nu}_{U_1} \oplus^3 \mathrm{Bel}^{\nu}_{U_2}\right)(T) = \left(\mathrm{Bel}_{\overline{\overline{U}}_1} \oplus \mathrm{Bel}_{\overline{\overline{U}}_2}\right)(T) \tag{5.28}$$

holds for each $T \subset S$ supposing that the believeability functions in question are defined.

Proof. Let A, B be arbitrary subsets of S, let $x \in \overline{\overline{A}} \cup \overline{\overline{B}} = (\bigcup\{A_1 : A_1 \in \nu(A)\}) \cap (\bigcup\{B_1 : B_1 \in \nu(B)\})$. Then there exists $A_1 \in \nu(A)$ and $B_1 \in \nu(B)$ such that $x \in A_1$ and $x \in B_2$ hold simultaneously, hence, $x \in A_1 \cap B_1$ and, consequently,

$$x \in \bigcup\{A_1 \cap B_1 \cdot A_1 \in \nu(A),\, B_1 \in \nu(B)\} = \overline{\nu(A) \sqcap \nu(B)}. \tag{5.29}$$

Let $x \in \overline{\nu(A) \sqcap \nu(B)}$. Then there exist $A_1 \in \nu(A)$, $B_1 \in \nu(B)$ such that $x \in A_1 \cap B_1$, hence, $x \in A_1$ and $x \in B_1$, so that $x \in \overline{\nu(A)} \cap \overline{\nu(B)} = \overline{\overline{A}} \cap \overline{\overline{B}}$. Applying this result to $U_1(\omega)$ and $U_2(\omega)$ we obtain that the equality $\overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) = \overline{\nu(U_1(\omega)) \sqcap \nu(U_2(\omega))}$ holds for each $\omega \in \Omega$.

Let $\emptyset \notin \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))$. Hence, for each $C \in \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))$, the inequality $C \neq 0$ holds and, as $U_1(\omega) \cap U_2(\omega) \in \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))$ is trivially valid due to the definition of the operation $\sqcap$, we may conclude that $U_1(\omega) \cap U_2(\omega) \neq \emptyset$. As $U_i(\omega) \subset \overline{\overline{U}}_i(\omega)$ holds trivially for both $i = 1, 2$, we obtain that $U_1(\omega) \cap U_2(\omega) \subset \overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega)$, hence, $\overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) \neq \emptyset$.

The inverse implication does not hold in general, but it can be proved under the condition that ν is consistence preserving as assumed above. Let $\overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) \neq \emptyset$, let $x \in S$ be such that

$$x \in \overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) = \left(\bigcup\{A : A \in \nu(U_1(\omega))\}\right) \cap \left(\bigcup\{B : B \in \nu(U_2(\omega))\}\right). \tag{5.30}$$

Then there exist $A \in \nu(U_1(\omega))$ and $B \in \nu(U_2(\omega))$ such that $x \in A$ and $x \in B$, hence, $x \in A \cap B$, so that $A \cap B \neq \emptyset$. As ν is consistence preserving, we obtain immediately that $U_1(\omega) \cap U_2(\omega) \neq \emptyset$. Using the other properties of ν assumed in Theorem 5.1 (hence, also in Theorem 5.3), we can conclude that $\emptyset \notin \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))$. Combining the obtained results we arrive at the conclusion that, under the conditions of Theorem 5.3, $\emptyset \notin \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))$ holds iff $\overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) \neq \emptyset$.

The inclusions

$$\overline{\nu(U_1(\omega)) \sqcap \nu(U_2(\omega))} \subset T, \tag{5.31}$$

i.e.
$$\bigcup \{A \cap B : A \in \nu(U_1(\omega)),\ B \in \nu(U_2(\omega))\} \subset T, \tag{5.32}$$

hold iff $A \cap B \subset T$ holds for each $A \in \nu(U_1(\omega))$, $B \in \nu(U_2(\omega))$, and this is equivalent to the case when $\{A \cap B : A \in \nu(U_1(\omega)),\ B \in \nu(U_2(\omega))\} \subset \mathcal{P}(T)$ holds, hence, to the case when $\nu(U_1(\omega)) \sqcap \nu(U_2(\omega)) \subset \mathcal{P}(T)$ holds. Using the proved equivalence, we can complete the proof of Theorem 5.3 by the following easy calculation.

$$\begin{aligned}
&\left(\mathrm{Bel}_{\overline{\overline{U}}_1} \oplus \mathrm{Bel}_{\overline{\overline{U}}_2}\right)(T) = \qquad (5.33)\\
&= P\left(\{\omega \in \Omega : \overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) \subset T\} \,/\, \{\omega \in \Omega : \overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) \neq \emptyset\}\right) = \\
&= P\left(\{\omega \in \Omega : \overline{\nu(U_1(\omega)) \sqcap \nu(U_2(\omega))} \subset T\} \,/\, \{\omega \in \Omega : \overline{\overline{U}}_1(\omega) \cap \overline{\overline{U}}_2(\omega) \neq \emptyset\}\right) = \\
&= P\left(\{\omega \in \Omega : \nu(U_1(\omega)) \sqcap \nu(U_2(\omega)) \subset \mathcal{P}(T)\} \,/\, \{\omega \in \Omega : \emptyset \notin \nu(U_1(\omega)) \sqcap \nu(U_2(\omega))\}\right) = \\
&= \left(\mathrm{Bel}^{\nu}_{U_1} \oplus^3 \mathrm{Bel}^{\nu}_{U_2}\right)(T). \qquad \square
\end{aligned}$$

6. CONCLUSIONS

Having developed a generalized definition for believeability functions in the case when the sets of compatible states are not completely identifiable in the sense that their membership in certain classes of sets cannot be always decided, we considered, in more detail, the case when this membership question can be positively answered only if also some other sets of states, close to the tested one in the sense of a neighbourhood relation or of an equivalence relation, are in the class of sets in question. We have proved that when the case of an eventual inconsistence of the input data is strictly and effectively separable from all other cases, the generalized model can be reduced to the classical one, simply considering every set of states inseparable from the actual set of compatible states also as a state compatible with the data being at our disposal. So, if $\overline{\nu_1(A)} = \overline{\nu_2(A)}$ for two neigbouring operations ν_1, ν_2 and for all $A \subset S$, in particular, if $\overline{[A]_1} = \overline{[A]_2}$ for two equivalence relations and for all $A \subset S$, the corresponding generalized believeability functions will be identical. In other words said, the abilities of believeability functions to quantify the uncertainty arising from our limited abilities to identify the set of compatible states are purely extensional and rather restricted. On the other side, when the case of inconsistent data cannot be strictly and effectively separated from all other cases, generalized believeability functions can play the role of a new tool to quantify the uncertainty in a Dempster–Shafer-like style also in this case. However, it is the user who must deduce which kind of definition of data inconsistence will be accepted; the problem of various definitions and interpretations of data inconsistence should deserve a more detailed investigation.

A possible and perhaps fruitful path how to achieve some results in this direction would be to abandon the assumption of *closed world* accepted in this paper, according to which S is the *exhaustive* list of *all* possible internal states of the investigated system (under the interpretation introduced in Chapter 1), in favour of the idea of *open world* when S is the list of *some, but not necessarily all*, such states (cf. [8]

and [9] for more details). Under this interpretation, the case when *no* state from S is compatible with the input data $X(\omega)$, i.e., the case when $U(X(\omega)) = \emptyset$, is understood in such a way that the data are true or at least consistent, but the actual internal state of the system in question is out of the set S of states. Consequently, it is not necessary to eliminate this case from our considerations as something undesirable and to re-normalize the probabilities used when believeability functions are defined. The only what is needed is to introduce a new state $s^* \notin S$ of the system, compriming all the possible states of the system not listed in S, and to conclude, if $U(X(\omega)) = \emptyset$, that the investigated system is in the state s^*. This approach seems to be hopeful from the point of view of mathematical technique, as we have seen, that the greatest part of technical problems with generalized believeability functions arise when defining and processing conditional probabilities resulting from such a re-normalization, however, the theoretical limits of this approach should be carefully investigated. Because of the limited extent of this paper we shall postpone such an investigation till another occasion.

Another possible and perhaps interesting way of generalization of the ideas and results presented above is to abandon the simplifying conditions that the set S of state is finite and that just the maximum σ-field $\mathcal{P}(\mathcal{P}(S))$ of subsets of $\mathcal{P}(S)$ is taken into consideration when the measurability of all the mappings taken their values in $\mathcal{P}(S)$ is defined. The necessary modifications resulting from such a generalization would not bring too much new from the basic philosophical and methodological point of view as applied above, but the technical apparatus would be so space and time consuming that it seems also better to investigate these matters separately.

Monographies [2], [5], and [6] in the list of references presented below can serve as introductory guides to the Dempster–Shafer theory of evidence.

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REFERENCES

[1] A. P. Dempster: Upper and lower probabilities induced by a multivalued mapping. Ann. Math. Statist. *38* (1967), 325–339.

[2] P. Hájek, T. Havránek and R. Jiroušek: Uncertain Information Processing in Expert Systems. CRC Press, California 1991.

[3] I. Kramosil: An intensional interpretation and generalization of Dempster–Shafer approach to uncertainty processing. In: Logica 93 – Proceedings of the 7th International Symposium (P. Kolář and V. Svoboda, eds.), Filosofia – Publishing House of the Institute of Philosophy of the Academy of Sciences of the Czech Republic, Prague 1994, pp. 105–122.

[4] E. L. Lehmann: Testing Statistical Hypotheses. Wiley, New York 1947.

[5] J. Pearl: Probabilistic Reasoning in Intelligent Systems – Networks of Plausible Inference. Morgan Kaufmann Publishers, Inc., San Matteo, California 1988.

[6] G. Shafer: A Mathematical Theory of Evidence. Princeton Univ. Press, Princeton, New Jersey 1976.

[7] P. Smets: Belief functions. In: Nonstandard Logics for Automated Reasoning (P. Smets, A. Mamdani, D. Dubois and H. Prade, eds.), Academic Press, London 1988, pp. 253–286.

[8] P. Smets: The nature of the unnormalized beliefs encountered in the transferable belief model. In: Uncertainty in AI 92 (D. Dubois, M. P. Wellman, B. d'Ambrosio and P. Smets, eds.), Morgan Kaufmann, San Matteo, California 1992, pp. 292–297.
[9] P. Smets: Belief functions: the disjunctive rule of combination and the generalized bayesian theorem. Internat. J. Approx. Reason. 7 (1993), 1–35.
[10] A. Wald: Sequential Analysis. Wiley, New York 1947.

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