Kybernetika

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Treshold logic unit optimization by linear programming

Threshold Logic Unit Optimization by Linear Programming

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Threshold logic unit is the main part of some perceptron-like systems for pattern recognition, whose training period ends by finding a weight vector satisfying certain system of inequalities. The determining of individual components of the weight vector can be accomplished by linear programming methods, which ensure — in case of linear separability of the given situation — finding the optimal weights which maximize admissible tolerances of both the threshold and the individual components of the weight vector. This method can be applied also to the pattern recognition for more categories, when characteristic vectors of individual categories are to be determined.

Simple perceptron and ADALINE are typical examples of pattern recognition learning systems based on the following principle: before the beginning of the training period finite sets of sample patterns of various categories are presented; during the training period adjustable weights of a threshold logic unit (TLU) are adjusted so that after the finishing of the training period (provided that the training ends) all sample patterns are correctly recognized.

If the situation is separable, i.e. if there exists such an adjustment of the adjustable weights of the TLU which provides correct classification of all sample patterns, there always exist more such solutions than one. Each of them will be called admissible solution or admissible weight vector. It is useful to compare the reliability of classification under threshold changes for various admissible weight vectors. An attempt will be made to find such a weight vector which maximizes the tolerance of the threshold value within which erroneous classification of sample patterns does not occur. The maximization will be done subject to the constraint that the modules of the adjustable weights of the TLU are not to exceed determined values.

Also the reliability of classification under changes of both threshold and weight vector components can be compared for various admissible weight vectors. Once more, an attempt will be made to find such a weight vector which maximizes the

tolerance of both the threshold value and the values of the weight vector components within which erroneous classification of sample patterns does not occur.

The importance of the investigation of this kind is evident mainly in the designing and realizing of perceptron-like systems, where the values of weights and threshold may differ from the computed ones as a consequence of the tolerances of the elements used in the system or as a consequence of the value changes of the elements in time.

Since every sample pattern defines unambiguously a point in an n-dimensional Euclidean space, speaking henceforth of sample patterns we always mean n-dimensional vectors or points in an n-dimensional space. Denoting parameters of the i-th sample pattern of the j-th category

$$^j z_i^1, \ldots, {}^j z_i^n,$$

we define sample patterns by column-vectors (T denotes transposition)

$$^j\mathbf{Z}_i = \left({}^j z_i^1, \ldots, {}^j z_i^n\right)^{\mathsf{T}}, \quad j = 1, \ldots, k, \ i = 1, \ldots, p_j.$$

First of all we shall investigate the recognition of patterns of two categories only, the generalization of the problem to the case of k categories will be done later.

PATTERN RECOGNITION FOR TWO CATEGORIES

Let

$$\left\{{}^1\mathbf{Z}_i\right\}_{i=1,\ldots,p_1} \quad \text{and} \quad \left\{{}^2\mathbf{Z}_i\right\}_{i=1,\ldots,p_2}$$

be two sets of sample patterns of categories σ_1 and σ_2. Denoting $w_1, \ldots, w_n$ the values of adjustable weights of the TLU with fixed threshold h, we define weight vector

$$\mathbf{W} = \left(w_1, \ldots, w_n\right)^{\mathsf{T}}.$$

The objective of the TLU training is to find weight vector $\mathbf{W}$ satisfying the following system of inequalities:

(1)
$$^1\mathbf{Z}_i^{\mathsf{T}}\mathbf{W} > h, \quad i = 1, \ldots, p_1,$$
$$^2\mathbf{Z}_i^{\mathsf{T}}\mathbf{W} < h, \quad i = 1, \ldots, p_2.$$

Introducing new denotation

$$\mathbf{A}_1 = \begin{bmatrix} {}^1\mathbf{Z}_1^{\mathsf{T}} \\ \vdots \\ {}^1\mathbf{Z}_{p_1}^{\mathsf{T}} \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} {}^2\mathbf{Z}_1^{\mathsf{T}} \\ \vdots \\ {}^2\mathbf{Z}_{p_2}^{\mathsf{T}} \end{bmatrix},$$

$$\mathbf{H} = h\mathbf{U}_{p_1}, \quad \mathbf{H}_2 = h\mathbf{U}_{p_2},$$

where $\mathbf{U}_m$ is the m-dimensional column-vector with all components equal to one,

(1) can be written in the following way:

(2)
$$A_1 W > H_1,$$
$$A_2 W < H_2.$$

Let Δh denote the change of the threshold value. Since correct recognition is required even under any change of the threshold value such that $|\Delta h| < \tau$ (for $\tau > 0$), (2) has to be replaced by the following system of inequalities:

(3)
$$A_1 W \geqq H_1 + \tau U_{p_1},$$
$$A_2 W \leqq H_2 - \tau U_{p_2}.$$

Since weight vector W is to satisfy (3), maximize τ and simultaneously have none of its components greater than a corresponding determined value

(4)
$$|w_i| \leqq d_i, \quad i = 1, \ldots, n,$$

it is useful to re-formulate the problem into a form more suitable for solving by linear programming. Since the components of the weight vector W can take both positive and negative values and since the linear programming method requires only nonnegative values of the solution vector, we introduce auxiliary vectors ^+W and ^-W with all components nonnegative so that

$$W = {}^+W - {}^-W.$$

It can be, therefore, required

$$0 \leqq {}^+w_i \leqq d_i,$$
$$0 \leqq {}^-w_i \leqq d_i, \quad i = 1, \ldots, n.$$

The method of linear programming further requires nonnegative constants on the right-hand sides of equations and inequalities. Starting from (3) and introducing

$$C_1 = \begin{bmatrix} A_1 & | & -A_1 & | & -U_{p_1} \end{bmatrix},$$
$$C_2 = \begin{bmatrix} A_2 & | & -A_2 & | & U_{p_2} \end{bmatrix},$$
$$G = ({}^+W^T, {}^-W^T, \tau)^T,$$

we obtain

$$C_1 G \geqq H_1,$$
$$C_2 G \leqq H_2.$$

This system of inequalities together with constraints (4) defines (under the condition that τ is maximized) the problem of linear programming, which can be solved by introducing slack variables and transforming thus inequalities into equations. Let

us denote the total number of sample patterns $p = p_1 + p_2$. We introduce $(2n + p)$- dimensional vector of slack variables $s_1, \ldots, s_{2n+p}$

$$S = (s_1, \ldots, s_{2n+p})^{\mathsf{T}} ;$$

further, for the sake of simplicity, we denote E_m m-by-m unit matrix and $N_{m_1 m_2}$ zero matrix of the type m_1-by-m_2. Relations (3) and (4) can be then expressed in the following way:

$$\begin{bmatrix} C_1 & -E_{p_1} & N_{p_1 p_2} & \\ \hline C_2 & N_{p_2 p_1} & E_{p_2} & N_{p,2n} \\ \hline E_{2n} & N_{2n,p+1} & & E_{2n} \end{bmatrix} \begin{bmatrix} G \\ \hline \\ S \end{bmatrix} = \begin{bmatrix} H_1 \\ \hline H_2 \\ \hline D \end{bmatrix},$$

where

$$D = (d_1, \ldots, d_n, -d_1, \ldots, -d_n)^{\mathsf{T}},$$

or, in a more compact form,

(5)
$$\begin{bmatrix} C_1 & \\ \hline C_2 & M \\ \hline E_{2n} & N_{2n,1} \end{bmatrix} \begin{bmatrix} G \\ \hline \\ S \end{bmatrix} = \begin{bmatrix} H_1 \\ \hline H_2 \\ \hline D \end{bmatrix},$$

where matrix M is equal to matrix E_{2n+p} with the exception of the first p_1 diagonal elements, which are multiplied by minus one. This system can be solved by the simplex method of linear programming [1], by means of which it can be already in the first stage decided whether there exists such a weight vector W, defining hyperplane

$$X^{\mathsf{T}} W = h,$$

which either separates correctly sample patterns of the respective categories or which passes through some of the sample patterns, but with the exception of them separates correctly all the remaining sample patterns. If such a weight vector does not exist, the given situation is not linearly separable and the simplex method ends in the first stage. If such vector exists, the simplex method continues in the second stage by maximizing τ. Starting from solution $\tau = 0$ (i.e. from the situation, when the hyperplane passes through some of the sample patterns and correctly separates the others), τ is gradually increased till its maximal value. If there exists for $\tau > 0$ no weight vector whose corresponding hyperplane separates correctly all sample patterns,

the given situation is not linearly separable. If such weight vector exists, the simplex method ensures reaching the solution after a finite number of steps.

Example. There are given two sample patterns of category σ_1

$$^1Z_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad ^1Z_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

and three sample patterns of category σ_2

$$^2Z_1 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \quad ^2Z_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \quad ^2Z_3 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

The classification of the two categories is to be performed using a TLU with fixed threshold $h = 0$ and with adjustable weights constrained so that their modules are not to exceed 10, i.e.

$$|w_1| \leqq 10, \quad |w_2| \leqq 10.$$

We are to find such values of the adjustable weights which admit the maximal tolerance of the threshold value without the occurrence of an erroneous recognition of sample patterns. Matrices A_1 and A_2 take the following form:

$$A_1 = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ -2 & 1 \end{bmatrix}$$

and relation (5)

$$\begin{bmatrix} 1 & 0 & -1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & -2 & -1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & 1 & 2 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} ^+w_1 \\ ^+w_2 \\ ^-w_1 \\ ^-w_2 \\ \tau \\ s_1 \\ . \\ . \\ s_9 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 10 \\ 10 \\ 10 \\ 10 \end{bmatrix}.$$

The solving by the simplex method yields

$$^+w_1 = 10, \quad ^+w_2 = 5,$$

$$^-w_1 = 0, \quad ^-w_2 = 10,$$

$$\tau = 10,$$

and hence

$$W = \begin{bmatrix} 10 \\ -5 \end{bmatrix}.$$

The adjustment of the weights, defined by the weight vector W, ensures the correct classification under any change of the threshold value Δh (for $|\Delta h| < 10$) and there cannot be reached a better

reliability by some other weight vector. It has also to be mentioned that in some cases (and our example is one of them) there does not exist only one optimal weight vector, and that there exists a system of weight vectors, for which maximal possible τ is the same. The weight vector obtained by the simplex method of linear programming is one of them.

Let Δ_0 denote the threshold value change and $\Delta_1, \ldots, \Delta_n$ the changes of weight vector components. If (for $\tau > 0$) the correct classification of all sample patterns is required even under any changes of the threshold value and the values of the weight vector components such that

$$\max_{i=0,1,\ldots,n} |\Delta_i| < \tau ,$$

(2) must be replaced by the following system of inequalities:

$$A_1 W - \tau B_1 U_n \geqq H_1 + \tau U_{p_1} ,$$

$$A_2 W + \tau B_2 U_n \leqq H_2 - \tau U_{p_2} ,$$

where B_m is the matrix of the same type as matrix A_m and its elements are equal to the modules of the corresponding elements of matrix A_m ($m = 1, 2$). Introducing new vectors

$$V_j = \left(v_{j,1}, \ldots, v_{j,p_j} \right)^{\mathsf{T}} , \quad j = 1, 2 ,$$

where

$$v_{ji} = 1 + \sum_{r=1}^{n} \left| {}^j z_i^r \right| , \quad j = 1, 2, \; i = 1, \ldots, p_j ,$$

we may write

(6)
$$A_1 W \geqq H_1 + \tau V_1 ,$$

$$A_2 W \leqq H_2 - \tau V_2 .$$

If we want to find the optimal weight vector (with bounded components) maximizing the tolerance (of the threshold value and the values of the weight vector components) which ensures correct classification of all sample patterns, we start from (6) and transform the problem into the standard form of linear programming. Defining

$$C_1^* = \left[A_1 \mid -A_1 \mid -V_1 \right] ,$$

$$C_2^* = \left[A_2 \mid -A_2 \mid V_2 \right] ,$$

we may write

(7)
$$
\begin{bmatrix}
C_1^* & \\
\hline
C_2^* & M \\
\hline
E_{2n} & N_{2n,1}
\end{bmatrix}
\begin{bmatrix}
G \\
\hline
S
\end{bmatrix}
=
\begin{bmatrix}
H_1 \\
\hline
H_2 \\
\hline
D
\end{bmatrix} .
$$

The solving of this system under the condition of maximizing τ can be performed by the simplex method.

> *Note.* A TLU with an adjustable threshold and n adjustable weights can be investigated as a TLU with a fixed threshold and $n + 1$ adjustable weights. It is sufficient to define new sample patterns by augmenting the old ones by setting the $(n + 1)$st component equal to one. If both the values of adjustable weights and the value of the threshold are to maximize the tolerances of the threshold and the weight vector components, the problem can be transformed into the problem of finding an $(n + 1)$-dimensional weight vector which maximizes the admissible tolerances of weights provided that the value of the threshold can be strictly kept. The corresponding necessary changes in individual formulas are evident.

PATTERN RECOGNITION FOR k CATEGORIES

The recognition of patterns of k categories can be performed by two different methods. The problem may be either transformed into solving a few problems of pattern recognition for two categories [2] (in this case a few adaptive TLU's are required together with a logic circuit comparing the results of the individual TLU's), or a system evaluating the maximal value out of the values of the type

$$X^T W_j, \quad j = 1, ..., k$$

may be used. Since the former method transforms immediately the problem into problems of pattern recognition for two categories, we shall investigate in brief only the latter method.

There are given the following sets

$$\{^j Z_i\} \quad i = 1, ..., p_j, \quad j = 1, ..., k$$

of sample patterns of categories $\sigma_1, ..., \sigma_k$. Denoting $W_1, ..., W_k$ the characteristic vectors of the individual categories, the objective of training consists in determining vectors satisfying the following system of inequalities:

(8) $\qquad ^r Z_i^T W_r > {}^r Z_i^T W_j, \quad r = 1, ..., k, j = 1, ..., k, j \neq r, i = 1, ..., p_r.$

Denoting

$$A_r = \begin{bmatrix} {}^r Z_i^T \\ \vdots \\ {}^r Z_{p_j}^T \end{bmatrix}, \quad r = 1, ..., k,$$

we may write (8) in the following form:

(9) $\qquad A_r W_r > A_r W_j, \quad r = 1, ..., k, j = 1, ..., k, j \neq r,$

or in a more convenient form

(10) $\qquad A^* W^* > 0,$

where
$$W^* = \left(W_1^\mathsf{T}, \ldots, W_k^\mathsf{T} \right)^\mathsf{T}$$

and A^* is a matrix of the type $\pi \times v$, where

$$\pi = (k - 1) \sum_{i=1}^{k} p_i, \quad v = kn.$$

This matrix can be decomposed into k matrices P_r $(r = 1, \ldots, k)$ of the type $\pi_r \times v$ where $\pi_r = (k - 1) p_r$. Each of these matrices can be further decomposed into $(k - 1) k$ matrices P_{rst} of the type $p_r \times n$, where

$$
\begin{aligned}
P_{rst} = \quad & A_r \quad s = 1, \ldots, k - 1, \, t = r, \\
= \; & -A_r \quad s = 1, \ldots, k - 1, \\
& \qquad \text{for} \quad s < r, \, t = s \\
& \qquad \text{for} \quad s \geqq r, \, t = s + 1, \\
= \; & N_{p_r, n} \quad \text{for all remaining matrices}.
\end{aligned}
$$

Since the forming of matrix A^* reflects the transformation of the recognition problem for k categories into the recognition problem for two categories only, certain analogy with the convergence proof of the iteration procedure for learning systems for recognition patterns of k categories (cf. Nilsson [3]) could be expected.

Let Δ_{ij} be the change of the j-th component of the i-th characteristic vector. If correct pattern classification is required even under the condition that the components of the characteristic vectors change so that

$$\max_{\substack{i = 1, \ldots, k \\ j = 1, \ldots, n}} \left| \Delta_{ij} \right| < \tau,$$

(9) has to be replaced by

$$(11) \quad A_r W_r - \tau B_r U_n \geqq A_r W_j + \tau B_r U_n, \quad r = 1, \ldots, k, \, j = 1, \ldots, k, \, j \neq r,$$

where B_r is the matrix of the same type as matrix A_r and its elements are equal to the modules of the corresponding elements of matrix A_r. Hence,

$$A_r \left(W_r - W_j \right) \geqq \tau V_r, \quad r = 1, \ldots, k, \, j = 1, \ldots, k, \, j \neq r,$$

where

$$V_r = \left(v_{r,1}, \ldots, v_{r,p_r} \right)^\mathsf{T}$$

and

$$v_{ri} = 2 \sum_{s=1}^{n} \left| {}^r z_i^s \right|.$$

Applying above defined matrix A^* and vector W^* and introducing new π-dimensional

vector

$$V^* = (\underbrace{V_1^T, \ldots, V_1^T}_{k-1}, \underbrace{V_2^T, \ldots, V_2^T}_{k-1}, \ldots, \underbrace{V_k^T, \ldots, V_k^T}_{k-1})^T \,,$$

we may write (11) in the following way:

(12) $$A^*W^* \geqq \tau V^* \,.$$

As in the recognition problem for two categories, the components of the characteristic vectors are constrained in the following way:

$$|w_{ij}| \leqq d_{ij} \,,$$

where d_{ij} are given constants $(i = 1, \ldots, k, j = 1, \ldots, n)$. Further, analogously the following vectors are introduced

$$G^* = ({}^+W^{*T}, {}^-W^{*T}, \tau)^T \,,$$

$$S^* = (s_1, \ldots, s_{2kn+\pi})^T \,,$$

$$D^* = (d_{11}, d_{12}, \ldots, d_{1n}, \ldots, d_{k1}, \ldots, d_{kn}, -d_{11}, -d_{12}, \ldots, -d_{kn})^T \,,$$

where $s_1, \ldots, s_{2kn+\pi}$ are nonnegative slack variables. Then the given problem can be solved by linear programming starting from the following standard form:

$$\begin{bmatrix} A^* & -A^* & -V^* & \\ \hdashline & & & -E_{2kn+\pi} \\ E_{2kn} & N_{2kn,1} & & \end{bmatrix} \begin{bmatrix} G^* \\ \hdashline S^* \end{bmatrix} = \begin{bmatrix} N_{\pi,1} \\ \hdashline D^* \end{bmatrix} .$$

In this way the recognition problem for k categories was transformed into the pattern recognition problem for two categories only. As (5) or (7) this problem can also be solved by the simplex method.

CONCLUSION

The application of the linear programming methods to the problem of determining weights of a TLU for pattern recognition is not new [4]; published works, however, were primarily concerned with the following problems:

1. deciding whether a given situation is linearly separable;
2. in case of linear separability, finding any admissible solution vector, possibly determining the admissible weight vector with the minimal sum of the modules of its components;
3. in case of nonseparability, determining the weight vector minimizing reasonably defined losses incurred during the classification.

Although the problem of finding the weights which maximize admissible tolerances is very interesting, it has not yet drawn much attention. The application of linear programming for solving the problem of pattern recognition taking into consideration the tolerance of the threshold value, or taking into consideration the tolerances of both the threshold value and the values of adjustable weights, is new and has not yet been published elsewhere.

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VÝTAH

Optimalizace adaptivního prahového prvku pomocí lineárního programování

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Adaptivní prahový prvek je základem některých perceptronových soustav na rozpoznávání obrazců, u kterých je cílem trénovací fáze nalezení váhového vektoru vyhovujícího určité soustavě nerovností. Určení jednotlivých složek váhového vektoru je možno provést metodami lineárního programování, které umožňují — v případě lineární separovatelnosti dané situace — určení optimálních vah, které maximalizují přípustné tolerance prahu a jednotlivých složek váhového vektoru. Tento postup je možno rozšířit na rozpoznávání obrazců více tříd, kdy určujeme charakteristické vektory jednotlivých tříd.

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