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WRAPPED EIGENSTRUCTURE OF CHAOS

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Nonlinear differential equations with chaotic, i.e. with bounded, nonvanishing and noncyclic solutions will be analyzed. Two tools of analysis will be introduced: *(i)* the pencil of affine and nonaffine vector fields of the right hand side of nonlinear differential equation, *(ii)* generalized eigenstructure with curved generalized eigenvectors, and even mutually wrapped unstable generalized eigenvectors.

The state model of linear control systems $dx/dt = F_K x$ where F_K is the state matrix parametrized by vector or matrix K, e.g. $F_K = F + GK$, resp. $F + KH$ have been used for any linear control system [16], [17]. The matrix K changes generally the right eigenstructure (V, S) of semisimple (diagonalizable) F_K, where V is the matrix of right eigenvectors $v_1, \dots, v_n$ as the columns, S is the diagonal matrix with the eigenvalues $s_1, \dots, s_n$: $F_K V = VS$. The control synthesis is done to make matrix exponential $\exp(F_K t)$ properly asymptotically tend to $0 \in R^n$, the unique (for invertible F_K) equilibrium point of $dx/dt = F_K x$. By the properness we understand that the eigenvalues are from the feasibility truncated cone of the left half plane, [16].

The state model of nonlinear control systems $dx/dt = f_K(x)$ where f_K is a (unique solution guaranteing, e.g. globally Lipschitzian) vector field parametrized by vector or matrix K, e.g. $f_K(.) = f(.) + g(K(.))$, resp. $f(.) + Kh(.)$ have been postulated for use for any nonlinear control system, [18] . The matrix K changes the generalized eigenstructure of $f_K(.)$, [18]. The control synthesis in the vicinity of equilibrium point is done to make generalized matrix exponential $\lim_{N \to \infty} [I + f_K(.)t/N]^N$ properly asymptotically tend to an equilibrium point of $dx/dt = f_K(x)$. Generally there are several equilibrium points, i.e. several solutions of $f_K(x) = 0$ for fixed K. For the dimension of the state space $n = 1$, the only possible asymptotic stabilization is the stabilization to an equilibrium point – the attractor of the dimension 0. For the dimension of the state space $n = 2$, there is possible even the stabilization on a cycle – the attractor of dimension 1. For the dimension of the state space $n = 3, 4, \dots$, there are some $f_K(.)$ for which it is possible the stabilization on the attractors – the cycles of dimensions $2, 3, \dots$ but also on the attractors of Hausdorff fractal dimensions $2+\varepsilon, 3+\varepsilon, \dots$ where $\varepsilon \in (0,1)$. The stabilization on the fractal attractors with enlarged above whole dimension by ε should have both interesting and important applications connected with mixing of solutions in the state space – with application to physiological health, [1], [3], [4], [6], [7], [8], [9], [13], [14], [18], [19]. We take as fundamental the change of paradigma from the negative to the positive valuation of chaos as the goal of behavior. The reasons for such new paradigma: *(i)* chaos makes possible better absorption of energy and movement, *(ii)* chaos is distinguished by spectral reserve with the spectrum of the type $1/f$ which is resistive with respect of resonance, *(iii)* chaos is substantially connected with healthy activity – at the difference to the periodic behavior, and the point attractor connected with the end of activity at all; with this is connected even the robustification of the behavior, *(iv)* chaos is very closely connected with the novel fundamental tools of the nature's description – the fractals, self-similarity, and renormalization groups: chaos acts mainly as an organizing principle. The specified reasons are mutually connected. Some our results in this field of control synthesis are presented in [2], [21], [22], [23], [24]. In this paper we shall limit ourselves on analysis and a scalar, real parameter K, in the spirit of classical Root Locus method of Bode and Evans and some of its nonlinear generalizations [15], [16].

Definition 1. We shall call

$$f_K(x) = c + Fx + Kg(x)$$
$$K \in R;\ c, x \in R^n;\ g(x) = o(x),\ \mathrm{d}x/\mathrm{d}t = f_K(x)$$

the K-pencil of vector fields, the first vector field being affine, the second nonaffine.

Definition 2. Let for $K = 1$, $f_K(x)$ has equilibria $e_1, e_2, \dots, e_m$, i.e. $f_1(e_i) = 0$ $(i = 1, 2, \dots, m)$. Then the K-loci, $K \geq 0$, of the equilibria, eigenvectors and eigenvalues at those equilibria (E_K, V_K, S_K), we shall call the eigenstructure of the K-pencil of vector fields.

Example 1. Let us consider three K-pencils of vector fields, introduced by Chua [12], Lorenz [11], and Roessler [10] – only for $K = 1$:

$$f_K^{chu}(x) = \begin{bmatrix} -\alpha & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + K \begin{bmatrix} -\alpha\phi(x_1) \\ 0 \\ 0 \end{bmatrix},$$

$$\phi(x_1) = ax_1 \text{ for } |x_1| \leq 1, = bx_1 - a + b \text{ for } x_1 \leq -1, = bx_1 + a - b \text{ for } x_1 \geq 1$$

$$f_K^{lor}(x) = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + K \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix}$$

$$f_K^{roe}(x) = \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ 0 & 0 & -c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + K \begin{bmatrix} 0 \\ 0 \\ x_1x_3 \end{bmatrix}$$

The equilibria parametrized by K:

$$e_1^{chu} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; e_{K,2}^{chu} = \begin{bmatrix} \frac{K(b-a)}{Kb+1} \\ 0 \\ -\frac{K(b-a)}{Kb+1} \end{bmatrix}, \; e_{K,3}^{chu} = -e_{K,2}^{chu}$$

$$e_1^{lor} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \; e_{K,2}^{lor} = \begin{bmatrix} \frac{R^{lor}}{K} \\ \frac{R^{lor}}{K} \\ \frac{r-1}{K} \end{bmatrix}, \; e_{K,3}^{lor} = \begin{bmatrix} -\frac{R^{lor}}{K} \\ -\frac{R^{lor}}{K} \\ \frac{r-1}{K} \end{bmatrix}, \; R^{lor} = \sqrt{(}b(r-1))$$

$$e_{K,1}^{roe} = \begin{bmatrix} \frac{c+R_K^{roe}}{2K} \\ -\frac{c+R_K^{roe}}{2aK} \\ \frac{c+R_K^{roe}}{2aK} \end{bmatrix}, \; e_{K,2}^{roe} = \begin{bmatrix} \frac{c-R_K^{roe}}{2K} \\ -\frac{c-R_K^{roe}}{2aK} \\ \frac{c-R_K^{roe}}{2aK} \end{bmatrix}, \; R_K^{roe} = \sqrt{(}c^2 - 4abK)$$

The state matrices linearized near equilibria parametrized by K:

$$F_{K,1}^{chu} = \begin{bmatrix} -\alpha(Ka+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix}, \; F_{K,2}^{chu} = \begin{bmatrix} -\alpha(Kb+1) & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{bmatrix} = F_{K,3}^{chu}$$

$$F_1^{lor} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{bmatrix},$$

$$F_2^{lor} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & -R^{lor} \\ R^{lor} & R^{lor} & -b \end{bmatrix}, \; F_3^{lor} = \begin{bmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & R^{lor} \\ -R^{lor} & -R^{lor} & -b \end{bmatrix}$$

$$F_{K,1}^{roe} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c+R_K^{roe}}{2a} & 0 & \frac{c+R_K^{roe}}{2} - c \end{bmatrix}, \; F_{K,2}^{roe} = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ \frac{c-R_K^{roe}}{2a} & 0 & \frac{c-R_K^{roe}}{2} - c \end{bmatrix}$$

The K-pencil of Chua vector field has K-fixed center equilibrium e_1^{chu} and K-dependent both the off-center equilibria $e_{K,2}^{chu}$, $e_{K,3}^{chu}$ and all the state matrices $F_{K,1}^{chu}$, $F_{K,2}^{chu}$, $F_{K,3}^{chu}$ with K-dependent eigenvalues and eigenvectors. Parameter values used by Chua, [12], are $a = -\frac{8}{7}$, $b = -\frac{5}{7}$, $\alpha = 9$, $\beta = 14$, $K = 1$. For those α, β, a, b, K : $s_{1,1} = 2.233$, $s_{1,2} = -0.9735{+}\mathrm{i}2.667$, $s_{1,3} = -0.9735{-}\mathrm{i}2.667$, $s_{k,1} = -3.957$, $s_{k,2} = 0.1928 + \mathrm{i}3.01$, $s_{k,3} = 0.1928 - \mathrm{i}3.01$ $(k = 2, 3)$. Each equilibrium has both eigenvalues with positive and negative real parts or each equilibrium is *hyperbolic* – with its neighborhood expanding in some lines (for real eigenvectors) or a plane (for complex conjugate eigenvectors defining the plane by the real and imaginary part of an eigenvectors). Changing our K, some equilibria lose their hyperbolicity, see Fig. 1. The state space projector to horizontal axis is $[\ 1\ 0\ 0\ ]$, the projector to the vertical axis is $[\ 0\ 1\ \frac{1}{4}\ ]$. At any equilibrium there sits the triple of abssisas representing the three eigenvectors. The off-center equilibria and eigenvectors triples are K-dependent.

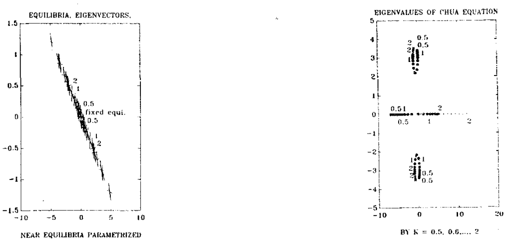

Fig. 1.

The K-pencil of Lorenz vector field has simply K-dependent the off-center equilibria: $e^{lor}_{K,2} = e^{lor}_{1,2}/K$, $e^{lor}_{K,3} = e^{lor}_{1,3}/K$ and K-fixed both the center equilibrium e^{lor}_1 and the state matrices F^{lor}_1, F^{lor}_2, F^{lor}_3. So even their eigenvalues and eigenvectors are K-fixed. Parameter values used by Lorenz, [11], are $\sigma = 10, b = \frac{8}{3}, r = 28$ and $K = 1$. For those σ, b, r and for $K > 0$: $s_{1,1} = -22.83$, $s_{1,2} = 11.83$, $s_{1,3} = -\frac{8}{3}$; $s_{k,1} = -13.85$, $s_{k,2} = 0.09396 + i10.19$, $s_{k,2} = 0.09396 - i10.19$ $(k = 2, 3)$. Even now each equilibrium is *hyperbolic*.

The K-pencil of Roessler vector field has K-dependent both equilibria $e^{roe}_{K,1}, e^{roe}_{K,2}$, and the state matrices $F^{roe}_{K,1}$, $F^{roe}_{K,2}$, and even their eigenvalues and eigenvectors are K-dependent. Parameter values used by Roessler, [10], are $a = 0.4$, $b = 2$, $c = 4$ and $K = 1$. For those parameters: $s_{1,1} = -0.07691 + i3.23$, $s_{1,2} = -0.07691 - i3.23$, $s_{1,3} = 0.3427$, $s_{2,1} = 0.1323 + i0.9807$, $s_{2,2} = 0.1323 - i0.9807$, $s_{2,3} = -3.654$. Again, even now each equilibrium is *hyperbolic*. Changing K, some equilibria lose their hyperbolicity.

Note 1. We were considering the eigenlocus of the K-pencil of the affine and nonaffine parts of the vector field. It can be easily generalized to the p-sheaf, for the differential topological notion of the sheaf see [5], [17], [25], taking as the parameter

p any of the parameters: α, β, a, b, K ; σ, b, r, K ; a, b, c, K. In fact we computed, [20], all such p-eigenloci of f_p^{chu}, f_p^{lor}, f_p^{roe} – generally with the result of losing the hyperbolicity of the equilibrium point in some vicinity of the nominal value of the parameter p.

In [18] we introduced the generalized eigenstructure as based on a refinement of the stable and unstable manifolds of the central manifold theorem. In following we shall introduce the curved eigenstructure based on the K-pencil of solutions of differential equations with the right hand side being previously introduced K-pencil of the affine and nonaffine vector fiels.

Definition 2. The curved eigenstructure of the vector field $f_1(x)$ with the hyperbolic equilibria points, is the limit point of the continuation on $(0, 1]$ of the eigenstructure of $\lim_{K \to 0} f_K(x)$.

Algorithm. To find the curved eigenstructure of the vector field from the Definition 2:
1. Compute the equilibria points.
2. Compute the eigenvectors and eigenvalues at those equilibria points.
3. Choose the point on the real eigenvector or either on the real or imaginary part of the one of complex conjugate eigenvectors near the equilibrium points.
4. For the real eigenvector with positive (negative) eigenvalue integrate the differential equation with the vector field from Definition 1 starting in an initial point from the step 3 for positive (negative) time. The solution is the curved real eigenvector of the curved eigenstructure.
5. For the real or imaginary parts of the complex eigenvector from the step 4 with positive (negative) real part of the eigenvalue, integrate the differential equation with the vector field from Definition 1 starting in the initial point from the step 3 for positive (negative) time. The solution is the curve on the curved surface of the curved eigenstructure.

Note 2. On the opposite way, the eigenstructure has the meaning of the tangent vector or plane to the curved real eigenvector and or of the curved real surface with the meaning of the both real eigenvalues and the real and imaginary parts of the complex eigenvalues as tangent coefficients, [18].

Example 2. We shall demonstrate the curved eigenstructure of the Roessler equation near the two equilibria for Roessler nominal values of the parameters $a = 0.4$, $b = 2$, $c = 4$, $K = 1$, see Fig. 2. The projector of the state space to the horizontal axis is $[\ 1\ \frac{1}{3}\ 0\ ]$, and to the vertical axis is $[\ 0\ \frac{8}{9}\ -\frac{1}{4}\ ]$. The sphere-like segments with the center sitting at two equilibrium points are represented by the spiral sitting on these segments. The spiral near the bottom equilibrum is stable, so it was computed for negative time. The stable curved axis for the top equilibrium was computed also for negative time.

Example 3. The Lorenz equation which is the prototype differential equation with chaotic – bounded, nonvanishing, noncyclic behaviour, we shall use for the demonstration of the behaviour of *unstable* curved eigenstructure in large. For nominal values of the parameters $\sigma = 10$, $b = \frac{8}{3}$, $r = 28$, $K = 1$, see Fig. 3. The projector to the horizontal axis is $[1\ \frac{1}{4}\ 0\ ]$, and to the vertical axis is $[0\ 1\ -\frac{1}{4}\ ]$. Now the unstable curved eigenvectors get *wrapped*. Similar wrapped unstable eigenstructure of chaos we have computed for the Chua and Roessler equations, like for the Lorenz equation not only for nominal values of parameter but for the sets of the parameters α, β, a, b, K; σ, b, r, K; a, b, c, K sets containing the nominal parameters, [20]. For the both the Lorenz and the Chua equation the curved unstable eigenvectors get wrapped changing their positions from the vicinity of one of the hyperbolic equilibria points to the the other; for the Roessler equation, the unstable eigenvector from the bottom hyperbolic equilibrium point, after reaching the vicinity ot the top hyperbolic equilibrium point, remain in that vicinity.

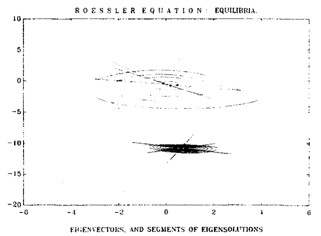

Fig. 2.

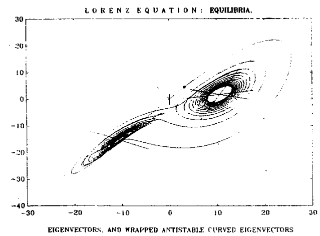

Fig. 3.

Note 3. The eigenvectors of simple (diagonalizable) linear vector field span the solutions. This they do in some vicinity (by global linearization) even the curved eigenvectors. This property is lost in large: the curved unstable eigenvectors settle in fractal attractors.

Conclusion. The K-pencil of eigenstructure of matrices was generalized to K-pencil of the eigenstructure of the affine and nonaffine parts of the vector field. The linear eigenstructure of the matrix was generalized to the curved eigenstructure of the vector field. It was demonstrated that its unstable eigenvectors are wrapped for the chaotic solution of Lorenz, Chua and Roessler equations.

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