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NORMAL COVARIANCES

JIŘÍ MICHÁLEK

The main goal of this paper is to characterize the class of normal covariances for random processes. This notion generalizes a known weakly stationary case and can be a suitable mathematical tool for describing real random processes which are not weakly stationary.

The notion of a normal covariance was introduced by the author in the paper [1]. Characterization of normal covariances for random sequences is given in [2]. The class of normal covariances was discovered by studying of locally stationary covariances which were introduced by Silverman in [3]. The name – normal covariances – is based on a close connection with the theory of normal operators in Hilbert spaces.

Definition 1. Let $R(\cdot, \cdot)$ be a covariance defined in the whole plane \mathbb{R}_2 . The covariance $R(\cdot, \cdot)$ is called normal if it can be written in the form

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{i\lambda(s+t)} e^{i\mu(s-t)} ddF(\lambda, \mu), \quad s \in \mathbb{R}_1, \quad t \in \mathbb{R}_1$$

where $F(\cdot, \cdot)$ is a two-dimensional distribution function with finite variation.

The class of normal covariances is sufficiently large because every continuous weakly stationary covariance and every continuous symmetric covariance (for details see [1]) are normal. Their product is a normal covariance too. Let $\{Z(s)\}$, $s \in \mathbb{R}_1$, be a complex valued random process with everywhere vanishing expected value and with a normal covariance. Then by use of the Karhunen theorem such a process can be expressed in the form of a stochastic integral

$$Z(s) = \iint_{-\infty}^{+\infty} e^{s\lambda} e^{i\mu s} dd\zeta(\lambda, \mu)$$

understood in the quadratic mean sense where $\zeta(\cdot, \cdot)$ is a plane martingale satisfying $E\{\zeta(\lambda_1, \mu_1) \overline{\zeta(\lambda_2, \mu_2)}\} = F(\min(\lambda_1, \lambda_2), \min(\mu_1, \mu_2))$.

Theorem 1. Every normal covariance function is continuous at the whole plane.

Proof. Let $R(\cdot, \cdot)$ be a normal covariance, let $(s, t) \in \mathbb{R}_2$ be quite arbitrary. Let h_1, h_2 be real numbers. We must estimate the difference $|R(s + h_1, t + h_2) - R(s, t)|$. Let $z = \lambda + i\mu$, then

$$\begin{aligned} |R(s + h_1, t + h_2) - R(s, t)| &= \left| \int \int_{-\infty}^{+\infty} (e^{(s+h_1)z} e^{(t+h_2)z} - e^{sz} e^{tz}) ddF(\lambda, \mu) \right| = \\ &= \left| \int \int_{-\infty}^{+\infty} e^{sz} e^{tz} (e^{h_1 z} e^{h_2 z} - 1) ddF(\lambda, \mu) \right| \leq \\ &\leq \int \int_{-\infty}^{+\infty} e^{(s+t)\lambda} |e^{\lambda(h_1+h_2)} e^{i\mu(h_1-h_2)} - 1| ddF(\lambda, \mu) = \int \int_{-\infty}^{+\infty} e^{\lambda(s+t)} f(\lambda, \mu, h_1, h_2) ddF(\lambda, \mu), \end{aligned}$$

where

$$f(\lambda, \mu, h_1, h_2) = |e^{\lambda(h_1+h_2)} \cos((h_1 - h_2)\mu) + i e^{\lambda(h_1+h_2)} \sin((h_1 - h_2)\mu) - 1|$$

and

$$\lim_{\substack{h_1 \rightarrow 0 \\ h_2 \rightarrow 0}} f(\lambda, \mu, h_1, h_2) = 0 \quad \text{for every } (\lambda, \mu) \in \mathbb{R}_2.$$

Further $f(\lambda, h_1, h_2) \leq 2 e^{\lambda(h_1+h_2)} + 1$. When $h_1 + h_2 \geq 0$ and $\lambda \geq 0$ then $e^{\lambda(h_1+h_2)} \leq e^{2\lambda\varepsilon}$, if $\lambda \leq 0$ then $e^{\lambda(h_1+h_2)} \leq 1$ (it is possible to consider $|h_1| < \varepsilon$, $|h_2| < \varepsilon$ because of $h_1 \rightarrow 0, h_2 \rightarrow 0$). In case that $h_1 + h_2 < 0$ the situation is quite analogous. For every $\lambda \in \mathbb{R}_1$ and every h_1, h_2 with $|h_1| < \varepsilon, |h_2| < \varepsilon$

$$e^{\lambda(s+t)} e^{\lambda(h_1+h_2)} \leq \max(1, e^{\lambda(s+t+2\varepsilon)})$$

that is an integrable majorant function as we assume the existence of integral $\int \int_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} ddF(\lambda, \mu)$ for every (s, t) . Using the Lebesgue dominated theorem we immediately obtain that $F(\cdot, \cdot)$ is continuous at (s, t) . \square

Further properties of normal covariances are the following:

1. For every $(s, t) \in \mathbb{R}_2$

$$|R(s, t)| \leq \left(\int_{-\infty}^{+\infty} e^{2s\lambda} dF_1(\lambda) \right)^{1/2} \left(\int_{-\infty}^{+\infty} e^{2t\lambda} dF_1(\lambda) \right)^{1/2} = R^{1/2}(s, s) \cdot R^{1/2}(t, t)$$

where $F_1(\cdot)$ is the first marginal of $F(\cdot, \cdot)$, i.e.

$$F_1(\lambda) = \int_{-\infty}^{+\infty} dF(\lambda, \mu).$$

2. The function $R_1(s) = R(s, s) = \int_{-\infty}^{+\infty} e^{2s\lambda} dF_1(\lambda)$ is a nonnegative definite kernel with respect to sum, i.e.

$$R_1(\tau_1 + \tau_2) = \int_{-\infty}^{+\infty} e^{2\lambda(\tau_1 + \tau_2)} dF_1(\lambda)$$

is a symmetric covariance in $(\tau_1, \tau_2) \in \mathbb{R}_2$.

3. Similarly, the function $R_2(t) = R(t, -t) = \int_{-\infty}^{+\infty} e^{2i\mu t} dF_2(\mu)$, where $F_2(\cdot)$ is the second marginal of $F(\cdot, \cdot)$, is a weakly stationary covariance.
4. Without loss of generality, we can put $R(0, 0) = 1$ that means the function $F(\cdot, \cdot)$ will be a probability distribution function in the plane.

The following theorem will characterize normal covariances as functions that are in some sense nonnegative definite.

Theorem 2. A covariance function $R(\cdot, \cdot)$ defined at the whole plane is normal if and only if

- 1) $R(0, 0) = 1$
- 2) $R(\cdot, \cdot)$ is continuous
- 3) there exists a function $S(\cdot, \cdot)$ such that for every $(s, t) \in \mathbb{R}_2$ $R(s, t) = S(s + t, s - t)$ and for every finite collection $(\alpha_1, \alpha_2, \dots, \alpha_n)$ of complex numbers and every real numbers $u_1, \dots, u_n, v_1, \dots, v_n$

$$\sum_i \sum_j \alpha_i \bar{\alpha}_j S(u_i + u_j, v_i - v_j) \geq 0.$$

Proof. First, we shall construct a suitable Hilbert space. Let L be the linear set of all complex valued functions that are everywhere vanishing except a finite number of points in the plane, i.e. $f(\cdot, \cdot) \in L$ if and only if there exist $(u_i, v_i) \in \mathbb{R}_2, i = 1, 2, \dots, n$ such that $f(u_i, v_i) \neq 0$ and $f(\cdot, \cdot) = 0$ otherwise. We can in L define an Hermite bilinear form $\langle f, g \rangle, f, g \in L$ by the relation

$$\langle f, g \rangle = \sum_{u,v} \sum_{x,y} f(u, v) \overline{g(x, y)} S(u + x, v - y).$$

According to our assumption $\|f\|^2 \geq 0$ and hence $\|\cdot\|$ is a seminorm in L . Let $h \in \mathbb{R}_1$ and let us define a shift-operator T_h in L in the following way

$$T_h f(u, v) = f(u - h, v - h).$$

Let $N_0 \subset L, N_0 = \{f: \|f\| = 0\}$ and let us consider a factor space L/N_0 . Then the bilinear form defined above is a scalar product and $\|\cdot\|$ is a norm. Let H be a completion of L/N_0 with respect the norm to $\|\cdot\|$. Then H is our underlying Hilbert space. As every T_h maps N_0 into N_0 , there is a possibility to translate every operator T_h from L into H . The definition domain $\mathcal{D}(T_h)$ of every T_h will be the linear set L/N_0 in H , $\mathcal{D}(T_h)$ is thus everywhere dense in H . Let us consider for every operator T_h its adjoint operator T_h^* and let us prove that $\mathcal{D}(T_h^*) \supset L/N_0$. Let $f, g \in L/N_0$ then

$$\begin{aligned} \langle T_h f, g \rangle &= \sum_{u,v} \sum_{x,y} f(u - h, v - h) \overline{g(x, y)} S(u + x, v - y) = \\ &= \sum_{u,v} \sum_{x,y} f(u, v) \overline{g(x, y)} S(u + (x + h), v - (y - h)) = \\ &= \sum_{u,v} \sum_{x,y} f(u, v) \overline{g(x - h, y + h)} S(u + x, v - y) = \langle f, S_h g \rangle \end{aligned}$$

where $S_h g(x, y) = g(x - h, y + h)$. As this equality holds for every $f \in L/N_0$ that is everywhere dense in H the element $S_h g$ equals $T_h^* g$. We proved that $\mathcal{D}(T_h^*) \supset L/N_0$. It means the every operator T_h can be closed, in other words, for every T_h there exists a closed operator \bar{T}_h in $H, T_h \subset \bar{T}_h$. Now, we shall show that $T_h T_h^* = T_h^* T_h$ on L/N_0 . When $f \in L/N_0$, then $T_h T_h^* f(u, v) = T_h f(u - h, v + h) = f(u - 2h, v) = T_h^* f(u - h, v - h) = T_h^* T_h f(u, v)$. This fact implies, further, that for every pair $f, g \in L/N_0$

$$\langle T_h f, T_h g \rangle = \langle T_h^* T_h f, g \rangle = \langle T_h T_h^* f, g \rangle = \langle T_h^* f, T_h^* g \rangle$$

because $T_h = T_h^{**}$ on L/N_0 . At this moment we can construct a closed enlargement \overline{T}_h of T_h . An element $f \in H$ belongs to $\mathcal{D}(\overline{T}_h)$ if there exists a sequence $\{f_n\}_{n=1}^\infty \subset L/N_0$ such that $f_n \rightarrow f$ and $\{T_h f_n\}_{n=1}^\infty$ is convergent too. If $f_1 = \lim_{n \rightarrow \infty} T_h f_n$ then we put $f_1 = \overline{T}_h f$. There is no problem to prove that \overline{T}_h is in this way defined unambiguously. Let $\{g_n\}_{n=1}^\infty \subset L/N_0$ be another sequence converging to f , but $T_h g_n \rightarrow g_1 \neq f_1$. Then for any $g \in L/N_0$

$$\begin{aligned} \langle g, f_1 - g_1 \rangle &= \langle g, \lim_{n \rightarrow \infty} T_h(f_n - g_n) \rangle = \lim_{n \rightarrow \infty} \langle g, T_h(f_n - g_n) \rangle = \\ &= \lim_{n \rightarrow \infty} \langle T_h^* g, f_n - g_n \rangle = \langle T_h^* g, f - f \rangle = 0. \end{aligned}$$

Thanks to the fact that L/N_0 is dense in H $g_1 = f_1$. As $\|T_h f\| = \|T_h^* f\|$ for every $f \in L/N_0$, we can prove that $\mathcal{D}(T_h^*) = \mathcal{D}(\overline{T}_h)$. Further, $T_h^* = (\overline{T}_h)^*$ and because of closeness of \overline{T}_h $\overline{T}_h^{**} = \overline{T}_h$. It remains to prove that $\overline{T}_h^{**} \overline{T}_h = \overline{T}_h^{**} \overline{T}_h^*$ and after this we can state that \overline{T}_h is a normal operator in H . First, we must prove that $\mathcal{D}(\overline{T}_h^{**} \overline{T}_h) = \mathcal{D}(\overline{T}_h \overline{T}_h^*)$. Let $f \in \mathcal{D}(\overline{T}_h^{**} \overline{T}_h)$, i.e. $\overline{T}_h f \in \mathcal{D}(\overline{T}_h^*)$ and simultaneously $f \in \mathcal{D}(\overline{T}_h)$. At the same moment $f \in \mathcal{D}(\overline{T}_h^*) = \mathcal{D}(T_h^*)$ and we can consider $T_h^* f$. Let $g \in L/N_0$ be quite arbitrary then $\langle T_h g, T_h^* f \rangle = \langle g, T_h^{**} T_h^* f \rangle = \langle g, \overline{T}_h T_h^* f \rangle$ that means that $T_h^* f \in \mathcal{D}(\overline{T}_h)$. We have proved that $\mathcal{D}(\overline{T}_h^{**} \overline{T}_h) \subset \mathcal{D}(\overline{T}_h \overline{T}_h^*)$. Quite analogously we can prove the opposite inclusion. We see that for every operator T_h there exists a normal enlargement \overline{T}_h , $T_h = \overline{T}_h$ on L/N_0 and $\{\overline{T}_h^j, h \in \mathbb{R}_1\}$, forms a group on the linear set L/N_0 . For every \overline{T}_h there exists a resolution of the identity in $H \{P_z^h\}$, $z \in \mathbb{C}$ such that

$$\overline{T}_h = \int \int_{-\infty}^{+\infty} z dP_z^h.$$

Let $\delta(\cdot, \cdot)$ be the element in L/N_0 defined as $\delta(0, 0) = 1$, $\delta(u, v) = 0$ otherwise. Let us calculate $\langle T_{h_1} \delta(\cdot, \cdot), T_{h_2} \delta(\cdot, \cdot) \rangle$. Thus

$$\begin{aligned} \langle T_{h_1} \delta(\cdot, \cdot), T_{h_2} \delta(\cdot, \cdot) \rangle &= \\ &= \sum_{u,v} \sum_{x,y} T_{h_1} \delta(u, v) \overline{T_{h_2} \delta(x, y)} S(u+x, v-y) = \\ &= \sum_{u,v} \sum_{x,y} \delta(u-h_1, v-h_1) \delta(x-h_2, y-h_2) S(u+x, v-y) = \\ &= \sum_{u,v} \sum_{x,y} \delta(u, v) \delta(x, y) S(h_1+h_2, h_1-h_2) = R(h_1, h_2). \end{aligned}$$

By use of the polar coordinates every operator \overline{T}_h can be expressed as

$$\overline{T}_h = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} e^{i\mu} e^{i\nu} dE^h(\lambda, \mu)$$

where $\{E^h(\cdot, \cdot)\}$ is another resolution of the identity in H . At this moment we put $h = 1/n$ and the group property $T_{h_1}(T_{h_2}) = T_{h_2}(T_{h_1}) = T_{h_1+h_2}$ enables that for every integer $j \in \mathbb{Z}$

$$T_{1/n}^j = T_{j/n}.$$

Then we can write

$$R(j/n, k/n) = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} e^{i(j+k)\alpha} e^{i\beta(j-k)} d\langle E_{(\alpha, \beta)}^{1/n} \delta(\cdot, \cdot), \delta(\cdot, \cdot) \rangle$$

using properties of the resolution of identity in H . We can continue and express

$$R(j|n, k|n) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(j+k)/n]x} e^{i\beta[(j-k)/n]} d\langle E_{(\alpha/n, \beta/n)}^{1/n} \delta(\cdot, \cdot), \delta(\cdot, \cdot) \rangle.$$

Under the choice of suitable j, k such that $j|n \rightarrow s, k|n \rightarrow t$ if n tends to $+\infty$ the continuity of $R(\cdot, \cdot)$ gives that

$$R(s, t) = \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(j+k)/n]x} e^{i\beta[(j-k)/n]} ddF_{(\alpha/n, \beta/n)}^{1/n}$$

if we denoted $\langle E_{(\alpha/n, \beta/n)}^{1/n} \delta(\cdot, \cdot), \delta(\cdot, \cdot) \rangle = F_{(\alpha/n, \beta/n)}^{1/n}$. Let

$$R_n(s, t) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i(s+t)x} e^{i\beta(s-t)} ddF_{(\alpha/n, \beta/n)}^{1/n};$$

we see that $R_n(\cdot, \cdot)$ is a normal covariance and we shall prove that $R_n(s, t) \rightarrow R(s, t)$ as $n \rightarrow +\infty$. First, we estimate

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\alpha(s+t)} e^{i\beta(s-t)} ddF_{(\alpha/n, \beta/n)}^{1/n} - \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\alpha(j+k)/n} e^{i\beta[(j-k)/n]} ddF_{(\alpha/n, \beta/n)}^{1/n} \right| \leq \\ & \leq \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\alpha(j+k)/n} (e^{i\beta(s-t)} - e^{i\beta[(j-k)/n]}) ddF_{(\alpha/n, \beta/n)}^{1/n} \right| + \\ & + \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\beta(s-t)} (e^{i\alpha(s+t)} - e^{i\alpha(j+k)/n}) ddF_{(\alpha/n, \beta/n)}^{1/n} \right| \leq \\ & \leq \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\alpha(j+k)/n} |e^{i\beta(s-t-(j-k)/n)} - 1| ddF_{(\alpha/n, \beta/n)}^{1/n} + \\ & + \left| \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\alpha(j+k)/n} (e^{i\alpha(s+t-(j+k)/n)} - 1) ddF_{(\alpha/n, \beta/n)}^{1/n} \right|. \end{aligned}$$

We can choose $j, k \in \mathbb{Z}$ that $j|n \rightarrow s, k|n \rightarrow t$ when $n \rightarrow +\infty$ and $0 \leq s - j|n < 1/n, 0 \leq t - k|n < 1/n$. Then, the first term can be estimated as

$$\begin{aligned} & \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i\alpha(j+k)/n} |e^{i\theta_n \beta} - 1| ddF_{(\alpha/n, \beta/n)}^{1/n} \leq \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2i\alpha(j+k)/n} ddF_{(\alpha/n, \beta/n)}^{1/n} \right)^{1/2} \\ & \left(\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} |e^{i\theta_n \beta} - 1|^2 ddF_{(\alpha/n, \beta/n)}^{1/n} \right)^{1/2} = \\ & = R^{1/2}((j+k)|n, (j+k)|n) \left(\int_{-\infty}^{+\infty} 2(1 - \cos(\theta_n \beta)) dF_{2(\beta/n)}^{1/n} \right)^{1/2} \end{aligned}$$

where $\theta_n = s - t - j|n + k|n$ and $F_{2(\beta/n)}^{1/n} = \int_{-\infty}^{+\infty} dF_{(\alpha/n, \beta/n)}^{1/n}$. Thanks to the continuity of $R(\cdot, \cdot)$, $R((j+k)|n, (j+k)|n) \rightarrow R(s+t, s+t)$ as $n \rightarrow +\infty$. Since $-1/n < \theta_n < 1/n$, $\cos(\theta_n \beta) \geq \cos(\beta)$ for every natural n and hence

$$\begin{aligned} & \int_{-\infty}^{+\infty} (1 - \cos(\theta_n \beta)) dF_{2(\beta/n)}^{1/n} = \int_{-\infty}^{+\infty} (1 - \cos(\theta_n \beta)) dF_{2(\beta)}^{1/n} \leq \\ & \leq \int_{-\infty}^{+\infty} (1 - \cos(\beta)) dF_{2(\beta)}^{1/n} = 1 - \operatorname{Re} \varphi_{1/n}(1), \end{aligned}$$

where

$$\varphi_{1/n}(u) = \int_{-\pi}^{+\pi} e^{i\beta u} dF_{2(\beta)}^{1/n} = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} e^{i\beta u} ddF_{(\alpha, \beta)}^{1/n} = R_n(u/2n, -u/2n).$$

It means we must prove that $\lim_{n \rightarrow \infty} R_n(1/2n, -1/2n) = 1$. Every operator $T_{1/n}$ can be expressed as $T_{1/n} = A_{1/n} U_{1/n}$, where $A_{1/n}$ is a positive self-adjoint operator and $U_{1/n}$ is unitary. In our case

$$A_{1/n} = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} e^{\lambda} dE_{(\lambda, \mu)}^{1/n}, \quad U_{1/n} = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} e^{i\mu} dE_{(\lambda, \mu)}^{1/n}.$$

Then we can write that $U_{1/n}f(u, v) = f(u, v - 1/n)$ and $\langle U_{1/n}\delta(\cdot, \cdot), \delta(\cdot, \cdot) \rangle = S(0, 1/n) = R(1/2n, -1/2n)$. On the other hand,

$$\langle U_{1/n}\delta(\cdot, \cdot), \delta(\cdot, \cdot) \rangle = \int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} e^{i\mu} d\langle E_{(\lambda, \mu)}^{1/n} \delta(\cdot, \cdot), \delta(\cdot, \cdot) \rangle = \int_{-\pi}^{+\pi} e^{i\mu} dF_{2(\mu)}^{1/n} = \varphi_{1/n}(1).$$

As we suppose the continuity of $R(\cdot, \cdot)$ we obtain $\lim_{n \rightarrow \infty} \varphi_{1/n}(1) = 1$. The second term

$$\int_{-\infty}^{+\infty} \int_{-\pi}^{+\pi} e^{\alpha[(j+k)/n]} (e^{\alpha[s+t-(j+k)/n]} - 1) ddF_{(\alpha/n, \beta/n)}^{1/n}$$

can be estimated in the following way:

$$\begin{aligned} & \left| \int_{-\infty}^{+\infty} e^{\alpha[(j+k)/n]} (e^{\alpha[s+t-(j+k)/n]} - 1) dF_{1(\alpha/n)}^{1/n} \right| \leq \\ & \leq \left(\int_{-\infty}^{+\infty} e^{2\alpha[(j+k)/n]} dF_{1(\alpha/n)}^{1/n} \right)^{1/2} \left(\int_{-\infty}^{+\infty} (e^{\alpha\varrho_n} - 1)^2 dF_{1(\alpha/n)}^{1/n} \right)^{1/2} = \\ & = \left(R\left(\frac{j+k}{n}, \frac{j+k}{n}\right) \right)^{1/2} \left(\int_{-\infty}^{+\infty} (e^{\alpha\varrho_n} - 1)^2 dF_{1(\alpha/n)}^{1/n} \right)^{1/2}, \quad 0 \leq \varrho_n = s+t - \frac{j+k}{n} < \frac{2}{n}. \end{aligned}$$

The last inequality implies that

$$(e^{2\alpha/n} - 1)^2 \geq (1 - e^{\alpha\varrho_n})^2$$

for every α and hence

$$\begin{aligned} & \int_{-\infty}^{+\infty} (e^{2\alpha\varrho_n} - 1)^2 dF_{1(\alpha/n)}^{1/n} \leq \int_{-\infty}^{+\infty} (e^{2(\alpha/n)} - 1)^2 dF_{1(\alpha/n)}^{1/n} = \\ & = \int_{-\infty}^{+\infty} (e^{4(\alpha/n)} - 2e^{2(\alpha/n)} + 1) dF_{1(\alpha/n)}^{1/n} = \int_{-\infty}^{+\infty} (e^{4\alpha} - 2e^{2\alpha} + 1) dF_{1(\alpha)}^{1/n} = \\ & = R(2/n, 2/n) - 2R(1/n, 1/n) + 1 \end{aligned}$$

and thanks to the continuity of $R(\cdot, \cdot)$ we can state that

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} (e^{2\alpha\varrho_n} - 1)^2 dF_{1(\alpha/n)}^{1/n} = 0.$$

We have proved that

$$|R_n(j/n, k/n) - R_n(s, t)| \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{where } j/n \rightarrow s, \quad k/n \rightarrow t.$$

As $R_n(j/n, k/n) = R(j/n, k/n)$ and $R(\cdot, \cdot)$ is continuous we obtained that

$$\lim_{n \rightarrow \infty} R_n(s, t) = R(s, t) \quad \text{for every } (s, t) \in \mathbb{R}_2.$$

Further, we shall prove that the sequence of the first marginals $\{F_{1(\alpha)}^{1/n}\}_{n=1}^{\infty}$ is compact in the sense that there exists a subsequence converging to a probability distribution function. We know that for every $j \in \mathbb{Z}$

$$R_n(j, j) = S_n(2j, 0) = R(j, j) = \int_{-\infty}^{+\infty} e^{2j\alpha} dF_{1(\alpha/n)}^{1/n}$$

is not depending on $n \in \mathbb{N}$. Thus, for every $j > 0$

$$R(j, j) \geq \int_{-\infty}^{2j} e^{2j\alpha} dF_{1(\alpha/n)}^{1/n} + (1 - F_{1(\alpha_1/n)}^{1/n})$$

that means

$$F_{1(\alpha_1/n)}^{1/n} \geq 1 - R(j, j) e^{-2\alpha_1 j}.$$

When ε is chosen quite arbitrarily we can find $\alpha_1 = \alpha_1(\varepsilon)$ such that for every $n \in \mathbb{N}$

$$F_{1(\alpha_1/n)}^{1/n} > 1 - \varepsilon.$$

Similarly, one can prove that there exists a suitable $\alpha_0 = \alpha_0(\varepsilon)$ such that for every $n \in \mathbb{N}$

$$F_{1(\alpha_0/n)}^{1/n} < \varepsilon$$

because $R(j, j) = \int_{-\infty}^{+\infty} e^{2\alpha j} dF_{1(\alpha/n)}^{1/n}$ holds for negative $j \in \mathbb{Z}$ too. This fact shows that the sequence $\{F_{1(\alpha)}^{1/n}\}_{n=1}^{\infty}$ is compact, i.e. there exists a subsequence $\{F_{1(\alpha_k)}^{1/n_k}\}_{k=1}^{\infty}$ converging to $F_1(\cdot)$ at all points of continuity. Without loss of generality we can assume that $\{F_{1(\alpha)}^{1/n}\}$ is just convergent to $F_1(\cdot)$.

In the case of the sequence $\{F_{2(\alpha)}^{1/n}\}_{n=1}^{\infty}$ we shall consider the sequence of the corresponding characteristic functions $\{\varphi_{1/n(\alpha)}\}_{n=1}^{\infty}$ where

$$\varphi_{1/n}(t) = \int_{-n\pi}^{+n\pi} e^{it\beta} dF_{2(\beta/n)}^{1/n}.$$

We know that

$$\varphi_{1/n}(2k/n) = \int_{-n\pi}^{+n\pi} e^{2i(k/n)\beta} dF_{2(\beta/n)}^{1/n} = R(k/n, -k/n) = S_n(0, 2k/n).$$

We can choose $k \in \mathbb{Z}$ in such a way that $k/n \rightarrow t$ as $n \rightarrow +\infty$ and $0 < t - k/n < 1/n$ for every $n \in \mathbb{N}$. First, we shall prove that

$$\varphi_{1/n}(k/n) - \varphi_{1/n}(t) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

As $\varphi_{1/n}(t) = \int_{-n\pi}^{+n\pi} e^{it\beta} dF_{2(\beta/n)}^{1/n}$ then

$$\begin{aligned} |\varphi_{1/n}(k/n) - \varphi_{1/n}(t)| &= \left| \int_{-n\pi}^{+n\pi} (e^{it\beta} - e^{i(k/n)\beta}) dF_{2(\beta/n)}^{1/n} \right| = \\ &= \left| \int_{-n\pi}^{+n\pi} e^{i(k/n)\beta} (e^{i(t-(k/n))\beta} - 1) dF_{2(\beta/n)}^{1/n} \right| \leq \left(\int_{-n\pi}^{+n\pi} |e^{i(t-(k/n))\beta} - 1|^2 dF_{2(\beta/n)}^{1/n} \right)^{1/2} = \\ &= \sqrt{2} \left(\int_{-n\pi}^{+n\pi} (1 - \cos(t - k/n)\beta) dF_{2(\beta/n)}^{1/n} \right)^{1/2} \leq \\ &\leq \sqrt{2} \left(\int_{-n\pi}^{+n\pi} (1 - \cos(\beta)) dF_{2(\beta)}^{1/n} \right)^{1/2} = \sqrt{2} (1 - \operatorname{Re} \varphi_{1/n}(1))^{1/2} \rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned}$$

as was shown sooner. The assumptions of the theorem yield immediately that the function $R(t/2, -t/2) = S(0, t)$ is a characteristic function, i.e.

$$S(0, t) = \int_{-\infty}^{+\infty} e^{it\beta} dF_2(\beta).$$

As

$\varphi_{1/n}(2k/n) = S(0, 2k/n) = \int_{-\infty}^{+\infty} e^{i(2k/n)\beta} dF_2(\beta)$ then $\varphi_{1/n}(2t) \rightarrow S(0, 2t)$ as $n \rightarrow \infty$; we have proved that

$$F_{2(\beta/n)}^{1/n} \rightarrow F_2(\beta)$$

at all points of continuity.

Now, we can estimate the measure of $K = [\alpha_0, \alpha_1] \times [\beta_0, \beta_1]$ in the plane

$$\iint_K ddF_{(\alpha/n, \beta/n)}^{1/n}.$$

As

$$\begin{aligned} \iint_K ddF_{(\alpha/n, \beta/n)}^{1/n} &= \iint_{-\infty}^{+\infty} \psi_{[\alpha_0, \alpha_1]}(\alpha) \psi_{[\beta_0, \beta_1]}(\beta) ddF_{(\alpha/n, \beta/n)}^{1/n} = \\ &= \int_{\alpha_0}^{\alpha_1} \int_{-\infty}^{+\infty} ddF_{(\alpha/n, \beta/n)}^{1/n} - \int_{\alpha_0}^{\alpha_1} \int_{-\infty}^{\beta_0} ddF_{(\alpha/n, \beta/n)}^{1/n} - \int_{\alpha_0}^{\alpha_1} \int_{\beta_1}^{+\infty} ddF_{(\alpha/n, \beta/n)}^{1/n} \end{aligned}$$

then

$$\begin{aligned} \iint_{\mathbb{K}} ddF_{(\alpha/n, \beta/n)}^{1/n} &\geq (F_{1(\alpha_1/n)}^{1/n} - F_{1(\alpha_0/n)}^{1/n}) - \int_{-\infty}^{+\infty} \int_{-\infty}^{\beta_0} ddF_{(\alpha/n, \beta/n)}^{1/n} - \\ &- \int_{-\infty}^{+\infty} \int_{\beta_1}^{\infty} ddF_{(\alpha/n, \beta/n)}^{1/n} = (F_{1(\alpha_1/n)}^{1/n} - F_{1(\alpha_0/n)}^{1/n}) - (F_{2(\beta_0/n)}^{1/n} + 1 - F_{2(\beta_1/n)}^{1/n}) > 1 - 2\varepsilon. \end{aligned}$$

because $F_{1(\cdot)}^{1/n} \rightarrow F_1(\cdot)$ and $F_{2(\cdot)}^{1/n} \rightarrow F_2(\cdot)$ as $n \rightarrow \infty$ and $F_1(\cdot), F_2(\cdot)$ are probability distribution functions. This inequality proves that the sequence $\{F_{(\cdot/n, \cdot/n)}^{1/n}\}_{n=1}^{\infty}$ is compact. Hence, there exists a subsequence $\{F_{(\cdot/n_k, \cdot/n_k)}^{1/n_k}\}_{k=1}^{\infty}$ that is convergent to a probability distribution function $F(\cdot, \cdot)$. It remains to prove that

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{\lambda(s+t)} e^{i\mu(s-t)} ddF(\lambda, \mu).$$

We proved that

$$R(s, t) = \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} \int_{-n\pi}^{+n\pi} e^{\alpha(s+t)} e^{i\beta(s-t)} ddF_{(\alpha/n, \beta/n)}^{1/n}.$$

At this moment we need possibility to change the order between integration and convergence. This change is possible under uniform integration of $|e^{\alpha(s+t)} e^{i\beta(s-t)}| = e^{\alpha(s+t)}$ with respect to the sequence $\{F_{(\cdot/n_k, \cdot/n_k)}^{1/n_k}\}_{k=1}^{\infty}$. We know that

$$R((s+t)/2, (s+t)/2) = S(s+t, 0) = \lim_{n \rightarrow \infty} \int_{-\infty}^{+\infty} e^{\alpha(s+t)} dF_{1(\alpha/n)}^{1/n}$$

is a continuous covariance function because the function $S(\cdot, 0)$ forms a nonnegative definite kernel with respect to the sum. Every function of these properties can be expressed as a bilateral Laplace transform

$$S(s+t, 0) = \int_{-\infty}^{+\infty} e^{\alpha(s+t)} dG_1(\alpha)$$

where $G_1(\cdot)$ is a probability distribution function because $S(0, 0) = 1$. Sooner we proved that $F_{1(\alpha/n)}^{1/n} \rightarrow F_1(\cdot)$ as $n \rightarrow \infty$, and on the basis of the convergence of moments we state that $G_1(\cdot) = F_1(\cdot)$. We have proved

$$\lim_{k \rightarrow \infty} \iint_{-\infty}^{+\infty} |e^{\alpha(s+t)} e^{i\beta(s-t)}| ddF_{(\alpha/n_k, \beta/n_k)}^{1/n_k} = \iint_{-\infty}^{+\infty} |e^{\alpha(s+t)} e^{i\beta(s-t)}| ddF(\alpha, \beta)$$

i.e. the function $e^{\alpha(s+t)}$ is uniformly integrable with respect to the sequence $\{F_{(\cdot/n_k, \cdot/n_k)}^{1/n_k}\}_{k=1}^{\infty}$. Further, the convergence $F_{(\cdot/n_k, \cdot/n_k)}^{1/n_k} \rightarrow F(\cdot, \cdot)$ as $k \rightarrow \infty$ implies the existence of $\iint_{-\infty}^{+\infty} e^{\alpha(s+t)} e^{i\beta(s-t)} ddF(\alpha, \beta)$. Together, we can change order between integration and convergence and we can write

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{\alpha(s+t)} e^{i\beta(s-t)} ddF(\alpha, \beta).$$

On the contrary, let $R(\cdot, \cdot)$ be normal. Let

$$S(u, v) = \iint_{-\infty}^{+\infty} e^{\alpha u} e^{i\beta v} ddF(\alpha, \beta).$$

Then for arbitrary $\alpha_1, \alpha_2, \dots, \alpha_n \in \mathbb{C}$ and arbitrary $u_1, u_2, \dots, u_n, v_1, v_2, \dots, v_n \in \mathbb{R}_1$ surely

$$\sum_j \sum_k \alpha_j \bar{\alpha}_k S(u_j + u_k, v_j - v_k) \geq 0$$

because

$$\sum_j \sum_k \alpha_j \bar{\alpha}_k e^{\alpha(u_j + u_k)} e^{i\beta(v_j - v_k)} = \left| \sum_j \alpha_j e^{\alpha u_j} e^{i\beta v_j} \right|^2 \geq 0. \quad \square$$

The class of normal covariances can be also described by use of the corresponding reproducing kernel Hilbert space (RKHS).

Theorem 3. Let $R(\cdot, \cdot)$ be a continuous covariance function defined in the whole plane \mathbb{R}_2 . $R(\cdot, \cdot)$ is normal if and only if

$$R(s + h, t) = \langle R(s, u); R(t, u + h) \rangle$$

holds for every real h , $\langle \cdot, \cdot \rangle$ is the scalar product in RKHS due to the covariance function $R(\cdot, \cdot)$.

Proof. Let $R(\cdot, \cdot)$ be normal. Then

$$R(s, t) = \iint_{-\infty}^{+\infty} e^{\alpha(s+\tau)} e^{i\beta(s-\tau)} ddF(\alpha, \beta), \quad (s, t) \in \mathbb{R}_2.$$

Let h be any real number, thus

$$\begin{aligned} R(s + h, t) &= \iint_{-\infty}^{+\infty} e^{\alpha(s+h+\tau)} e^{i\beta(s+h-\tau)} ddF(\alpha, \beta) = \\ &= \iint_{-\infty}^{+\infty} e^{\alpha s} e^{i\beta s} e^{\alpha(\tau+h)} e^{i\beta(\tau-h)} ddF(\alpha, \beta). \end{aligned}$$

Similarly,

$$\begin{aligned} R(t, u + h) &= \iint_{-\infty}^{+\infty} e^{\alpha(t+u+h)} e^{i\beta(t-u-h)} ddF(\alpha, \beta) = \\ &= \iint_{-\infty}^{+\infty} e^{\alpha(t+h)} e^{i\beta(t-h)} e^{\alpha u} e^{i\beta u} ddF(\alpha, \beta) \end{aligned}$$

and

$$R(s, u) = \iint_{-\infty}^{+\infty} e^{\alpha s} e^{i\beta s} e^{\alpha u} e^{i\beta u} ddF(\alpha, \beta).$$

Now, by use of the "reproducing property" $m(s) = \langle m(\cdot); R(s, \cdot) \rangle$ holding for every $m(\cdot) \in \text{RKHS}$ we obtain immediately

$$R(s + h, t) = \langle R(s, u); R(t, u + h) \rangle.$$

On the contrary, let for every $h \in \mathbb{R}_1$ the covariance function $R(\cdot, \cdot)$ satisfy

$$R(s + h, t) = \langle R(s, u); R(t, u + h) \rangle$$

in the corresponding RKHS. Let us define the shift-operator T_h in the RKHS by the relation

$$T_h R(s, \cdot) = R(s + h, \cdot).$$

The definition domain of every T_h is formed by all linear combinations $\sum_{i=1}^n \alpha_i R(s_i, \cdot)$,

where α_i , $i = 1, \dots, n$, are complex numbers. The construction of the RKHS gives that the definition domain $\mathcal{D}(T_h)$ of T_h is everywhere dense linear subset in the RKHS. Let T_h^* be the adjoint operator to T_h in the RKHS. Let us prove that $\mathcal{D}(T_h) \subset \mathcal{D}(T_h^*)$. Let $m(\cdot) \in \mathcal{D}(T_h)$. By definition of T_h $y(\cdot) \in \text{RKHS}$ belongs to $\mathcal{D}(T_h^*)$ if and only if for every $x(\cdot) \in \mathcal{D}(T_h)$

$$\langle T_h x(\cdot); y(\cdot) \rangle = \langle x(\cdot); T_h y(\cdot) \rangle.$$

When $x(\cdot) = \sum_i \alpha_i R(s_i, \cdot)$, $m(\cdot) = \sum_j \beta_j R(t_j, \cdot)$, then $T_h x(\cdot) = \sum_i \alpha_i R(s_i + h, \cdot)$ and

$$\begin{aligned} \langle T_h x(\cdot); m(\cdot) \rangle &= \sum_i \sum_j \alpha_i \beta_j \langle R(s_i + h, u); R(t_j, u) \rangle = \\ &= \sum_i \sum_j \alpha_i \beta_j R(s_i + h, t_j) = \sum_i \sum_j \alpha_i \beta_j \langle R(s_i, u); R(t_j, u + h) \rangle = \langle x(\cdot); m^*(\cdot) \rangle \end{aligned}$$

where $m^*(\cdot) = \sum_j \beta_j R(t_j, \cdot) + h = T_h^* m(\cdot)$. In other words, the adjoint operator T_h^* is represented by the shift operator in the argument in the RKHS, i.e.

$$T_h^* R(s, \cdot) = R(s, (\cdot) + h).$$

It implies that T_h^* is everywhere densely defined and hence the operator T_h possesses a closed enlargement \bar{T}_h in the RKHS. There is no problem to show that for every $m_1(\cdot), m_2(\cdot) \in \mathcal{D}(T_h)$

$$\langle T_h m_1(\cdot); T_h m_2(\cdot) \rangle = \langle T_h^* m_1(\cdot); T_h^* m_2(\cdot) \rangle$$

and since this moment we can follow the proof of Theorem 2. We shall prove that every operator T_h in the RKHS is normal and by means of their spectral resolutions one can show in the same way as was used in the proof of Theorem 2 that the covariance function $R(\cdot, \cdot)$ is normal. \square

Remark. In this part we shall consider two covariances. The first one

$$R_1(s, t) = e^{c(s+t)^2/D} e^{-a(s-t)^2/D} e^{-i b(s^2-t^2)/D}, \quad D = 4ac - b^2 > 0,$$

is normal, the other one

$$R_2(s, t) = e^{-\gamma(s^2+t^2)}, \quad \gamma > 0,$$

is not normal. The covariance $R_1(s, t)$ is normal because

$$R_1(s, t) = \iint_{-\infty}^{+\infty} e^{a(s+t)} e^{i\theta(s-t)} e^{-(ax^2+bx\beta+c\beta^2)} dx d\beta$$

In case when $b = 0$ we obtain a locally stationary covariance because then $R_1(s, t) = S_1[(s+t)/2] S_2(s-t)$ where $S_1(u) = e^{(4c/D)u^2} > 0$, $S_2(t) = e^{-a t^2/D}$ is a characteristic function. Further, this case is interesting also because the correlation function corresponding to $R_1(s, t)$ for $b = 0$

$$\rho(s, t) = \frac{R_1(s, t)}{R_1^{1/2}(s, s) R_1^{1/2}(t, t)} = e^{-[(a+c)/D](s-t)^2}$$

is depending on $s - t$ only.

On the other hand, the covariance $R_2(s, t)$ is locally stationary also because

$$R_2(s, t) = e^{-2\gamma[(s+t)/2]^2} e^{\gamma(s-t)^2/2},$$

although the first term $e^{-2\gamma[(s+t)/2]^2}$ is not covariance. When we put, in the case of $R_1(\cdot, \cdot)$ $a = c = \sqrt{2}/2$ we have

$$R_1(s, t) = e^{2st} = e^{(s+t)^2/2} e^{-(s-t)^2/2},$$

and $\gamma = 1$ for the case of $R_2(\cdot, \cdot)$, then

$$R_2(s, t) = e^{-(s^2+t^2)} = e^{-[(s+t)/2]^2} e^{-(s-t)^2/2}.$$

The function $e^{[(s+t)/2]^2}$ is a covariance but $e^{-[(s+t)/2]^2}$ is not covariance. If we consider the shift-operator T_h in the RKHS due to the covariance e^{2st} then

$$R_1(s+h, t) = e^{2(s+k)t} = e^{2st} \cdot e^{2kt} = R_1(s, t) \cdot R_1(h, t).$$

This fact yields that for every $m(\cdot) \in \mathcal{D}(T_h)$

$$T_h(m \cdot) = m(\cdot) R_1(h, \cdot),$$

that shows $\langle T_h m(\cdot); x(\cdot) \rangle$ is a continuous linear functional on $\mathcal{D}(T_h)$ and hence $\mathcal{D}(T_h^*) \supset \mathcal{D}(T_h)$. On the other hand, a similar resolution in the case of $R_2(\cdot, \cdot)$ is not possible because

$$\begin{aligned} R_2(s+h, t) &= e^{-(s+h)^2} e^{-t^2} = e^{-(s^2+t^2)} e^{-2hs} \cdot e^{-h^2} = \\ &= R_2(s, t) R_2\left(\frac{h\sqrt{2}}{2}, \frac{h\sqrt{2}}{2}\right) e^{-2hs}. \end{aligned}$$

Thus

$$T_h m(\cdot) = R_2\left(\frac{h\sqrt{2}}{2}, \frac{h\sqrt{2}}{2}\right) \sum_{i=1}^n \alpha_i R_2(s_i, \cdot) e^{-2hs_i}$$

when $m(\cdot) = \sum_{i=1}^n \alpha_i R(s_i, \cdot)$ and the assumption $m_n(\cdot) \rightarrow 0$ in $\mathcal{D}(T_h)$ need not imply, in general, that $\langle T_h m_n(\cdot); x(\cdot) \rangle \rightarrow 0$. This fact causes that the adjoint operator T_h^* is not well defined ($(\mathcal{D}(T_h^*) \not\supset \mathcal{D}(T_h))$) and T_h cannot be normal in the RKHS.

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