Kybernetika

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On the group pulse processes. II. Power spectrum of the processes with independent points

Terms of use:

ON THE GROUP PULSE PROCESSES

II. Power Spectrum of the Processes with Independent Points

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Power spectrum of the nth order group pulse process with independent reference points occurrence times (i.e., of the process LA ... A) is derived. For a fast determination of the power spectrum formula a matrix method is introduced. At the end the structure of the derived power spectrum is shortly discussed.

1. INTRODUCTION

In the previous paper [1] the process LA ... A was defined to be the group pulse process of the order n, which is of the type A (i.e., with independent reference points occurrence times) on the levels $1, \ldots, n$. The type of the process on the level zero remains to be specified; it may be either of the type A or D or B.

The processes LA ... A in their simpler forms belong among the well known group pulse processes. For example, the one-pulse periodic process DA^1, which may be viewed at as a "degenerated" form of the 1st order periodic group pulse process with constant number of pulses in groups, was studied as early as in the late forties by McFarlane [2], Lawson and Uhlenbeck [3] and several others. The power spectra of the processes DA^D and DA^P were derived in the paper [4], the power spectrum of the process BA may be found in [5], and the process DAA is treated in [6]. In this work we want to proceed a bit further. We shall attempt to find a general formula for the power spectrum of the process LA ... A for arbitrary n.

Though a direct derivation of the power spectrum of the process LA ... A is possible it would be rather lengthy. Therefore the following procedure will be adopted here. First of all the power spectrum of the process AA will be derived step by step in the following section. This will partially duplicate the work done in [4] and [5], but, on the other hand, it will give the present paper a self-sufficiency. The procedure introduced in Section 2 will be generalized in Section 3 to obtain the power spectrum of the process LA ... A. Finally, in Section 4 a matrix method will be introduced that will allow a fast determination of the power spectrum formula in practical situations.

2. POWER SPECTRUM OF THE PROCESS AA

Let us consider a random pulse process $\xi(t)$ of the type AA (Fig. 1). Using the usual notation [1] the number of pulses in the ith group will be denoted K_i and the time the ith group reference point occurs τ_i. The occurrence of the kth pulse in the ith
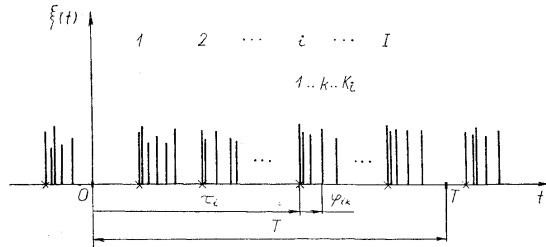

Fig. 1. Group pulse process of the type AA.

group with respect to the ith group reference point will be denoted φ_{ik}. The process $\xi(t)$ may be written in the form

$$(1) \qquad \xi(t) = \sum_{i=-\infty}^{\infty} \sum_{k=1}^{K_i} f(t - \tau_i - \varphi_{ik}, \boldsymbol{a}_{ik}).$$

Let us now consider an interval $(0, T)$ and let us suppose that there are exactly I pulse groups in the interval. In agreement with the definition of the process AA [1] the random variables τ_i will be supposed to be mutually independent and uniformly distributed in the interval $(0, T)$, that is, their probability density will be of the form

$$(2) \qquad w_1(\tau) = \left\langle \begin{array}{ll} 1/T & \tau \in (0, T), \\ 0 & \tau \notin (0, T). \end{array} \right.$$

Similarly, the random variables φ_{ik} will also be supposed to be mutually independent and all having the same distribution $w_1(\varphi)$. Finally, according to the definition, the random variables τ_i, φ_{ik}, K_i and the random vector $\boldsymbol{a}_{ik}$ are all considered to be mutually independent.

Now, let us suppose that the process $\xi(t)$ equals zero outside the interval $(0, T)$. This truncated realization, consisting of I pulse groups, will be denoted $\xi_{TI}(t)$ and may be written as

$$(3) \qquad \xi_{TI}(t) = \sum_{i=1}^{I} \sum_{k=1}^{K_i} f(t - \tau_i - \varphi_{ik}, \boldsymbol{a}_{ik}).$$

The spectrum of the process $\xi_{TI}(t)$ will be determined by taking the Fourier transform of (3). Denoting the spectrum of the pulse $f(t, \boldsymbol{a})$ by $s(\omega, \boldsymbol{a})$ we obtain

$$(4) \qquad S_{TI}(\omega) = \sum_{i=1}^{I} \sum_{k=1}^{K_i} s(\omega, \boldsymbol{a}_{ik}) \, e^{-j\omega(\tau_i + \varphi_{ik})}.$$

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The power spectrum of the process $\xi(t)$ will be determined from the definition formula

$$(5) \qquad \mathscr{W}(\omega) = \lim_{T \to \infty} \frac{1}{T} \left\langle \left| S_T(\omega) \right|^2 \right\rangle .$$

For this purpose the modulus squared of the spectrum (4) will be found first. Denoting the complex conjugate by an asterisk we have

$$(6) \qquad \left| S_{TI}(\omega) \right|^2 = S_{TI}(\omega)\, S_{TI}^*(\omega) =$$

$$= \sum_{i=1}^{I} \sum_{k=1}^{K_i} \sum_{l=1}^{I} \sum_{m=1}^{K_l} s(\omega, \boldsymbol{a}_{ik})\, \mathrm{e}^{-\mathrm{j}\omega(\tau_i + \varphi_{ik})} s^*(\omega, \boldsymbol{a}_{lm})\, \mathrm{e}^{\mathrm{j}\omega(\tau_l + \varphi_{lm})} .$$

From the four-fold sum we may take out the terms for which $i = l$ and $k = m$ simultaneously and the terms for which $i = l$ but $k \neq m$. We obtain

$$(7) \qquad \left| S_{TI}(\omega) \right|^2 = \sum_{i=1}^{I} \sum_{k=1}^{K_i} \left| s(\omega, \boldsymbol{a}_{ik}) \right|^2 +$$

$$+ \sum_{i=1}^{I} \sum_{\substack{k=1 \\ k \neq m}}^{K_i} \sum_{m=1}^{K_i} s(\omega, \boldsymbol{a}_{ik})\, s^*(\omega, \boldsymbol{a}_{im})\, \mathrm{e}^{-\mathrm{j}\omega(\varphi_{ik} - \varphi_{im})} +$$

$$+ \sum_{i=1}^{I} \sum_{k=1}^{K_i} \sum_{\substack{l=1 \\ i \neq l}}^{I} \sum_{m=1}^{K_l} s(\omega, \boldsymbol{a}_{ik})\, s^*(\omega, \boldsymbol{a}_{lm})\, \mathrm{e}^{-\mathrm{j}\omega(\tau_i - \tau_l)}\, \mathrm{e}^{-\mathrm{j}\omega(\varphi_{ik} - \varphi_{lm})} .$$

Now the expectation of (7) on a set of all truncated realizations consisting of just I pulse groups will be found. With respect to the postulated independence of all random variables we have

$$(8) \qquad \left\langle \left| S_{TI}(\omega) \right|^2 \right\rangle = I\langle K \rangle \left\langle \left| s(\omega, \boldsymbol{a}) \right|^2 \right\rangle + I(\langle K^2 \rangle - \langle K \rangle) \left\langle \left| s(\omega, \boldsymbol{a}) \right| \right\rangle^2 \left| \chi_\varphi(\omega) \right|^2 +$$

$$+ (I^2 - I) \langle K \rangle^2 \left\langle \left| s(\omega, \boldsymbol{a}) \right| \right\rangle^2 \left| \chi_\varphi(\omega) \right|^2 \left| \chi_\tau(\omega) \right|_T^2 .$$

Here the symbol $\langle \ \rangle$ denotes the expectation, $\chi_\varphi(\omega)$ is a characteristic function of the random variable φ and $\chi_\tau(\omega)$ is a charactetristic function of the random variable τ. It follows from (2) that the characteristic function $\chi_\tau(\omega)$ depends on the length of the interval $(0, T)$. This was denoted by the subscript T.

Expression (8) represents the expectation on the set of those truncated realizations that contain iust I pulse groups. The expectation on the ensemble of all truncated realizations can be found as

$$(9) \qquad \left\langle \left| S_T(\omega) \right|^2 \right\rangle = \sum_{I=0}^{\infty} P(I) \left\langle \left| S_{TI}(\omega) \right|^2 \right\rangle .$$

Here $P(I)$ is a probability of an event that just I pulse groups occur in the interval $(0, T)$. In the case of the process AA the probability $P(I)$ is given by the Poisson

distribution [1]. Hence

$$(10) \qquad \langle|S_T(\omega)|^2\rangle = \langle I\rangle \langle K\rangle \langle|s(\omega, \boldsymbol{a})|^2\rangle +$$
$$+ \langle I\rangle (\langle K^2\rangle - \langle K\rangle) \langle|s(\omega, \boldsymbol{a})|\rangle^2 |\chi_\varphi(\omega)|^2 +$$
$$+ \langle I\rangle^2 \langle K\rangle^2 \langle|s(\omega, \boldsymbol{a})|\rangle^2 |\chi_\varphi(\omega)|^2 |\chi_\tau(\omega)|_T^2 \,.$$

Now, if the length of the interval $(0, T)$ is increased, the number of groups occurring in the interval, I, will also increase. According to the elementary theorem for the renewal process [7], it will hold that

$$(11) \qquad \lim_{T\to\infty} \frac{\langle I\rangle}{T} = \langle v\rangle \,.$$

Here $\langle v\rangle$ is a mean density of the pulse groups. It can also be shown that [8]

$$(12) \qquad \lim_{T\to\infty} [T|\chi_\tau(\omega)|_T^2] = 2\pi\,\delta(\omega) \,.$$

Substituting (10) into (5) and using (11) and (12) we finally obtain

$$(13) \qquad \mathcal{W}(\omega) = \langle v\rangle \langle K\rangle \langle|s(\omega, \boldsymbol{a})|^2\rangle + \langle v\rangle (\langle K^2\rangle - \langle K\rangle) \langle|s(\omega, \boldsymbol{a})|\rangle^2 |\chi_\varphi(\omega)|^2 +$$
$$+ \langle v\rangle^2 \langle K\rangle^2 \langle|s(0, \boldsymbol{a})|\rangle^2 \, 2\pi\,\delta(\omega) \,.$$

This is the power spectrum of the process AA. The same formula could be obtained by appropriate rearrangement of the expression (18) in [5]. However, here the power spectrum of the process AA was derived by the direct method and the procedure used may serve as a sample one for the derivation that will be carried out in the following section.

3. POWER SPECTRUM OF THE PROCESS LA ... A

In the preceding section the power spectrum of the process AA was derived step by step. In this section the procedure will be generalized to obtain the power spectrum of the process LA ... A. Because of experience gained it will be possible to shorten the derivation to a considerable degree. In case of doubts it is recommended to consult the previous section. It is also assumed that the meaning of all symbols and subscripts was made clear enough so that the simplified notation introduced in [1] may be applied. From now on the following shortened notation will be used:

$$s(\omega, \boldsymbol{a}_{i,k_1,\cdots,k_n}) \quad \text{will be replaced by } s \,,$$
$$K_{i,k_1,\cdots,k_p} \quad \text{will be replaced by } K_p,$$
$$\varphi_{i,k_1,\cdots,k_p} \quad \text{will be replaced by } \varphi_p,$$

and $\qquad\qquad \chi_{\varphi_p}(\omega) \qquad$ will be replaced by χ_p.

Here and elsewhere in this paper it is assumed that $1 \leqq p \leqq n$.

As a point of departure for derivation of the power spectrum of the process

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LA ... A the analytical expression for $\xi(t)$ (expression (1) in [1], cf. also (1) here) may be used again. At first a realization $\xi_{TI}(t)$, truncated in an principal interval $(0, T)$, will be considered. A spectrum $S_{TI}(\omega)$ of this realization may be obtained by taking the Fourier transform of $\xi_{TI}(t)$. In the next step the modulus squared of this spectrum may be determined using the well known procedure (cf. the preceding section). This expression for $|S_{TI}(\omega)|^2$ will contain $2(n + 1)$-fold sum. From this $2(n + 1)$-fold sum $(n + 1)$ simpler units may be taken out. The structure of these units will be as follows:

(0)   The unit zero will contain those terms from the $2(n + 1)$-fold sum for which it holds simultaneously that

$$i = l, \quad k_1 = m_1, \quad \ldots, \quad k_n = m_n.$$

The unit zero will be formed by the $(n + 1)$-fold sum of the terms having the form

$$|s|^2 .$$

(1)   The first unit will contain those terms from the $2(n + 1)$-fold sum for which it holds simultaneously that

$$i = l, \quad k_1 = m_1, \quad \ldots, \quad k_{n-1} = m_{n-1}, \quad k_n \neq m_n .$$

The first unit will be formed by the $(n + 2)$-fold sum of the terms having the form

$$ss^{*\prime} \exp\left[-j\omega(\varphi_n - \varphi_n')\right] .$$

Here we use an apostrophe to distinguish the variables which have different subscripts, e.g., k_n and m_n, because in the simplified notation we use now the difference between these variables is wiped out.

(2)   The second unit will contain those terms from the $2(n + 1)$-fold sum for which it holds simultaneously that

$$i = l, \quad k_1 = m_1, \quad \ldots, \quad k_{n-2} = m_{n-2}, \quad k_{n-1} \neq m_{n-1} .$$

This second unit will be formed by the $(n + 3)$-fold sum of the terms having the form

$$ss^{*\prime} \exp\left[-j\omega(\varphi_n - \varphi_n')\right] \exp\left[-j\omega(\varphi_{n-1} - \varphi_{n-1}')\right] .$$

$\vdots$

(n)   The nth unit will contain those terms from the $2(n + 1)$-fold sum for which it holds simultaneously that

$$i = l, \quad k_1 \neq m_1 .$$

The nth unit will be formed by the $(2n + 1)$-fold sum of the terms having the form

$$ss^{*\prime} \exp\left[-j\omega(\varphi_n - \varphi_n')\right] \ldots \ldots \exp\left[-j\omega(\varphi_1 - \varphi_1')\right] .$$

$(n + 1)$ The rest of the $2(n + 1)$-fold sum will contain the terms for which it holds that

$$i \neq l .$$

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This last $(n + 1)$th unit will be formed by the terms having the form

$$ss^{*\prime} \exp\left[-j\omega(\varphi_n - \varphi_n')\right] \dots \exp\left[-j\omega(\varphi_1 - \varphi_1')\right] \exp\left[-j\omega(\tau_i - \tau_i')\right].$$

Now it is possible to determine the expectation of the expression $\left|S_{TI}(\omega)\right|^2$. This will be performed on the set of those truncated realizations that contain just I clusters of the order zero first and then on the ensemble of all truncated realizations. With respect to the postulated independence of the respective variables (the process LA ... A) we obtain

(14)
$$\left\langle\left|S_T(\omega)\right|^2\right\rangle = \langle I\rangle \langle K_1\rangle \dots \langle K_n\rangle \left\langle\left|s\right|^2\right\rangle +$$
$$+ \langle I\rangle \langle K_1\rangle \dots (\langle K_n^2\rangle - \langle K_n\rangle) \langle|s|\rangle^2 |\chi_n|^2 +$$
$$+ \langle I\rangle \langle K_1\rangle \dots (\langle K_{n-1}^2\rangle - \langle K_{n-1}\rangle) \langle K_n\rangle^2 \langle|s|\rangle^2 |\chi_n|^2 |\chi_{n-1}|^2 + \dots$$
$$\dots + \langle I\rangle (\langle K_1^2\rangle - \langle K_1\rangle) \langle K_2\rangle^2 \dots \langle K_n\rangle^2 \langle|s|\rangle^2 |\chi_n|^2 \dots |\chi_1|^2 +$$
$$+ \langle K_1\rangle^2 \dots \langle K_n\rangle^2 \langle|s|\rangle^2 |\chi_n|^2 \dots |\chi_1|^2 \langle X_0(\omega, I)\rangle.$$

Here $\langle X_0(\omega, I)\rangle$ is a function that according to the kind of the process on the level zero (that is according to the distribution of the random variables τ or ϑ) has the form (cf. [4] and [5]):

for a homogeneous process

(15)
$$\langle X_0(\omega, I)\rangle = \langle I^2\rangle \left|\chi_\tau(\omega)\right|_T^2,$$

for a periodic process

(16)
$$\langle X_0(\omega, I)\rangle = \sum_{i=-I}^{I} \sum_{\substack{l=-I \\ i \neq l}}^{I} \exp\left[-j\omega\, T_0(i - l)\right],$$

and for quasiperiodic and aperiodic processes

(17)
$$\langle X_0(\omega, I)\rangle = \left\langle \begin{array}{ll} 2\,\mathrm{Re}\left\{\dfrac{\chi_\vartheta(\omega)}{1 - \chi_\vartheta(\omega)}\left[\langle I\rangle - \langle B_\vartheta(\omega, I)\rangle\right]\right\} & \omega \neq 0, \\[2mm] \langle I^2\rangle - \langle I\rangle & \omega = 0, \end{array}\right.$$

The power spectrum of the process LA ... A may be determined in the case of the quasiperiodic, aperiodic or homogeneous processes from the definition formula (5) and in the case of the periodic process from the formula (4) in [4]. We obtain

(18)
$$\mathscr{W}(\omega) = \langle v\rangle \langle K_1\rangle \dots \langle K_n\rangle \left\langle\left|s\right|^2\right\rangle +$$
$$+ \langle v\rangle \langle K_1\rangle \dots \langle K_{n-1}\rangle (\langle K_n^2\rangle - \langle K_n\rangle) \langle|s|\rangle^2 |\chi_n|^2 +$$
$$+ \langle v\rangle \langle K_1\rangle \dots \langle K_{n-2}\rangle (\langle K_{n-1}^2\rangle - \langle K_{n-1}\rangle) \langle K_n\rangle^2 \langle|s|\rangle^2 |\chi_n|^2 |\chi_{n-1}|^2 + \dots$$
$$\dots + \langle v\rangle (\langle K_1^2\rangle - \langle K_1\rangle) \langle K_2\rangle^2 \dots \langle K_n\rangle^2 \langle|s|\rangle^2 |\chi_n|^2 \dots |\chi_1|^2 +$$
$$+ \langle v\rangle^2 \langle K_1\rangle^2 \dots \langle K_n\rangle^2 \langle|s|\rangle^2 |\chi_n|^2 \dots |\chi_1|^2 \, Y_0(\omega).$$

Here $Y_0(\omega)$ is a function that according to the kind of the process on the level zero has the form (cf. [4] and [5]):

for the homogeneous process

(19)
$$Y_0(\omega) = Y_A(\omega) = 2\pi\,\delta(\omega),$$

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for the periodic process

$$(20) \qquad Y_C(\omega) = Y_D(\omega) = -1/v_0 + 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0),$$

and at last for the quasiperiodic and the aperiodic processes

$$(21) \qquad Y_0(\omega) = Y_B(\omega) = -1/\langle v \rangle + Z_\vartheta(\omega)/\langle v \rangle + 2\pi \, \delta(\omega).$$

The formula (18) consists of $(n + 2)$ terms (units). In accordance with the introduced notation these terms will be referred to as the term zero, the first term, ..., the $(n + 1)$th term.

4. PROCEDURE FOR FAST DETERMINATION OF THE POWER SPECTRUM

In the previous section the general formula (18) for the power spectrum of the process LA ... A was derived. Though this formula is fully valid, its direct use for the determination of the power spectrum of a certain process may be inconvenient. Therefore a more suitable procedure will be introduced now.

Let us suppose we are to determine the formula for the power spectrum of the process LA ... A, which is of the order n. It is convenient to start by drawing a square matrix* of the order $(n + 2)$ and by numbering the columns and rows. For reasons that will be made clear elsewhere [9] it is also recommended to write the structural formula of the process above the matrix. Now the elements should be filled in. It is convenient to begin with the corner elements, then to fill up the first and the last columns, the subsidiary diagonal and finally the remaining columns or rows. The resulting matrix is shown below:

	L 0	A 1	A 2	... ...		A $n-1$	A n	$n+1$
0	$\langle v \rangle$	$\langle K_1 \rangle$	$\langle K_2 \rangle$	...		$\langle K_{n-1} \rangle$	$\langle K_n \rangle$	$\langle |s|^2 \rangle$
1	$\langle v \rangle$	$\langle K_1 \rangle$	$\langle K_2 \rangle$	...		$\langle K_{n-1} \rangle$	a_n'	$\langle |s| \rangle^2$
2	$\langle v \rangle$	$\langle K_1 \rangle$	$\langle K_2 \rangle$	...		a_{n-1}'	a_n''	$\langle |s| \rangle^2$
	⋮	⋮	⋮		.	⋮	⋮	⋮
					.			
	⋮	⋮	⋮	.		⋮	⋮	⋮
$n-1$	$\langle v \rangle$	$\langle K_1 \rangle$	a_2'	...		a_{n-1}''	a_n''	$\langle |s| \rangle^2$
n	$\langle v \rangle$	a_1'	a_2''	...		a_{n-1}''	a_n''	$\langle |s| \rangle^2$
$n+1$	$\langle v \rangle^2$	a_1''	a_2''	...		a_{n-1}''	a_n''	$\langle |s| \rangle^2 \, Y_0(\omega)$

* Similarly as in algebra, the word "matrix" is used here for a system of elements ordered in columns and rows. The way the elements are recorded in the matrix and then combined to give the power spectrum formula is explained in the text.

Each row in the matrix represents one term in the power spectrum formula. This formula will be obtained from the matrix by mutual multiplying of the elements in each row, adding all the rows, replacing a'_p and a''_p by expressions

(22) $$a'_p = (\langle K_p^2 \rangle - \langle K_p \rangle) \, |\chi_p|^2 \, ,$$

(23) $$a''_p = \langle K_p \rangle^2 \, |\chi_p|^2 \, ,$$

and substituting $Y_0(\omega)$ by the appropriate function from equations $(19)-(21)$.

Let us illustrate this procedure by an example and let us suppose that we are to determine a formula for the power spectrum of the process $\mathrm{DA^P A^P}$.

The matrix of the process LAA is given below:

	L 0	A 1	A 2	3
0	$\langle v \rangle$	$\langle K_1 \rangle$	$\langle K_2 \rangle$	$\langle |s|^2 \rangle$
1	$\langle v \rangle$	$\langle K_1 \rangle$	a'_2	$\langle |s| \rangle^2$
2	$\langle v \rangle$	a'_1	a''_2	$\langle |s| \rangle^2$
3	$\langle v \rangle^2$	a''_1	a''_2	$\langle |s| \rangle^2 \, Y_0(\omega)$

Now the elements of each row will be mutually multiplied and all rows added to give

$$\mathscr{W}(\omega) = \langle v \rangle \, \langle K_1 \rangle \, \langle K_2 \rangle \, \langle |s|^2 \rangle + \langle v \rangle \, \langle K_1 \rangle \, a'_2 \langle |s| \rangle^2 + $$
$$+ \langle v \rangle \, a'_1 \, a''_2 \langle |s| \rangle^2 + \langle v \rangle^2 \, a''_1 \, a''_2 \langle |s| \rangle^2 \, Y_0(\omega) \, .$$

After substituting v_0 for $\langle v \rangle$, $Y_D(\omega)$ for $Y_0(\omega)$ and using equations (20), (22) and (23) we obtain

$$\mathscr{W}(\omega) = v_0 \langle K_1 \rangle \, \langle K_2 \rangle \, \langle |s|^2 \rangle + v_0 \langle K_1 \rangle \, (\langle K_2^2 \rangle - \langle K_2 \rangle) \, \langle |s| \rangle^2 \, |\chi_2|^2 + $$
$$+ v_0 (\langle K_1^2 \rangle - \langle K_1 \rangle) \, \langle K_2 \rangle^2 \, \langle |s| \rangle^2 \, |\chi_2|^2 \, |\chi_1|^2 + $$
$$+ v_0^2 \langle K_1 \rangle^2 \, \langle K_2 \rangle^2 \, \langle |s| \rangle^2 \, |\chi_1|^2 \, |\chi_2|^2 \, \Big[-1/v_0 + 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \Big] \, .$$

Since the points on the first and second level are Poisson distributed, we may use the relation

$$\langle K_p^2 \rangle - \langle K_p \rangle = \langle K_p \rangle^2 \, ,$$

so that after slight rearrangements we finally have

$$\mathscr{W}(\omega) = v_0 \langle K_1 \rangle \, \langle K_2 \rangle \, \langle |s|^2 \rangle + v_0 \langle K_1 \rangle \, \langle K_2 \rangle^2 \, \langle |s| \rangle^2 \, |\chi_2|^2 + $$
$$+ v_0^2 \langle K_1 \rangle^2 \, \langle K_2 \rangle^2 \, \langle |s| \rangle^2 \, |\chi_2|^2 \, |\chi_1|^2 \, 2\pi \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) \, .$$

For the sake of clarity the derivation was shown in great detail. Naturally, after acquiring some practice several steps in this procedure may be skipped, which will further speed up the process of writing down the desired formula. Versatility and

usefulness of this matrix method will be evident first of all in the connection with mixed processes which are of different types (i.e., A, B or D) on different levels [9].

5. DISCUSSION

In Section 3 we have derived the formula for the power spectrum of the nth order group pulse process. This formula consisted of $(n + 2)$ terms. It is evident that the number of terms is a matter of agreement to some extent because it depends on what will be considered as one term. For example, if the function $Y_0(\omega)$ is broken down into its components and the formula is partially rearranged, the resulting number of terms will be naturally different. However, here we shall distinguish and number the terms according to the basic configuration given in (18).

On any level of clustering the group pulse process may be characterized by a certain time constant Θ_p, whose value is closely related to the magnitudes of the means and dispersions of the random variables φ_p and ε_p. With respect to the way the levels of clustering were defined in [1] it follows that the magnitude of Θ_p will be greater on higher levels then on lower ones. This is, generally speaking, due to the fact that there is usually a greater distance between points on higher levels.

Without going into details we may say that the prevailing majority of functions we are dealing with here (e.g., $s(\omega)$, $\chi_p(\omega)$, etc.) are square-integrable. Let us consider one such square-integrable function that will be denoted, for the sake of generality, as $g_p(\omega)$. Then, for certain C $(0 < C < 1)$ ω_p may be found so that the following equality holds:

$$(24) \qquad C \int_0^\infty |g_p(\omega)|^2 \, d\omega = \int_0^{\omega_p} |g_p(\omega)|^2 \, d\omega \,.$$

The parameter ω_p will be referred to as the upper limiting frequency (angular). Thus for given C, using relation (24) we may find the corresponding ω_p for any function $\langle|s|^2\rangle$, $\langle|s|\rangle^2$ and $|\chi_p|^2$. The magnitudes of ω_p and Θ_p are evidently inversely proportional: the higher is the level of clustering, the greater is Θ_p and the lower is the respective ω_p. It is convenient to define this relation in the following way

$$(25) \qquad \Theta_p = 1/\omega_p \,.$$

Since each term in (18) equals the product of some constants and functions, the upper limiting frequency of any term will equal approximately the upper limiting frequency of that function which has the lowest ω_p. And this function with the lowest ω_p in the pth term is just $|\chi_p|^2$. Hence, the upper limiting frequency of the terms in (18) gradually falls with growing p; it is highest for the term zero and lowest for the nth term. This situation is schematically shown in Fig. 2.

The first $(n + 1)$ terms in formula (18) represent a continuous part of the power spectrum. Beside the continuous part the power spectrum usually contains also discrete components which are represented in formula (18) by the term $(n + 2)$.

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In the case of an aperiodic process the power spectrum may (but need not) have a "dc" component, in the case of a periodic or quasiperiodic process it may contain a number of discrete or quasidiscrete components, i.e., the basic and the higher harmonics.
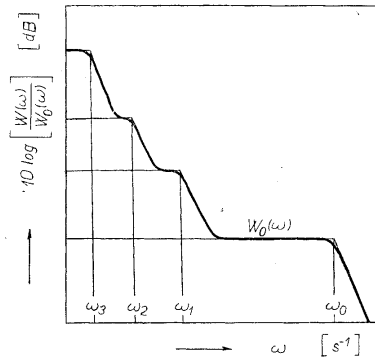

Fig. 2. Power spectrum of the process LA ... A (schematically). Frequency axis is logarithmic.

As it was mentioned elsewhere [1] the random pulse processes are often used to model signals and noises. Several of these models are also included in the process LA ... A. These are, for example, the processes DA^1 and DA^P [4], AA^P [5] and DA^PA^P [6]. However, it can be expected that the general results obtained here will be applicable to even broader class of problems.

One of the motives that led to this work was an intention to investigate whether under certain conditions the process LA ... A can give the power spectrum of the form $1/f$. The idea that an increase of the power spectrum at the lowest frequencies is due to the pulse multilevel clustering is very tempting. However, we have not obtained any usable results in this respect so far.

Note added in proof. During a recent search for the literature the author has become aware of two articles that are related to the topic discussed here. Gilchrist and Thomas [10] derived an expression for the power spectrum of the process AA that is basically the same as the one obtained in this work. Some interesting results regarding the process AA can also be found in the Rice's paper [11].

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