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A NEW METHOD FOR THE NONLINEAR APPROXIMATION OF SIGNALS

Part I: The optimal damping factor

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The method for the nonlinear signal approximation proposed in [9] is extended to the case with a strong nonlinearity, i.e. to the case which is important for the analysis of large scale systems. The effectivity of the suggested method is based on the computation of the optimal damping factor in each iteration step.

1. INTRODUCTION

The signal approximation is of a current interest for the analysis and synthesis of control systems ([1], [8], [12]). Especially the solution of problems connected with large scale systems must be based on models which are an approximation of real subsystems ([1], [8]). The signal approximation is a complicated nonlinear problem which has no sufficiently supporting basis in the classical approximation theory ([4], [11]). Therefore different heuristic methods ([2], [5], [11]) must be used for treating this problem. In the paper [9] a modification of the Gauss-Newton method, called the Damped Nonlinear Least Squares method, was derived. This Damped Nonlinear Least Squares (DNLS) method has for the signal approximation some advantages over other known modifications of the Gauss-Newton method:

(a) It is closely connected with the classical linear approximation theory. A suitable linear regression function, which is solved simultaneously with the nonlinear problem, can be introduced. This linear solution allows to demarcate the region in which the linearization of the pertinent nonlinear function can be used.

(b) The condition for the existence and uniqueness of the global minimum can be derived.

(c) It was proved (see Proposition 4.4 in [9]) that the global minimum of the nonlinear case and the unique minimum of the pertinent linear case are identical.

In [9] the case with a week nonlinearity, i.e. the case with the starting function near tne global minimum, was analyzed. We expect the use of the DNLS method in identification problems and here starting functions are mostly far from the global minimum. Deriving the procedure, which solves the case with a strong nonlinearity, is therefore desirable. In the present paper we shall deal above all with the following problems:

(i) Relations between the linear and nonlinear parts of the DNLS method for the damping factor ${}^j\mu < 1$ will be derived.

(ii) Two procedures for deriving the optimal damping factor ${}^j\mu_{opt}$ will be proposed.

(iii) Convergence of the DNLS method will be proved (in Part II).

(iv) The pertinent algorithm will be given (in Part II).

2. SIGNAL DESCRIPTION

We shall consider original signals $y(t) \in V$ resp. substitute signals $\bar{y}(t) \in \bar{V}$ ($\bar{V} \subset V$) which are the outputs of the stable single input-single output systems S resp. $\bar{S}$ to the same deterministic input signal $u(t) \in V_u$ for the zero initial conditions. The signal approximation is based on the output signals and so the external description of the systems is more useful. The original system S pertinent to the original signal $y(t) \in V$ is then given by

$$S: y(s) = F(s)\, u(s) = \frac{M(s)}{N(s)}\, u(s)\,, \tag{2.1}$$

where s is the complex variable, $y(s)$ resp. $u(s)$ are the Laplace transforms of $y(t)$ resp. $u(t)$ and $M(s)$ resp. $N(s)$ are Hurwitz polynomials of degree m resp. n without common factor.

The substitute system $\bar{S}$ pertinent to the substitute signal $\bar{y}(t) \in \bar{V}$ is given by

$$\bar{S}: \bar{y}(s) = \bar{F}(s, \boldsymbol{b}, \boldsymbol{a})\, u(s) = \frac{\bar{M}(s)}{\bar{N}(s)}\, u(s)\,, \tag{2.2}$$

where $\bar{M}(s) = 1 + \sum_{k=1}^{\bar{m}} b_k s^k$ and $\bar{N}(s) = \sum_{i=0}^{\bar{n}} a_i s^i$ are Hurwitz polynomials without common factor with $\bar{m} < m$ and $\bar{n} < n$.

The substitute signal $\bar{y}(t, \boldsymbol{b}, \boldsymbol{a}) \in \bar{V}$ which is decisive for the signal approximation, is nonlinear only in the coefficients a_i ($i = 0, 1, \ldots, \bar{n}$) (cf. [9]). For the sake of simplicity we shall consider only the nonlinear part of the approximation problem, i.e. $\bar{F}(s)$ with $\bar{M}(s) = 1$

$$\bar{F}(s, \boldsymbol{a}) = \frac{1}{\sum_{i=0}^{\bar{n}} a_i s^i}\,. \tag{2.3}$$

This case is more illustrative for explanation and more useful for applications. Later (in Section 8) we shall show how the signal approximation starting from the substitute transfer function (2.2) can be simply transformed in the considered case.

3. THE DNLS METHOD

First let us formulate the signal approximation in the continuous case, i.e. in the Hilbert space $L_2(0, \infty)$ with the norm $\|\bar{y}\| = [\int_0^\infty \bar{y}^2(t)\, dt]^{1/2}$ and the scalar product $(\bar{y}, z) = \int_0^\infty \bar{y}(t)\, z(t)\, dt$, in the following way:

To the given $y(t) \in V$ find the substitute function $\bar{y}(t, \boldsymbol{a}) \in \bar{V}(\bar{V} \subset V)$ such that

$$Q(\boldsymbol{a}) = \|y - \bar{y}\|^2 = \int_0^\infty [y(t) - \bar{y}(t, \boldsymbol{a})]^2\, dt \tag{3.1}$$

takes the minimum value.

We consider the continuous case and in this way we obtain more comprehensive and more illustrative results. The time interval $t \in (0, \infty)$ allows to calculate the norms and the scalar products only from the coefficients of the transfer functions (see e.g. [7]) and makes easy to verify the derived algorithms (for $y(\infty) = 0$ and $\bar{y}(\infty) = 0$).

Further we assume that the degree $\bar{n}$ of the polynomial $\bar{N}(s)$ is known.

The DNLS method was derived and its relation to the other modifications of the Gauss-Newton method was discussed in [9]. Here we shall present only the idea of this method and the resulting relations. Let us start with the gradient vector of the nonlinear part of the DNLS method. Let us denote the first partial derivatives of $\bar{y}(t, \boldsymbol{a})$ by

$$\partial_i\, \bar{y}(t, \boldsymbol{a}) = \frac{\partial}{\partial a_i}\, \bar{y}(t, \boldsymbol{a}) \quad (i = 0, 1, \ldots, \bar{n}) . \tag{3.2}$$

Then the gradient vector $\boldsymbol{g}(t, \boldsymbol{a})$ of $\bar{y}(t, \boldsymbol{a})$ at $\boldsymbol{a}$ is defined by

$$\boldsymbol{g}(t, \boldsymbol{a}) = [\partial_0\, \bar{y}(t, \boldsymbol{a}), \partial_1(\bar{y}(t, \boldsymbol{a}), \ldots, \partial_{\bar{n}}\, \bar{y}(t, \boldsymbol{a})]^{\mathrm{T}} = \tag{3.3}$$
$$= [-v^{(0)}(t), -v^{(1)}(t), \ldots, -v^{(\bar{n})}(t)]^{\mathrm{T}} .$$

The gradient vector $\boldsymbol{g}_L(t, \boldsymbol{a}) = -\boldsymbol{g}(t, \boldsymbol{a})$ pertinent to the linear part of the DNLS method follows directly from the signal $\bar{y}(t, \boldsymbol{a})$ written in the form of the regression function, i.e.

$$\bar{y}(t, \boldsymbol{a}) = \sum_{i=0}^{\bar{n}} a_i\, v^{(i)}(t) . \tag{3.4}$$

This can be easily verified by the pertinent Laplace transform

$$\mathscr{L}\{\bar{y}(t, \boldsymbol{a})\} = \mathscr{L}\{\sum_{i=0}^{\bar{n}} a_i\, v^{(i)}(t)\} = \frac{\sum_{i=0}^{\bar{n}} a_i s^i}{\bar{N}^2(s)}\, u(s) = \frac{u(s)}{\bar{N}(s)}, \tag{3.5}$$

where

$$\mathscr{L}\{v^{(i)}\}(t)\} = \frac{s^i}{\bar{N}^2(s)}\, u(s) .$$

The DNLS method is now given by two relations

$$ {}^{j+1}\boldsymbol{a} = {}^{j}\boldsymbol{a} + {}^{j}\mu \boldsymbol{G}^{-1}({}^{j}\boldsymbol{a}) \int_{0}^{\infty} \boldsymbol{g}(t, {}^{j}\boldsymbol{a}) \left[y(t) - {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) \right] \mathrm{d}t \,, \tag{3.6} $$

$$ {}^{j+1}\hat{\boldsymbol{a}} = {}^{j}\boldsymbol{a} + {}^{j}\mu \boldsymbol{G}^{-1}({}^{j}\boldsymbol{a}) \int_{0}^{\infty} \boldsymbol{g}_L(t, {}^{j}\boldsymbol{a}) \left[y(t) - {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) \right] \mathrm{d}t \,, \tag{3.7} $$

with $\boldsymbol{g}_L(t, {}^{j}\boldsymbol{a}) = -\boldsymbol{g}(t, {}^{j}\boldsymbol{a})$ and with the nonsingular matrix

$$ \boldsymbol{G}({}^{j}\boldsymbol{a}) = \int_{0}^{\infty} \boldsymbol{g}(t, {}^{j}\boldsymbol{a}) \, \boldsymbol{g}^{\mathrm{T}}(t, {}^{j}\boldsymbol{a}) \, \mathrm{d}t \,. $$

Superscripts on the left indicate the iteration steps and ${}^{j}\mu \in (0, 1)$ is a damping factor.

Let us discuss the relevancy of the damping factor ${}^{j}\mu$. All modifications of the Gauss-Newton method can be used for the solution of problems in which the following condition approximately holds

$$ \boldsymbol{g}(t, {}^{j+1}\boldsymbol{a}) \doteq \boldsymbol{g}(t, {}^{j}\boldsymbol{a}) \,. \tag{3.8} $$

The fulfilment of this condition can be governed with the help of the damping factor ${}^{j}\mu$, e.g. for the damping factor ${}^{j}\mu \to 0$ condition (3.8) always holds. Therefore the damping factor must be selected as large as possible to get a rapid convergence and on the other side condition (3.8) must be fulfilled. The fulfilment of condition (3.8) can be tested by the linear part of the DNLS method given by relation (3.7). This linear case serves as an etalon for the solution of the pertinent nonlinear part given by relation (3.6).

Let us remark that the DNLS method in discrete version can be used for the solution of a more general problem – for the approximation from a curve of the function $y(t)$. This fact opens the way for the use of the DNLS method in identification problems in which the transfer function (2.1) is unknown.

4. MAIN PROPOSITIONS OF THE DNLS METHOD

If we multiply relations (3.6) and (3.7) by the gradient vector $\boldsymbol{g}_L(t, {}^{j}\boldsymbol{a})$, we obtain the following functions

$$ {}^{j}\bar{\bar{y}}(t, {}^{j}\boldsymbol{a}, {}^{j+1}\boldsymbol{a}, {}^{j}\mu) = {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) - {}^{j}\mu \Delta \, {}^{j}\bar{y}(t) = \sum_{i=0}^{\bar{n}} {}^{j+1}a_i \, {}^{j}v^{(i)}(t) \,, \tag{4.1} $$

$$ {}^{j}z(t, {}^{j}\boldsymbol{a}, {}^{j+1}\hat{\boldsymbol{a}}, {}^{j}\mu) = {}^{j}\bar{y}(t, {}^{j}\boldsymbol{a}) + {}^{j}\mu \Delta \, {}^{j}\bar{y}(t) = \sum_{i=0}^{\bar{n}} {}^{j+1}\hat{a}_i \, {}^{j}v^{(i)}(t) \,, \tag{4.2} $$

where

$$ \Delta \, {}^{j}\bar{y}(t) = \sum_{i=0}^{\bar{n}} \Delta \, {}^{j}a_i \, {}^{j}v^{(i)}(t) = \sum_{i=0}^{\bar{n}} ({}^{j+1}\hat{a}_i - {}^{j}a_i) \, {}^{j}v^{(i)}(t) \,. $$

As it was shown in [9] relation (3.7) pertinent to the linear part of the DNLS method can be written in the following simpler form (for ${}^j\mu = 1$) and $\boldsymbol{G}_L({}^j\boldsymbol{a}) = \boldsymbol{G}({}^j\boldsymbol{a})$

$$(4.3) \qquad {}^{j+1}\hat{\boldsymbol{a}} = \boldsymbol{G}_L^{-1}({}^j\boldsymbol{a}) \int_0^\infty \boldsymbol{g}_L(t, {}^j\boldsymbol{a})\, y(t)\, \mathrm{d}t\,.$$

This relation is well known from the linear approximation theory (cf. [4], [6]). The minimal solution can be obtained in this case by an orthogonal projection, i.e. from the equations

$$(4.4) \qquad (y - {}^jz, {}^jv^{(i)}) = 0 \quad (i = 0, 1, \ldots, \bar{n})\,.$$

Supposing ${}^j\mu = 1$ the error of this linear approximation is then given by the relation (cf. [4])

$$(4.5) \qquad {}^j\delta^2 = \|y - {}^jz\|^2 = (y, y) - (y, {}^jz) = (y, y) - \sum_{i=0}^{\bar{n}} {}^{j+1}\hat{a}_i(y, {}^jv^{(i)})\,.$$

This relation follows from equations (4.4) multiplied by the coefficients ${}^{j+1}\hat{a}_i$

$$(4.6) \qquad \sum_{i=0}^{\bar{n}} {}^{j+1}\hat{a}_i(y - {}^jz, {}^jv^{(i)}) = (y - {}^jz, {}^jz) = (y, {}^jz) - ({}^jz, {}^jz) = 0\,.$$

Similarly we obtain for ${}^j\boldsymbol{a}$ resp ${}^{j+1}\boldsymbol{a}$ (for ${}^j\mu = 1$)

$$(4.7) \qquad (y - {}^jz, {}^j\bar{y}) = (y, {}^j\bar{y}) - ({}^j\bar{y}, {}^jz) = 0$$

resp.

$$(4.8) \qquad (y - {}^jz, {}^j\bar{y}) = (y, {}^j\bar{y}) - ({}^jz, {}^j\bar{y}) = 0\,.$$

The considered functionals can be used if the pertinent systems are stable. The following definition must be thus introduced:

Definition 4.1. The vector of the coefficients pertinent to the substitute system with the transfer function ${}^j\bar{F}(s, \boldsymbol{a}) = {}^j\bar{N}^{-1}(s)$ is an element of the subset $\Omega_{\boldsymbol{a}}$ of stable vectors $\boldsymbol{a}$, if the polynomial ${}^j\bar{N}(s) = \sum_{i=0}^{\bar{n}} {}^ja_is^i$ fulfils Routh-Hurwitz conditions of stability.

Let us emphasize that the control stability is the sufficient condition for the existence of the nonsingular matrix $\boldsymbol{G}_L({}^j\boldsymbol{a}) = \boldsymbol{G}({}^j\boldsymbol{a})$. This follows from the backwards analysis. Each stable signal given by relation (2.3) can be written in the form of regression function (3.4) and therefore the backwards linear approximation formulated by relation (4.3) must exist.

4.1. Basic parameters

Using the DNLS method we must distinguish two groups of parameters. The functionals or parameters, which do not depend on the damping factor, belong to the first group. The second group will be analyzed in paragraph 4.2. So we can directly calculate from the given vector of coefficients ${}^j\boldsymbol{a}$ the total error

(4.9) $${}^j\eta = \|y - {}^j\bar{y}\|$$

and the pertinent functionals $\|{}^jy\|^2$ and $(y, {}^jy)$. The most important is the total error which is composed from two errors (Proposition 4.2 in [9])

(4.10) $${}^j\eta^2 = {}^j\delta^2 + {}^j\varphi^2\,,$$

where ${}^j\delta$ is the error of the linear solution given by (3.7) resp. (4.5) and ${}^j\varphi = \|\Delta\,{}^jy\|$ is the error of the sensitivity functions given by (4.1) resp. (4.2). These two components of the total error can be computed after solving the system of linear equations according to (3.6) and (3.7).

In [9] it was shown that the functional ${}^j\varrho = ({}^j\bar{y}, \Delta\,{}^j\bar{y})$ is useful for the classification of initial vectors ${}^0\boldsymbol{a}$, i.e. for the classification of substitute systems. Its connection to the other functionals is given by the following relations

(4.11) $${}^j\varrho = ({}^j\bar{y}, \Delta\,{}^j\bar{y}) = ({}^j\bar{y}, {}^jz - {}^j\bar{y}) = (y, {}^j\bar{y}) - \|{}^j\bar{y}\|^2$$

resp.

$${}^j\varrho + {}^j\varphi^2 = (y, \Delta\,{}^j\bar{y}) = (y, {}^jz - {}^j\bar{y}) = (y, {}^jz) - (y, {}^j\bar{y})$$

which were proved with regard to relations (4.2), (4.6) and (4.7) in [9] for ${}^j\mu = 1$.

4.2. The influence of the damping factor

The square of the norms of functions (4.1) resp. (4.2) can be now written in the following form

(4.12) $$\|{}^j\bar{y}\|^? = \|{}^j\bar{y}\|^2 - 2\,{}^j\mu\,{}^j\varrho + {}^j\mu^2\,{}^j\varphi^2$$

resp.

(4.13) $$\|{}^jz\|^2 = \|{}^j\bar{y}\|^2 + 2\,{}^j\mu\,{}^j\varrho + {}^j\mu^2\,{}^j\varphi^2\,.$$

Let us analyze the relations which are connected with the linear part of the DNLS method. The following proposition will be useful.

Proposition 4.1. If $\{{}^j\boldsymbol{a}\} \in \Omega_a$ then for the functions $y \in V$, ${}^jz \in \bar{\bar{V}}(\bar{\bar{V}} \subset V)$ and ${}^j\bar{y} \in \bar{V}(\bar{V} \subset V)$ the following relations hold

(4.14) $$(y, {}^jz) - \|{}^jz\|^2 = (1 - {}^j\mu)({}^j\varrho + {}^j\mu\,{}^j\varphi^2)\,,$$

(4.15) $$(y, {}^j\bar{y}) - ({}^jz, {}^j\bar{y}) = (1 - {}^j\mu)\,{}^j\varrho\,,$$

(4.16) $$\|{}^jz\|^2 - (y, {}^j\bar{y}) = {}^j\mu^2\,{}^j\varphi^2 - (1 - 2\,{}^j\mu)\,{}^j\varrho\,,$$

(4.17) $$\|y - {}^jz\|^2 = {}^j\delta^2 + (1 - {}^j\mu)^2\,{}^j\varphi^2\,.$$

Proof. With regard to (4.2), (4.11) and (4.13) we can write (4.14) resp. (4.15) in the form

$$(y, {}^j\bar{y} + {}^j\mu\Delta\,{}^j\bar{y}) - \|{}^jz\|^2 = (y, {}^j\bar{y}) + {}^j\mu(y, \Delta\,{}^j\bar{y}) - \|{}^jz\|^2 =$$
$$= (y, {}^j\bar{y}) + {}^j\mu({}^j\varrho + {}^j\varphi^2) - \|{}^j\bar{y}\|^2 - 2{}^j\mu\,{}^j\varrho - {}^j\mu^2\,{}^j\varphi^2 = (1 - {}^j\mu)({}^j\varrho + {}^j\mu\,{}^j\varphi^2)$$

resp.

$$(y, {}^j\bar{y}) - \|{}^j\bar{y}\|^2 - {}^j\mu({}^j\bar{y}, \Delta\, {}^j\bar{y}) = (1 - {}^j\mu)\, {}^j\varrho\,.$$

Relation (4.16) follows directly from (4.13) and relation (4.17) can be obtained from relations (4.5), (4.11), (4.13), and from the relation

$$\|y\|^2 - (y, {}^j\bar{y}) = {}^j\delta^2 + {}^j\varphi^2 + {}^j\varrho \tag{4.18}$$

which was proved in [9]. □

The relations connected with the nonlinear part of the DNLS method are proved in the following proposition:

Proposition 4.2. If $\{{}^ja\} \in \Omega_a$ then for the functions $y \in V$, ${}^jz, {}^j\bar{y} \in \bar{V}$ $(\bar{V} \subset V)$ and $\bar{y} \in \bar{V}(\bar{V} \subset V)$ the following relations hold:

$$\|{}^j\bar{y} - {}^jz\|^2 = 4\, {}^j\mu^2\, {}^j\varphi^2\,, \tag{4.19}$$

$$\|y - {}^j\bar{y}\|^2 = {}^j\delta^2 + (1 + {}^j\mu)^2\, {}^j\varphi^2\,, \tag{4.20}$$

$$(y, {}^j\bar{y}) - (y, {}^j\bar{y}) = {}^j\mu({}^j\varrho + {}^j\varphi^2)\,, \tag{4.21}$$

$$\|{}^j\bar{y}\|^2 - (y, {}^j\bar{y}) = {}^j\mu\, {}^j\varphi^2 - (1 - {}^j\mu)\, {}^j\varrho\,, \tag{4.22}$$

$$({}^j\bar{y}, {}^j\bar{y}) - (y, {}^j\bar{y}) = {}^j\mu\, {}^j\varphi^2 - {}^j\varrho\,. \tag{4.23}$$

Proof. Relation (4.19) resp. (4.21) follows directly from relation (4.1) resp. (4.11). Relation (4.20) can be written in the form

$$\|y - {}^j\bar{y}\|^2 = \|y\|^2 - 2(y, {}^j\bar{y}) + \|{}^j\bar{y}\|^2 =$$
$$= \|y\|^2 - 2(y, {}^j\bar{y}) + 2\, {}^j\mu(y, \Delta\, {}^j\bar{y}) + \|{}^j\bar{y}\|^2$$

and with respect to (4.11) and (4.12) we obtain (4.20).

Relation (4.22) resp. (4.23) can be arranged

$$\|{}^j\bar{y}\|^2 - (y, {}^j\bar{y}) = \|{}^j\bar{y}\|^2 - (y, {}^j\bar{y}) + {}^j\mu(y, \Delta\, {}^j\bar{y})$$

resp.

$$({}^j\bar{y}, {}^j\bar{y}) - (y, {}^j\bar{y}) = \|{}^j\bar{y}\|^2 - {}^j\mu({}^j\bar{y}, \Delta\, {}^j\bar{y}) - (y, {}^j\bar{y}) + {}^j\mu(y, \Delta\, {}^j\bar{y})$$

and with respect to (4.11) we obtain (4.22) resp. (4.23). □

Let us discuss one property which is typical of all modifications of the Gauss-Newton method. Only the gradient vector $\boldsymbol{g}(t, {}^j\boldsymbol{a})$ is used in the jth iteration step. That is the difference to the Newton method. The change of the gradient vector $\boldsymbol{g}(t, {}^j\boldsymbol{a})$ into the new gradient vector $\boldsymbol{g}(t, {}^{j+1}\boldsymbol{a})$ is not under the computational control (cf. [9]). The new gradient vector $\boldsymbol{g}(t, {}^{j+1}\boldsymbol{a})$ is calculated directly from the new coefficients ${}^{j+1}a_i$ $(i = 0, 1, \ldots, \bar{n})$ at the beginning of the $(j + 1)$th step. Now let us show how the DNLS method respects this change of the gradient vectors. We start from the following reconstruction of the function ${}^{j+1}\bar{y}(t)$ from the functionals of the jth step based on relations (4.1), (4.2), (4.19) and (4.20) (cf. relation (3.12) in [9])

$$(4.24)\quad {}^{j+1}\bar{y}(t, {}^{j+1}\boldsymbol{a}) = \sum_{i=0}^{\bar{n}} {}^{j+1}a_i\, {}^{j+1}v^{(i)}(t) \doteq {}^{j}\bar{y}(t) + 2\,{}^{j}\mu \sum_{i=0}^{\bar{n}} \Delta\, {}^{j}a_i\, {}^{j}v^{(i)}(t) =$$

$$= \sum_{i=0}^{\bar{n}} {}^{j+1}a_i\, {}^{j}v^{(i)}(t) + 2\,{}^{j}\mu \sum_{i=0}^{\bar{n}} \Delta\, {}^{j}a_i\, {}^{j}v^{(i)}(t) = \sum_{i=0}^{\bar{n}} {}^{j+1}\hat{a}_i\, {}^{j}v^{(i)}(t) = {}^{j}z(t)\,.$$

Therefore the following basic relations can be written as

$$(4.25)\qquad (y, {}^{j+1}\bar{y}) = (y, {}^{j}z) + v_1({}^{j}\mu)\,,$$

$$(4.26)\qquad \|{}^{j+1}\bar{y}\|^2 = \|{}^{j}z\|^2 + v_2({}^{j}\mu)\,,$$

where errors $v_1({}^{j}\mu)$ and $v_2({}^{j}\mu)$ characterize the quality of the prediction of the function ${}^{j+1}\bar{y}(t)$ on the basis of the linear case given by (3.7) resp. (4.2). Now it is clear why the function ${}^{j}z(t)$ is considered as an etalon for deriving the function ${}^{j+1}\bar{y}(t)$. The fulfillment of relations (4.25) and (4.26), i.e. the magnitude of errors $v_1({}^{j}\mu)$ and $v_2({}^{j}\mu)$, can be governed by the damping factor ${}^{j}\mu$.

The advantage of the DNLS method over all other modifications of the Gauss-Newton method lies in the fact that two important functionals ${}^{j+1}\varphi^2$ and ${}^{j+1}\varrho$ can be predicted.

Proposition 4.3. If

(i) $\{{}^{j}\boldsymbol{a}\} \in \Omega_a$,

(ii) $y \in V$, ${}^{j}z$, ${}^{j}\bar{y} \in \bar{V} (\bar{V} \subset V)$, ${}^{j}\bar{y} \in \bar{V} (\bar{V} \subset V)$,

then the following relations hold

$$(4.27)\qquad {}^{j+1}\varrho = {}^{j+1}\varrho_P + v_1({}^{j}\mu) - v_2({}^{j}\mu)\,,$$

$$(4.28)\qquad {}^{j+1}\varphi^2 = {}^{j+1}\varphi_P^2 + v_2({}^{j}\mu) - 2v_1({}^{j}\mu) + {}^{j}\delta^2 - {}^{j+1}\delta^2 =$$

$$= {}^{j+1}\varphi_P^2 + v_3({}^{j}\mu) + {}^{j}\delta^2 - {}^{j+1}\delta^2\,,$$

where

$${}^{j+1}\varrho_P = (1 - {}^{j}\mu)({}^{j}\varrho + {}^{j}\mu\, {}^{j}\varphi^2)\,, \quad {}^{j+1}\varphi_P^2 = (1 - {}^{j}\mu)^2\, {}^{j}\varphi^2\,,$$

and

$$v_3({}^{j}\mu) = v_2({}^{j}\mu) - 2v_1({}^{j}\mu)\,.$$

Proof. Relation (4.27) resp. (4.28) can be arranged taking into account (4.11), (4.13), (4.25), (4.26) resp. (4.11), (4.18), (4.25), (4.27)

$${}^{j+1}\varrho = (y, {}^{j+1}\bar{y}) - \|{}^{j+1}\bar{y}\|^2 = (y, {}^{j}z) - \|{}^{j}z\|^2 + v_1({}^{j}\mu) - v_2({}^{j}\mu) =$$

$$= (y, {}^{j}\bar{y}) + {}^{j}\mu(y, \Delta\, {}^{j}\bar{y}) - \|{}^{j}\bar{y}\|^2 - 2\,{}^{j}\mu\, {}^{j}\varrho - {}^{j}\mu^2\, {}^{j}\varphi^2 + v_1({}^{j}\mu) -$$

$$- v_2({}^{j}\mu) = (1 - {}^{j}\mu)({}^{j}\varrho + {}^{j}\mu\, {}^{j}\varphi^2) + v_1({}^{j}\mu) - v_2({}^{j}\mu)$$

resp.

$${}^{j+1}\varphi^2 = \|y\|^2 - (y, {}^{j+1}\bar{y}) - {}^{j+1}\delta^2 - {}^{j+1}\varrho = \|y\|^2 - (y, {}^{j}\bar{y}) - {}^{j}\mu({}^{j}\varrho + {}^{j}\varphi^2) -$$

$$- v_1({}^j\mu) - {}^{j+1}\delta^2 - (1 - {}^j\mu)({}^j\varrho + {}^j\mu\, {}^j\varphi^2) - v_1({}^j\mu) + v_2({}^j\mu) =$$
$$= (1 - {}^j\mu)({}^j\varrho + {}^j\varphi^2) - (1 - {}^j\mu)({}^j\varrho + {}^j\mu\, {}^j\varphi^2) + v_3({}^j\mu) + {}^j\delta^2 - {}^{j+1}\delta^2 =$$
$$= (1 - {}^j\mu)^2\, {}^j\varphi^2 + v_3({}^j\mu) + {}^j\delta^2 - {}^{j+1}\delta^2 . \qquad \square$$

Relation (4.28) can be written in other forms

$$(4.29) \qquad {}^{j+1}\varphi^2 + {}^{j+1}\delta = {}^{j+1}\eta^2 = {}^j\delta^2 + {}^{j+1}\varphi_P^2 + v_3({}^j\mu)$$

or with the help of relation (4.27)

$$(4.30) \qquad {}^{j+1}\eta^2 = {}^j\delta^2 + {}^{j+1}\varphi_P^2 + {}^{j+1}\varrho_P - {}^{j+1}\varrho - v_1({}^j\mu) =$$
$$= {}^j\delta^2 + (1 - {}^j\mu)({}^j\varrho + {}^j\varphi^2) - v_4({}^j\mu) ,$$

where $v_4({}^j\mu) = {}^{j+1}\varrho + v_1({}^j\mu)$.

All predictions in Proposition 4.3 are related to the linear etalon ${}^jz(t)$. The change of this etalon must be tested to demarcate the region in which Proposition 4.3 can be efficiently used. Let us define the DNLS sequence.

Definition 4.2. If the sequence $\{{}^j\boldsymbol{a}\}$ generated from relation (3.6) satisfies the following properties

(i) $\{{}^j\boldsymbol{a}\} \in \Omega_{\bar{n}}\ (\Omega_{\bar{n}} \subset \Omega_a)$,
(ii) $y \in V,\ \ {}^jz, {}^j\bar{y} \in \overline{V}\ (\overline{V} \subset V),\ \ {}^{j+1}\bar{y}, {}^j\bar{y} \in \overline{V}\ (\overline{V} \subset V)$,
(iii) ${}^j\pi = ({}^j\varphi^2 - {}^{j+1}\varphi^2({}^j\mu))\,\big|{}^j\delta^2 - {}^{j+1}\delta^2({}^j\mu)\big| > 1$ for ${}^j\eta^2 > {}^{j+1}\eta^2({}^j\mu)$ and ${}^j\mu \in (0, 1)$,

then it is the DNLS sequence.

Hypothesis (iii) tests the reliability of the etalon ${}^jz(t)$. We can write according to relation (4.29)

$$(4.31) \qquad {}^j\eta^2 - {}^{j+1}\eta^2({}^j\mu) = {}^j\eta^2 - {}^j\delta^2 - {}^{j+1}\varphi_P^2 - v_3({}^j\mu) = {}^j\varphi^2 - {}^{j+1}\bar{\varphi}^2 ,$$

where ${}^{j+1}\bar{\varphi}^2 = {}^{j+1}\varphi_P^2 + v_3({}^j\mu)$ is the prediction for ${}^{j+1}\delta = {}^j\delta$. So we ask that it holds for ${}^j\delta > {}^{j+1}\delta$

$${}^j\varphi^2 - {}^{j+1}\bar{\varphi}^2 > 2\big|{}^j\delta^2 - {}^{j+1}\delta^2\big| .$$

4.3. The optimal solution

The condition for deriving the minimum of the total error can be written in two forms

$$(4.32) \qquad \frac{\mathrm{d}v_3({}^j\mu)}{\mathrm{d}{}^j\mu} = - \frac{\mathrm{d}\, {}^{j+1}\varphi_P^2}{\mathrm{d}\mu} = 2(1 - {}^j\mu)\, {}^j\varphi^2$$

or

$$(4.33) \qquad \frac{\mathrm{d}v_4({}^j\mu)}{\mathrm{d}{}^j\mu} = - ({}^j\varrho + {}^j\varphi^2) .$$

The second relation resp. the derivate of relation (4.30), i.e.

$$\text{(4.34)} \qquad \frac{\mathrm{d}({}^{j+1}\eta^2 + {}^{j+1}\varrho + v_1({}^j\mu))}{\mathrm{d}^j\mu} = -({}^j\varrho + {}^j\varphi^2),$$

illustrates the importance of the distance between the jth function and the global minimum. Both decisive parameters, i.e. ${}^j\varrho$ and ${}^j\varphi^2$, must go by the convergent sequence $\{{}^j\boldsymbol{a}\}$ to zero (cf. Proposition 4.4 in [9]).

The result of an iteration step can be delayed $(v_1({}^j\mu) < 0)$ or accelerated $(v_1({}^j\mu) > 0)$ with respect to the prediction given by Proposition 4.3. The second case gives better results as they are predicted by Proposition 4.3. It shows the following proposition. First let us introduce the notation $(\cdot)({}^j\mu = 1) = (\cdot)(1)$, e.g. ${}^{j+1}\eta^2({}^j\mu = 1) = {}^{j+1}\eta^2(1)$.

Proposition 4.4. If

(i) $\{{}^j\boldsymbol{a}\}$ is the DNLS sequence,

(ii) $v_1({}^j\mu) > 0$ and $\mathrm{d}v_1({}^j\mu)/\mathrm{d}^j\mu > 0$ for ${}^j\mu \in (0, 1)$,

(iii) ${}^j\varrho > 0$ and ${}^j\delta > {}^{j+1}\delta$,

then there exists a region of solutions which satisfy the condition

$$\text{(4.35)} \qquad {}^{j+1}\delta^2 < {}^{j+1}\eta^2({}^j\mu) < {}^j\delta^2 \quad \text{for} \quad {}^j\mu < 1\,.$$

Proof. The condition

$$\text{(4.36)} \qquad {}^{j+1}\eta(1) + {}^{j+1}\varrho(1) < {}^j\delta^2$$

respecting relation (4.30) and hypothesis (ii) shows that there exists an intersection of the trajectory of solutions for ${}^j\mu \in (0, 1)$ in coordinates $(y, {}^{j+1}\bar{y}({}^j\mu))$ and $\|{}^{j+1}\bar{y}({}^j\mu)\|^2$ with the line given by the ideal point $(y, {}^jz(1)) = \|{}^jz(1)\|^2$. Now the following relations can be written with respect to (4.5) and (4.18) resp. (4.11) and (4.18)

$$\text{(4.37)} \qquad \|y\|^2 - (y, {}^{j+1}\bar{y}({}^j\mu_a)) = {}^{j+1}\eta^2({}^j\mu_a) + {}^{j+1}\varrho({}^j\mu_a) = \\ = \|y\|^2 - (y, {}^jz(1)) = {}^j\delta^2 \quad \text{for} \quad {}^j\mu_a \in (0, 1)\,,$$

resp.

$$\text{(4.38)} \qquad \|y\|^2 - \|{}^{j+1}\bar{y}({}^j\mu_b)\|^2 = {}^{j+1}\eta({}^j\mu_b) + 2\,{}^{j+1}\varrho({}^j\mu_b) = \\ = \|y\|^2 - \|{}^jz(1)\|^2 = {}^j\delta^2 \quad \text{for} \quad {}^j\mu_b \in (0, 1)\,.$$

Therefore the relation

$$\text{(4.39)} \qquad {}^{j+1}\eta^2({}^j\mu) + {}^{j+1}\varrho({}^j\mu) < {}^j\delta^2$$

holds for ${}^j\mu \in ({}^j\mu_a, 1)$.
Further it holds with regard to (4.30) and (4.33) for ${}^j\mu \to {}^j\mu_{opt}$

$$\text{(4.40)} \qquad \frac{\mathrm{d}^{j+1}\varrho}{\mathrm{d}^j\mu} < -({}^j\varrho + {}^j\varphi^2) = \frac{\mathrm{d}}{\mathrm{d}^j\mu}({}^{j+1}\varrho_P + {}^{j+1}\varphi_P^2) < \frac{\mathrm{d}^{j+1}\varrho_P}{\mathrm{d}^j\mu}.$$

Relation (4.35) follows from (4.27), (4.38) to (4.40) for ${}^{j+1}\varrho_P({}^j\mu_b) > {}^{j+1}\varrho({}^j\mu_b) > 0$ $({}^{j+1}\varrho_P(0) = {}^{j+1}\varrho(0) = {}^j\varrho)$ and $v_1({}^j\mu_b) < v_2({}^j\mu_b) < v_1({}^j\mu_b) + {}^{j+1}\varrho_P({}^j\mu)$. □

The given propositions open a number of possibilities how to find the optimal damping factor ${}^j\mu_{opt}$. Two alternatives will be derived in the next section.

5. DERIVING THE OPTIMAL DAMPING FACTOR

It is clear from the given relations that the heuristic approach must be used by the signal approximation in the cases with a strong nonlinearity. The meaning of the term "heuristics" is vague in the literature. Two extreme groups of heuristic procedures must be distinguished. The heuristic procedures, which use a great number of experiments, e.g. to scan some region, belong to the first group. On the other side there are heuristic procedures which use a small number of experiments, e.g. only two experiments, and so they are near to deductive methods. Some deductive conception must exist and a few inductive steps are used to get the best strategy for solving the considered problem. Here we can speak about semiheuristic or semi-deductive procedures. This second case will be used by deriving the optimal damping factor. The deductive basis for so considered heuristics is given by relations of Section 4.

So that deriving the optimal damping factor must be based on some experiments — on some inductive steps. These experiments should be as simple as possible. The functionals $\|{}^{j+1}\bar{y}({}^j\mu)\|^2$, $(y, {}^{j+1}\bar{y}({}^j\mu))$, ${}^{j+1}\eta^2({}^j\mu)$ and ${}^{j+1}\varrho({}^j\mu)$ can be calculated very simply from the new coefficients ${}^{j+1}a_i({}^j\mu)$ $(i = 0, 1, \ldots, \bar{n})$. On the other side the functionals ${}^{j+1}\varphi$ and ${}^{j+1}\delta$ can be computed after deriving the new gradient vector $\boldsymbol{g}(t, {}^{j+1}\boldsymbol{a}, {}^j\mu)$ and after inverting the pertinent matrix $\boldsymbol{G}({}^{j+1}\boldsymbol{a}, {}^j\mu)$. In this section we shall consider only the procedures starting from the first group of the functionals.

5.1. Solution from two experiments

It can be expected that the error function $v_3({}^j\mu)$ will have the same form as the error ${}^{j+1}\varphi_P^2$, but with an opposite trend with respect to the damping factor ${}^j\mu$ (see relations (4.25), (4.26) and (4.28)); so

$$v_3({}^j\mu) = ({}^j\mu - \psi)^2 R \,, \tag{5.1}$$

where ψ and R can be calculated from two points — from two experiments for ${}^j\mu_1$ and ${}^j\mu_2$, i.e. ${}^{j+1}\eta^2({}^j\mu_1)$ and ${}^{j+1}\eta^2({}^j\mu_2)$. If we obtain with the help of (4.29) $v_3({}^j\mu_1) > 0$ and $v_3({}^j\mu_2) > 0$ for ${}^j\mu_2 > {}^j\mu_1$ and $v_3({}^j\mu_2) > v_3({}^j\mu_1)$, then

$$\psi = ({}^j\mu_1 q - {}^j\mu_2)/(q - 1) \,, \tag{5.2}$$

$$R = v_3({}^j\mu_1)/({}^j\mu_1 - \psi)^2 \,, \tag{5.3}$$

where

$$q = \sqrt{(v_3({}^j\mu_2)/v_3({}^j\mu_1))} \,.$$

Now we can formulate the following proposition:

Proposition 5.1. If

(i) $\{{}^j\boldsymbol{a}\}$ is the DNLS sequence,

(ii) $v_3({}^j\mu_1) > 0$ and $v_3({}^j\mu_2) > 0$ exist for ${}^j\mu_1, {}^j\mu_2 \in (0{\cdot}5, 1)\,({}^j\mu_2 > {}^j\mu_1)$,

then the optimal damping factor is given by the relation

$$(5.4) \qquad {}^j\mu_{opt} = \frac{{}^j\varphi^2 + \psi R}{{}^j\varphi^2 + R}$$

and the predicted error ${}^{j+1}\eta_P^2({}^j\mu_{opt})$ is given by

$$(5.5) \qquad {}^{j+1}\eta_P^2({}^j\mu_{opt}) = {}^j\delta^2 + \frac{{}^j\varphi^2 R(1 - \psi)^2}{{}^j\varphi^2 + R}.$$

Proof. Relation (5.4) follows from condition (4.32). The minimality is given with respect to (5.3) and hypothesis (ii) by

$$\frac{\mathrm{d}^2\,{}^{j+1}\eta^2({}^j\mu)}{\mathrm{d}\,{}^j\mu^2} = 2\,{}^j\varphi^2 + 2R = 0\,.$$

The predicted error can be written with the help of (4.29) in the form

$${}^{j+1}\eta_P^2({}^j\mu_{opt}) = {}^j\delta^2 + (1 - {}^j\mu_{opt})\,{}^j\varphi^2 + (\psi - {}^j\mu_{opt})\,\psi R$$

and then we obtain with respect to relation (5.4) expression (5.5) □

The suitability of the error approximation by expression (5.1) can be tested with the help of the predicted error. So we can ask (cf. Algorithm 7.2) that it holds

$$(5.6) \qquad |{}^{j+1}\eta^2({}^j\mu_{opt}) - {}^{j+1}\eta_P^2({}^j\mu_{opt})| < \varepsilon_2\,.$$

The proposed procedure can be used for cases with $v_3({}^j\mu) > 0$. But only the derivatives are decisive for deriving the optimal damping factor. So the curve of $v_3({}^j\mu)$ can be shifted and we obtain for $v_3({}^j\mu_1) < 0$ and $v_3({}^j\mu_2) > 0$

$$(5.7) \qquad v_3^*({}^j\mu) = ({}^j\mu - \psi^*)^2\, R^* = v_3({}^j\mu) - v_3({}^j\mu_1)$$

where $\psi^* = {}^j\mu_1$ and $R^* = v_3^*({}^j\mu_2)/({}^j\mu_2 - {}^j\mu_1)^2$. Now we can use the relations in Proposition 5.1. This procedure is sufficiently correct for the delayed cases, i.e. $v_1({}^j\mu) < 0$ for ${}^j\mu \in (0, 1)$.

5.2. The case $v_1({}^j\mu) > 0$

Proposition 4.4 shows that for the error function it holds

$$(5.8) \qquad v_3({}^j\mu) < -{}^{j+1}\varphi_P^2({}^j\mu)$$

for the values ${}^j\mu$ near the optimal damping factor and so the use of relation (5.7) can lead to significant errors. Here it is better to use another way and to start from relations (4.30) and (4.33) and above all from the fact that the parameter ${}^{j+1}\varrho({}^j\mu_{opt}) \to$

$\to 0$, as Proposition 4.4 for ${}^{j+1}\varrho({}^j\mu_c) = 0$ shows, i.e. ${}^{j+1}\eta({}^j\mu_c) < {}^j\delta^2$. It follows from relations (4.18), (4.37) and hypothesis (ii) in Proposition 4.4 (for ${}^j\mu_c > {}^j\mu_a$) that

$$v_1({}^j\mu_c) > v_1({}^j\mu_a) = (1 - {}^j\mu_a)({}^j\varrho + {}^j\varphi^2)\,. \tag{5.9}$$

So the value ${}^j\mu \in ({}^j\mu_a, {}^j\mu_c)$ can be used as the optimal damping factor.

The optimum of the total error is mostly flat and so we can fulfil simultaneously another condition connected with the next step $v_1({}^{j+1}\mu) > 0$ and in this way to obtain the sequence of steps with $v_1({}^j\mu) > 0$. Relations (4.30), (4.33) resp. (5.9) show that this favourable case can arise for ${}^j\varrho > {}^j\varphi^2$. Therefore for ${}^j\delta > {}^{j+1}\delta$ the optimal solution is given by the condition

$${}^{j+1}\varrho({}^j\mu_{opt}) \doteq 2\,{}^{j+1}\varphi_P^2({}^j\mu_{opt}) = 2(1 - {}^j\mu_{opt})^2\,{}^j\varphi^2\,. \tag{5.10}$$

Let us have the point computed for some damping factor ${}^j\mu_x$ with ${}^{j+1}\varrho({}^j\mu_x) > 0$ and ${}^{j+1}\eta^2({}^j\mu_x) < {}^j\delta^2$. Then the optimal damping factor is given by

$${}^j\mu_{opt} \doteq {}^j\mu_x + \frac{2\,{}^{j+1}\varphi_P^2({}^j\mu_x) - {}^{j+1}\varrho({}^j\mu_x)}{4(1 - {}^j\mu)\,{}^j\varphi^2 + \mathrm{d}\,{}^{j+1}\varrho/\mathrm{d}\,{}^j\mu}\,. \tag{5.11}$$

5.3. The case ${}^j\varphi < {}^j\delta$

The simple strategy, e.g. ${}^j\mu = 1$, can be used near the global minimum, i.e. in the region

$$\Phi_0 = \{{}^j\boldsymbol{a}: {}^{j+1}\delta \doteq {}^j\delta \doteq {}^*\delta,\ {}^j\varphi < {}^j\delta\}\,. \tag{5.12}$$

But the simple procedures starting from two experiments, e.g. from ${}^{j+1}\eta^2(1)$ and ${}^{j+1}\eta^2(0{\cdot}5)$

$${}^j\mu_H = 0{\cdot}5 + 0{\cdot}25\,\frac{{}^j\eta^2 - {}^{j+1}\eta^2(1)}{{}^j\eta^2 + {}^{j+1}\eta^2(1) - 2\,{}^{j+1}\eta^2(0{\cdot}5)}\,, \tag{5.13}$$

which can be found in the literature (see e.g. [5], [11]), are in many cases more effective.

Here the DNLS method offers a better solution. We can derive using Proposition 4.3 a simpler procedure which starts from one experiment only, i.e. from ${}^{j+1}\eta^2(1)$ and ${}^{j+1}\eta^2(0{\cdot}5)$ approximated by relation (4.28) for $v_3(0{\cdot}5) = 0$ where

$${}^{j+1}\eta^2(0{\cdot}5) \doteq {}^j\delta^2 + {}^{j+1}\varphi_P^2(0{\cdot}5)\,. \tag{5.14}$$

Then we obtain from relation (5.13)

$${}^j\mu_D = 0{\cdot}5 + 0{\cdot}25\,\frac{{}^j\eta^2 - {}^{j+1}\eta^2(1)}{{}^{j+1}\eta^2(1) + 0{\cdot}5\,{}^j\varphi^2 - {}^j\delta^2}\,. \tag{5.15}$$

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