Václav Soukup

Linear nonstationary system with discrete-time input

Kybernetika, Vol. 9 (1973), No. 1, (30)--41

Persistent URL: http://dml.cz/dmlcz/124650

KYBERNETIKA — VOLUME 9 (1973), NUMBER 1

Linear Nonstationary System with Discrete-Time Input

VÁCLAV SOUKUP

The work is concerned with a linear continuous system the parameters of which vary with time and an input is discrete-time. Two forms of a system discrete-time description and their mutual relations are investigated.

In linear stationary sampled-data systems a continuous part is usually described by its transfer function in the modified discrete Laplace transform or Z-transform. There are possibilities to use the corresponding difference equation either in the scalar or vector-matrix form ([1] and [2]). We can use several methods for the transformation of a continuous differential description into an ε-parameter difference form for stationary systems ([1], [2] and [3]).

Nonstationary systems can be practically investigated in a time domain only. We derive here the discrete-time form of state equations and the corresponding scalar difference equation of a linear nonstationary continuous-time system with discrete-time input. It is shown that both these forms can be obtained from a continuous-time description if the system transition matrix is known.

I. SYSTEM STATE EQUATIONS

Let us consider a single-input, single-output deterministic linear system described by two vector-matrix equations

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\,\mathbf{x}(t) + \mathbf{b}(t)\,u(t)\,, \tag{1a}$$

$$y(t) = \mathbf{c}(t)\,\mathbf{x}(t) \tag{1b}$$

where a system input and output are denoted by $u(t)$ and $y(t)$ respectively,

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_s(t) \end{bmatrix} \tag{2}$$

is an $(s \times 1)$ vector of state variables $x_i(t)$, $\boldsymbol{A}(t)$ is an $(s \times s)$ matrix, $\boldsymbol{b}(t)$ an $(s \times 1)$ column vector and $\mathbf{c}(t)$ an $(1 \times s)$ row vector.

Solving the equation (1a) we obtain [4]

(3) $$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\, \boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\, \boldsymbol{b}(\tau)\, u(\tau)\, \mathrm{d}\tau$$

where a system transition (fundamental) matrix $\boldsymbol{\Phi}(t, t_0)$ is the solution of the equation

(4) $$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{\Phi}(t, t_0) - \boldsymbol{A}(t)\, \boldsymbol{\Phi}(t, t_0) = \boldsymbol{0}$$

under condition

$$\boldsymbol{\Phi}(t_0, t_0) = \boldsymbol{I} .$$

$\boldsymbol{\Phi}(t, t_0)$ posesses the properties:

1. The elements of $\boldsymbol{\Phi}(t, t_0)$ are continuous functions of both t and t_0;
2.

(5) $$\boldsymbol{\Phi}(t, t) = \boldsymbol{I} ;$$

3.

(6) $$\boldsymbol{\Phi}(t, \tau)\, \boldsymbol{\Phi}(\tau, t_0) = \boldsymbol{\Phi}(t, t_0) .$$

Now assume that a discrete-time input signal is applied to the system described by (1). Let the constant time interval between two neighbouring input values (sampling period) be $T = 1$ here for simplicity without loss of generality, i.e., $t = n + \varepsilon$ where n ranges over the integers and ε is a continuous parameter, $0 \leqq \varepsilon \leqq 1$.

A dynamical behaviour of the continuous system (1) with a discrete-time input can be described in the discrete-time state space form as

(7a) $$\boldsymbol{x}(n + 1, \varepsilon) = \boldsymbol{F}(n, \varepsilon)\, \boldsymbol{x}(n, 0) + \boldsymbol{h}(n + 1, \varepsilon)\, u(n + 1) ,$$

(7b) $$y(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \boldsymbol{x}(n, \varepsilon)$$

where

(9) $$\boldsymbol{F}(n, \varepsilon) = \boldsymbol{\Phi}(n + 1, \varepsilon; n) ,$$

(10) $$\boldsymbol{h}(n, \varepsilon) = \mathbf{E}(n, \varepsilon)\, \boldsymbol{b}(n, 0)$$

if

(11) $$\mathbf{E}(n, \varepsilon) = \boldsymbol{\Phi}(n, \varepsilon; n)$$

with the property

(12) $$\mathbf{E}(n, 0) = \boldsymbol{I} .$$

The equation (7a) for $\varepsilon = 0$ has the form

(8a) $$\boldsymbol{x}(n + 1, 0) = \boldsymbol{F}(n, 0)\, \boldsymbol{x}(n, 0) + \boldsymbol{b}(n + 1, 0)\, u(n + 1)$$

and the equation (7b) can also be written as

$$y(n, \varepsilon) = \gamma(n, \varepsilon)\, \mathbf{x}(n, 0) \tag{8b}$$

where

$$\gamma(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \mathbf{E}(n, \varepsilon)\,. \tag{13}$$

Then equivalent state equations (8) can be used instead of the equations (7).

Note that the state equations (7) have the form

$$\mathbf{x}(nT + T, \varepsilon T) = \mathbf{F}(nT, \varepsilon T)\, \mathbf{x}(nT, 0) + \mathbf{h}(nT + T, \varepsilon T)\, u(nT + T)\,, \tag{14a}$$

$$y(nT, \varepsilon T) = \mathbf{c}(nT, \varepsilon T)\, \mathbf{x}(nT, \varepsilon T) \tag{14b}$$

if $T \neq 1$.

Proof. A discrete-time input signal can be represented in continuous-time domain by a modulated Dirac function sequence

$$u^*(t) = u(t) \sum_{k=0}^{\infty} \delta(t - k^+)\,. \tag{15}$$

Substituting the relation (15) into the equation (3) we get for $t_0 = k_0$

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t, k_0)\, \mathbf{x}(k_0^-) + \sum_{k=k_0}^{n} \boldsymbol{\Phi}(t, k)\, \mathbf{b}(k)\, u(k) = \tag{16a}$$

$$= \boldsymbol{\Phi}(t, k_0)\, \mathbf{x}(k_0^+) + \sum_{k=k_0+1}^{n} \boldsymbol{\Phi}(t, k)\, \mathbf{b}(k)\, u(k) \tag{16b}$$

where $t - 1 < n < t$.

For $k_0 = n$ we have

$$\mathbf{x}(t) = \boldsymbol{\Phi}(t, n)\, \mathbf{x}(n^-) + \boldsymbol{\Phi}(t, n)\, \mathbf{b}(n)\, u(n) = \tag{17a}$$

$$= \boldsymbol{\Phi}(t, n)\, \mathbf{x}(n^+)\,. \tag{17b}$$

Using the discrete version of time we can write

$$\mathbf{x}(t) = \mathbf{x}(n, \varepsilon)\,,$$

$$\mathbf{x}(k^+) = \mathbf{x}(k, 0)$$

and

$$\mathbf{x}(k^-) = \mathbf{x}(k - 1, 1)\,.$$

Then the equation (17b) can be written as

$$\mathbf{x}(n, \varepsilon) = \boldsymbol{\Phi}(n, \varepsilon; n)\, \mathbf{x}(n, 0) \tag{18}$$

and the equation (16b) for $k_0 = n$, $t = n + 1 + \varepsilon$

(19) $\mathbf{x}(n + 1, \varepsilon) = \boldsymbol{\Phi}(n + 1, \varepsilon; n)\, \mathbf{x}(n, 0) + \boldsymbol{\Phi}(n + 1, \varepsilon; n + 1)\, \mathbf{b}(n + 1)\, u(n + 1)\,.$

Obviously the equations (19) and (7a) are identical if the designations (9), (10) and (11) are used.

For $\varepsilon = 0$ the eq. (19) has the form

$$\mathbf{x}(n + 1, 0) = \boldsymbol{\Phi}(n + 1, 0; n)\, \mathbf{x}(n, 0) + \mathbf{b}(n + 1)\, u(n + 1)$$

and the validity of (8a) is proved.

The system output is given simply by the eq. (1b) as

$$y(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \mathbf{x}(n, \varepsilon) \tag{7b}$$

or with respect to (18)

$$y(n, \varepsilon) = \mathbf{c}(n, \varepsilon)\, \boldsymbol{\Phi}(n, \varepsilon; n)\, \mathbf{x}(n, 0)\,.$$

Using the relations (11) and (13) the validity of (8b) is evident.

II. SCALAR DIFFERENCE EQUATION

The results presented in the previous chapter make possible to determine a convenient, correct form of a scalar difference equation between a system input and output.

This difference equation can be written in two forms:

a)

$$y(n + s, \varepsilon) + \sum_{i=0}^{s-1} \alpha_i(n, \varepsilon)\, y(n + i, \varepsilon) = \sum_{j=1}^{s} \beta_j(n, \varepsilon)\, u(n + j)\,. \tag{20}$$

The coefficients $\alpha_i(n, \varepsilon)$ are the elements of the $(1 \times s)$ row vector

$$\begin{aligned} \boldsymbol{\alpha}(n, \varepsilon) &= [\alpha_0(n, \varepsilon)\, \alpha_1(n, \varepsilon) \ldots \alpha_{s-1}(n, \varepsilon)] = \\ &= -\gamma(n + s, \varepsilon)\, \boldsymbol{\Phi}(n + s, 0; n)\, \mathbf{Q}^{-1}(n, \varepsilon) \end{aligned} \tag{21}$$

and $b_j(n, \varepsilon)$ are the elements of the $(1 \times s)$ row vector

$$\begin{aligned} \boldsymbol{\beta}(n, \varepsilon) &= [\beta_1(n, \varepsilon)\, \beta_2(n, \varepsilon) \ldots \beta_s(n, \varepsilon)] = \\ &= \gamma(n + s, \varepsilon)\, \boldsymbol{\Phi}(n + s, 0; n) \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{R}(n, \varepsilon)\} \end{aligned} \tag{22}$$

where the $(s \times s)$ matrices $\mathbf{Q}(n, \varepsilon)$ and $\mathbf{B}(n, 0)$ are given by

$$\mathbf{Q}(n, \varepsilon) = \begin{bmatrix} \gamma(n, \varepsilon) \\ \gamma(n + 1, \varepsilon)\, \boldsymbol{\Phi}(n + 1, 0; n) \\ \vdots \\ \gamma(n + s - 1, \varepsilon)\, \mathbf{Q}(n + s - 1, 0; n) \end{bmatrix} \tag{23}$$

and

$$B(n, 0) = [\Phi^{-1}(n+1, 0; n)\, b(n+1, 0);\ \Phi^{-1}(n+2, 0; n)\, b(n+2, 0);\ \dots \\ \dots\ \Phi^{-1}(n+s, 0; n)\, b(n+s, 0)] \tag{24}$$

respectively, and the $(s \times s)$ matrix $R(n, \varepsilon)$ has the structure

$$R(n, \varepsilon) = \begin{bmatrix} 0 & 0 & 0 & \dots 0 & 0 \\ r_{11}(n, \varepsilon) & 0 & 0 & \dots 0 & 0 \\ r_{21}(n, \varepsilon) & r_{22}(n, \varepsilon) & 0 & \dots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{s-1,1}(n, \varepsilon) & r_{s-1,2}(n, \varepsilon) & r_{s-1,3}(n, \varepsilon) & \dots r_{s-1,s-1}(n, \varepsilon) & 0 \end{bmatrix} \tag{25}$$

where

$$r_{ij}(n, \varepsilon) = \gamma(n+i, \varepsilon)\, \Phi(n+i, 0; n+j)\, b(n+j, 0), \quad i, j = 1, 2, \dots, s-1. \tag{26}$$

Obviously the relation

$$\beta(n, \varepsilon) = \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, B(n, 0) + \alpha(n, \varepsilon)\, R(n, \varepsilon) \tag{27}$$

is valid between $\alpha_i(n, \varepsilon)$ and $\beta_j(n, \varepsilon)$.

b) A system difference equation can be written in the second form as

$$y(n+s, \varepsilon) + \sum_{i=0}^{s-1} \mu_i(n, \varepsilon)\, y(n+i, 0) = \sum_{j=1}^{s} v_j(n, \varepsilon)\, u(n+j) \tag{28}$$

where the coefficients $\mu_i(n, \varepsilon)$ are given by the row vector

$$\mu(n, \varepsilon) = [\mu_0(n, \varepsilon)\, \mu_1(n, \varepsilon) \dots \mu_{s-1}(n, \varepsilon)] = \\ = -\gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, Q^{-1}(n, 0) \tag{29}$$

and $v_j(n, \varepsilon)$ are the elements of the row vector

$$v(n, \varepsilon) = [v_1(n, \varepsilon)\, v_2(n, \varepsilon) \dots v_s(n, \varepsilon)] = \\ = \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \{B(n, 0) - Q^{-1}(n, 0)\, R(n, 0)\}. \tag{30}$$

Here

$$Q(n, 0) = Q(n, \varepsilon)_{\varepsilon=0} = \begin{bmatrix} c(n, 0) \\ c(n+1, 0)\, \Phi(n+1, 0; n) \\ \vdots \\ c(n+s-1)\, \Phi(n+s-1, 0; n) \end{bmatrix}, \tag{31}$$

$B(n, 0)$ is given by (24) and $R(n, 0)$ of the structure (25) has the elements

$$r_{ij}(n, 0) = c(n+i, 0)\, \Phi(n+i, 0; n+j)\, b(n+j). \tag{32}$$

Obviously

(33) $$\boldsymbol{\nu}(n, \varepsilon) = \gamma(n + s, \varepsilon)\, \Phi(n + s, 0; n)\, \mathbf{B}(n, 0) + \boldsymbol{\mu}(n, \varepsilon)\, \mathbf{R}(n, 0)\,.$$

The following relations can be used for the mutual transformation of the forms (a) and (b):

(34) $$\boldsymbol{\alpha}(n, \varepsilon)\, \mathbf{Q}(n, \varepsilon) = \boldsymbol{\mu}(n, \varepsilon)\, \mathbf{Q}(n, 0)\,,$$

(35)
$$\boldsymbol{\beta}(n, \varepsilon)\, \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{R}(n, \varepsilon)\}^{-1} = \boldsymbol{\nu}(n, \varepsilon)\, \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, 0)\, \mathbf{R}(n, 0)\}^{-1}\,,$$

(36) $$\alpha_i(n, 0) = \mu_i(n, 0)\,; \quad i = 0, 1, \ldots, s - 1$$

and

(37) $$\beta_j(n, 0) = \nu_j(n, 0)\,; \quad j = 1, 2, \ldots, s\,.$$

Proof. a) Gradually applying the operator

(38) $$V^i f(n, \varepsilon) = f(n + i, \varepsilon)\,; \quad i = 0, 1, \ldots, s\,,$$

to the eq. (8b) and using the eq. (8a) we have

(39)
$$\begin{aligned}
y(n, \varepsilon) &= \gamma(n, \varepsilon)\, \mathbf{x}(n, 0)\,, \\
y(n + 1, \varepsilon) &= \gamma(n + 1, \varepsilon)\, [\Phi(n + 1, 0; n)\, \mathbf{x}(n, 0) + \mathbf{b}(n + 1, 0)\, u(n + 1)]\,, \\
&\;\vdots \\
y(n + i, \varepsilon) &= \gamma(n + i, \varepsilon)\, [\Phi(n + i, 0; n)\, \mathbf{x}(n, 0) + \\
&\quad + \sum_{j=1}^{i} \Phi(n + i, 0; n + j)\, \mathbf{b}(n + j, 0)\, u(n + j)]\,, \\
&\;\vdots \\
y(n + s - 1, \varepsilon) &= \gamma(n + s - 1, \varepsilon)\, [\Phi(n + s - 1, 0; n)\, \mathbf{x}(n, 0) + \\
&\quad + \sum_{j=1}^{s-1} \Phi(n + s - 1, 0; n + j)\, \mathbf{b}(n + j, 0)\, u(n + j)]
\end{aligned}$$

and

(40)
$$\begin{aligned}
y(n + s, \varepsilon) &= \gamma(n + s, \varepsilon)\, [\Phi(n + s, 0; n)\, \mathbf{x}(n, 0) + \\
&\quad + \sum_{j=1}^{s} \Phi(n + s, 0; n + j)\, \mathbf{b}(n + j, 0)\, u(n + j)]\,.
\end{aligned}$$

Introducing the $(s \times 1)$ vectors

(41) $$\mathbf{y}(n, \varepsilon) = \begin{bmatrix} y(n, \varepsilon) \\ y(n + 1, \varepsilon) \\ \vdots \\ y(n + s - 1, \varepsilon) \end{bmatrix}$$

and

$$\mathbf{u}(n+1) = \begin{bmatrix} u(n+1) \\ u(n+2) \\ \vdots \\ u(n+s) \end{bmatrix} \tag{42}$$

the set of equations (39) can be written in the vector-matrix form

$$\mathbf{y}(n, \varepsilon) = \mathbf{Q}(n, \varepsilon)\, \mathbf{x}(n, 0) + \mathbf{R}(n, \varepsilon)\, \mathbf{u}(n+1) \tag{43}$$

where the $(s \times s)$ matrices $\mathbf{Q}(n, \varepsilon)$ and $\mathbf{R}(n, \varepsilon)$ are given by (23) and (25) with (26), respectively.

Solving the eq. (43) for $\mathbf{x}(n, 0)$ we get

$$\mathbf{x}(n, 0) = \mathbf{Q}^{-1}(n, \varepsilon)\, [\mathbf{y}(n, \varepsilon) - \mathbf{R}(n, \varepsilon)\, \mathbf{u}(n+1)] \tag{44}$$

provided $\mathbf{Q}(n, \varepsilon)$ is nonsingular.

Substituting now the relation (44) into the equation (40) we can write

$$\begin{aligned} y(n+s, \varepsilon) &= \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{Q}^{-1}(n, \varepsilon)\, [\mathbf{y}(n, \varepsilon) - \mathbf{R}(n, \varepsilon)\, \mathbf{u}(n+1)] + \\ &+ \mathbf{r}_s(n, \varepsilon)\, \mathbf{u}(n+1) \end{aligned} \tag{45}$$

where the $(1 \times s)$ row vector is given by

$$\mathbf{r}_s(n, \varepsilon) = \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{B}(n, 0) \tag{46}$$

if $\mathbf{B}(n, 0)$ has the form (24).

With respect to (46) the equation (45) can be written as

$$\begin{aligned} y(n+s, \varepsilon) &- \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{y}(n, \varepsilon) = \\ &= \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \{\mathbf{B}(n, 0) - \mathbf{Q}^{-1}(n, \varepsilon)\, \mathbf{R}(n, \varepsilon)\}\, \mathbf{u}(n+1) \end{aligned} \tag{47}$$

and the validity of the relations (20), (21) and (22) is evident.

b) Writing the equation (43) for $\varepsilon = 0$ we have

$$\mathbf{y}(n, 0) = \mathbf{Q}(n, 0)\, \mathbf{x}(n, 0) + \mathbf{R}(n, 0)\, \mathbf{u}(n+1) \tag{48}$$

where $\mathbf{Q}(n, 0)$ is given by (31) and $\mathbf{R}(n, 0)$ given by (25) has now the elements (32).

Substituting the solution $\mathbf{x}(n, 0)$ from the equation (48) into the equation (40) we get

$$\begin{aligned} y(n+s, \varepsilon) &= \gamma(n+s, \varepsilon)\, \Phi(n+s, 0; n)\, \mathbf{Q}^{-1}(n, 0)\, [\mathbf{y}(n, 0) - \mathbf{R}(n, 0)\, \mathbf{u}(n+1)] + \\ &+ \mathbf{r}_s(n, \varepsilon)\, \mathbf{u}(n+1)\,. \end{aligned} \tag{49}$$

Using (24)

(50) $$y(n+s,\varepsilon) - y(n+s,\varepsilon)\,\boldsymbol{\Phi}(n+s,0;n)\,\mathbf{Q}^{-1}(n,0)\,\mathbf{y}(n,0) = \\ = y(n+s,\varepsilon)\,\boldsymbol{\Phi}(n+s,0;n)\,\{\mathbf{B}(n,0) - \mathbf{Q}^{-1}(n,0)\,\mathbf{R}(n,0)\}\,\boldsymbol{u}(n+1)$$

and the validity of (28), (29) and (30) is proved.

The relations (34)–(37) results directly comparing the equations (20) and (28).

We can see that the coefficients $\mu_i(n,\varepsilon)$, $\nu_j(n,\varepsilon)$ can be obtained in general by simpler way than $\alpha_i(n,\varepsilon)$, $\beta_j(n,\varepsilon)$.

III. SIMPLIFIED CASES

1. Normalized canonical form

The above relations are much simpler if the system (1) can be described on definite time interval in normalized canonical form [5, pp. 317] as

(51a) $$\dot{\mathbf{x}}(t) = \boldsymbol{b}(t)\,u(t)\,,$$

(51b) $$y(t) = \mathbf{c}(t)\,\mathbf{x}(t)\,.$$

Then according to (4)

$$\boldsymbol{\Phi}(t,t_0) = \boldsymbol{I}$$

and the discrete-time version of state equations has the form

(52) $$\mathbf{x}(n+1,\varepsilon) = \mathbf{x}(n+1,0) = \mathbf{x}(n,0) + \boldsymbol{b}(n+1,0)\,u(n+1)\,, \\ y(n,\varepsilon) = \mathbf{c}(n,\varepsilon)\,\mathbf{x}(n,0)\,.$$

The scalar difference equation has the coefficients given for the form (a) by

(53) $$\boldsymbol{\alpha}(n,\varepsilon) = -\mathbf{c}(n+s,\varepsilon)\,\mathbf{Q}^{-1}(n,\varepsilon)$$

and

(54) $$\boldsymbol{\beta}(n,\varepsilon) = \mathbf{c}(n+s,\varepsilon)\,\{\mathbf{B}(n,0) - \mathbf{Q}^{-1}(n,\varepsilon)\,\mathbf{R}(n,\varepsilon)\}$$

where

(55) $$\mathbf{Q}(n,\varepsilon) = \begin{bmatrix} \mathbf{c}(n,\varepsilon) \\ \mathbf{c}(n+1,\varepsilon) \\ \vdots \\ \mathbf{c}(n+s-1,\varepsilon) \end{bmatrix},$$

(56) $$\mathbf{B}(n,0) = [\boldsymbol{b}(n+1,0)\,\boldsymbol{b}(n+2,0)\ldots\boldsymbol{b}(n+s,0)]$$

and $\mathbf{R}(n, \varepsilon)$ has the elements

$$r_{ij}(n, \varepsilon) = \mathbf{c}(n + i, \varepsilon)\, \mathbf{b}(n + j, 0)\,. \tag{57}$$

The coefficients $\mu(n, \varepsilon)$ and $\nu(n, \varepsilon)$ of the form (b) are simplified by similar way.

2. Stationary system

If the parameters of the system (1) do not vary with time then

$$\Phi(t, t_0) = \mathrm{e}^{\mathbf{A}(t-t_0)} \tag{58}$$

and the state equations are given by

$$\begin{aligned} \mathbf{x}(n + 1, \varepsilon) &= \mathrm{e}^{\mathbf{A}(1+\varepsilon)}\mathbf{x}(n, 0) + \mathrm{e}^{\mathbf{A}\varepsilon}\mathbf{b}\, u(n + 1)\,, \\ y(n, \varepsilon) &= \mathbf{c}\, \mathbf{x}(n, \varepsilon) \end{aligned} \tag{59}$$

or

$$\begin{aligned} \mathbf{x}(n + 1, 0) &= \mathrm{e}^{\mathbf{A}}\mathbf{x}(n, 0) + \mathbf{b}\, u(n + 1)\,, \\ y(n, \varepsilon) &= \mathbf{c}\mathrm{e}^{\mathbf{A}\varepsilon}\, \mathbf{x}(n, 0)\,. \end{aligned} \tag{60}$$

To find the coefficients of scalar difference equation we determine

$$\mathbf{Q}(\varepsilon) = \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathrm{e}^{\mathbf{A}} \\ \vdots \\ \mathbf{c}\mathrm{e}^{\mathbf{A}(s-1)} \end{bmatrix} \mathrm{e}^{\mathbf{A}\varepsilon}\,, \tag{61}$$

$$\mathbf{B} = [\mathrm{e}^{-\mathbf{A}}\mathbf{b};\, \mathrm{e}^{-2\mathbf{A}}\mathbf{b};\, \ldots;\, \mathrm{e}^{-s\mathbf{A}}\mathbf{b}] \tag{62}$$

and

$$r_{ij}(\varepsilon) = \mathbf{c}\mathrm{e}^{\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{A}(i-j)}\mathbf{b}\,. \tag{63}$$

Then

$$\boldsymbol{\alpha} = -\mathbf{c}\mathrm{e}^{\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{A}s}\, \mathbf{Q}^{-1}(\varepsilon) = -\mathbf{c}\mathrm{e}^{\mathbf{A}s} \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathrm{e}^{\mathbf{A}} \\ \vdots \\ \mathbf{c}\mathrm{e}^{\mathbf{A}(s-1)} \end{bmatrix}^{-1} \tag{64}$$

and

$$\boldsymbol{\beta}(\varepsilon) = \mathbf{c}\mathrm{e}^{\mathbf{A}\varepsilon}\mathbf{B} + \boldsymbol{\alpha}\, \mathbf{R}(\varepsilon)\,. \tag{65}$$

By similar way can be obtained

$$\boldsymbol{\mu}(\varepsilon) = -\mathbf{c}\mathrm{e}^{\mathbf{A}\varepsilon}\mathrm{e}^{\mathbf{A}s} \begin{bmatrix} \mathbf{c} \\ \mathbf{c}\mathrm{e}^{\mathbf{A}} \\ \vdots \\ \mathbf{c}\mathrm{e}^{\mathbf{A}(s-1)} \end{bmatrix}^{-1} \tag{66}$$

and

(67) $$\mathbf{v}(\varepsilon) = \mathbf{c}e^{\mathbf{A}\varepsilon}e^{\mathbf{A}s}\mathbf{B} + \mu(\varepsilon)\,\mathbf{R}\,.$$

CONCLUSIONS

Considering an application of discrete-time input to a linear nonstationary system given by continuous-time state equations

a) the state equations in the discrete-time form or the scalar difference equation can be obtained if system transition matrix $\Phi(t, t_0)$ is known;

b) in spite of $\Phi(t, t_0)$ cannot be obtained in general by analytical way the presented relations might be useful for numerical solution;

c) using the above relations the mutual transformation between vector-matrix and scalar discrete-time description is always possible.

EXAMPLE

A discrete-time signal is applied to the continuous system described by

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} e^t \\ 1 \end{bmatrix} u(t)\,,$$

$$y(t) = \begin{bmatrix} 1 & e^{-t} \end{bmatrix} \mathbf{x}(t)\,.$$

Obviously the matrix $\mathbf{A}$ is stationary in this case and the transition matrix can be obtained as

$$\Phi(t, t_0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(t-t_0)} \end{bmatrix}.$$

Let us determine the discrete-time state equations and the corresponding scalar difference equation of this system.

1. Using the relations (9)–(13) we have

$$\mathbf{F}(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(1+\varepsilon)} \end{bmatrix},$$

$$\mathbf{E}(\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-\varepsilon} \end{bmatrix},$$

$$\mathbf{h}(n, \varepsilon) = \begin{bmatrix} e^n \\ e^{-\varepsilon} \end{bmatrix}$$

and

$$\gamma(n, \varepsilon) = \begin{bmatrix} 1 & e^{-(n+2\varepsilon)} \end{bmatrix}.$$

The state equations are given by (7) as

$$\mathbf{x}(n+1,\varepsilon) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-(1+\varepsilon)} \end{bmatrix} \mathbf{x}(n,0) + \begin{bmatrix} e^{n+1} \\ e^{-\varepsilon} \end{bmatrix} u(n+1),$$

$$y(n,\varepsilon) = [1 \;\; e^{-(n+\varepsilon)}]\, \mathbf{x}(n,\varepsilon)$$

or according to (8)

$$\mathbf{x}(n+1,0) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-1} \end{bmatrix} \mathbf{x}(n,0) + \begin{bmatrix} e^{n+1} \\ 1 \end{bmatrix} u(n+1),$$

$$y(n,\varepsilon) = [1 \;\; e^{-(n+2\varepsilon)}]\, \mathbf{x}(n,0).$$

2. Using (23)–(26) we have

$$\mathbf{Q}(n,\varepsilon) = \begin{bmatrix} 1 & e^{-(n+2\varepsilon)} \\ 1 & e^{-(n+2+2\varepsilon)} \end{bmatrix},$$

$$\mathbf{B}(n,0) = \begin{bmatrix} e^{n+1} & e^{n+2} \\ e & e^2 \end{bmatrix}$$

and

$$\mathbf{R}(n,\varepsilon) = \begin{bmatrix} 0 & 0 \\ e^{n+1} + e^{-(n+1+2\varepsilon)} & 0 \end{bmatrix}.$$

From the relation (21) we get

$$\boldsymbol{\alpha}(n,\varepsilon) = \boldsymbol{\alpha} = [e^{-2}; -(1+e^{-2})]$$

and using (22) or (27)

$$\boldsymbol{\beta}(n,\varepsilon) = [-e^{n-1} - e^{-(n+1+2\varepsilon)};\, e^{n+2} + e^{-(n+2+2\varepsilon)}].$$

Using the relation (29) we have

$$\boldsymbol{\mu}(\varepsilon) = \frac{1}{1-e^{-2}} [e^{-2} - e^{-(4+2\varepsilon)};\, e^{-(4+2\varepsilon)} - 1]$$

and from (30) or (33)

$$\boldsymbol{\nu}(n,\varepsilon) = \left[\frac{e^{-(2+2\varepsilon)} - 1}{1-e^{-2}} (e^{n-1} + e^{-(n+1)});\, e^{n+2} + e^{-(n+2+2\varepsilon)} \right].$$

Acknowledgement. The author would like to thank Ing. J. Ježek CSc. (ÚTIA-ČSAV) for his helpful commets.

(Received March 15, 1972.)

[1] V. Strejc at al.: Syntéza regulačních obvodů s číslicovým počítačem. NČSAV, Praha 1965.
[2] H. Freeman: Discrete-Time Systems. John Wiley, New York 1965.
[3] J. Ježek: Algoritmus pro výpočet diskrétního přenosu lineární dynamické soustavy. Kybernetika *4* (1968), 3, 246—259.
[4] L. A. Zadeh, C. A. Desoer: Linear System Theory: The State Space Approach. Mc Graw Hill, New York 1963.
[5] R. E. Kalman, P. L. Falb, M. A. Arbib: Topics in Mathematical System Theory. Mc Graw Hill, New York 1969.

Ing. Václav Soukup, katedra řídicí techniky elektrotechnické fakulty ČVUT (Czech Technical University, Department of Automatic Control), Karlovo nám. 13, 121 35 Praha 2. Czechoslovakia.