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NECESSARY AND SUFFICIENT OPTIMALITY CONDITIONS FOR TWO-STAGE STOCHASTIC PROGRAMMING PROBLEMS

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The explicit form of the partial derivatives of the optimalized functions in two-stage stochastic programming problems was introduced in [5]. In this paper we shall employ this result to get some optimality conditions. More precisely, we shall present the necessary and sufficient optimality conditions in the concave case. Moreover, we shall apply the obtained results to some special cases.

1. INTRODUCTION

Necessary and sufficient optimality conditions for two-stage stochastic programming problems were already introduced in the literature. For example, general optimality conditions using the theory of saddle point were determined in [11]. The case in which the probability measure can depend on the solution is considered in [12]. In this paper we shall present another form of the optimality conditions for a special type of the two-stage stochastic programming problem.

Let

$X \subset E_n$, $Z_1 \subset E_{s_1}$, $Z_2 \subset E_{s_2}$, $U \subset E_r$, $n, s_1, s_2, r \geqq 1$ be non-empty sets,

$(\Omega, \mathscr{S}, P)$ be the probability space,
$\eta = \eta(\omega)$ and $\xi = \xi(\omega)$, be respectively s_1-dimensional and s_2-dimensional vectors defined on $(\Omega, \mathscr{S}, P)$ such that

$$P\{\omega\colon \eta(\omega) \in Z_1\} = P\{\omega\colon \xi(\omega) \in Z_2\} = 1 \tag{1}$$

(E_n, $n \geqq 1$ denotes an n-dimensional Euclidean space). It is assumed in (1) that $\{\omega\colon \eta(\omega) \in Z_1\} \in \mathscr{S}$ and $\{\omega\colon \xi(\omega) \in Z_2\} \in \mathscr{S}$. Moreover, let $h_i(u, z_2)$, $i = 0, 1, 2, \ldots$ $\ldots, l$ and $f_i(x, z_1)$, $i = 0, 1, 2, \ldots, l$ be real valued continuous functions defined on $E_r \times Z_2$ and $E_n \times Z_1$, respectively.

If the mapping $K(u, z_1, z_2)$ and the functions $\varphi(u, z_1, z_2)$, $\psi(u, z_1, z_2)$ are defined by

$$(2)\qquad \begin{aligned} K(u, z_1, z_2) &= \{x \in X: f_i(x, z_1) \leqq h_i(u, z_2),\ i = 1, 2, \ldots, l\}\,, \\ \varphi(u, z_1, z_2) &= \sup\{f_0(x, z_1): x \in K(u, z_1, z_2)\}\,, \\ \psi(u, z_1, z_2) &= \varphi(u, z_1, z_2) + h_0(u, z_2) \end{aligned}$$

and if $F_j(u)$, $j = 1, 2, \ldots, m$ are real valued continuous functions defined on E_r, then we can introduce the two-stage stochastic programming problem as a problem of two-stage decision to find

$$(3)\qquad \sup\{f_0(x, z_1): x \in K(u, z_1, z_2)\}$$

and further to find

$$(4)\qquad \sup\{\mathsf{E}\psi(u, \eta(\omega), \xi(\omega)): u \in U'\}\,,$$

where

$$U' = \{u \in U_r: u = (u_1, \ldots, u_r),\ F_j(u) \geqq 0,\ j = 1, 2, \ldots, m,\ u_i \geqq 0,\ i = 1, 2, \ldots, r\}\,.$$

(E denotes the operator of mathematical expectation).

The problem given by relations (2), (4) is similar to the well known stochastic programming problem with *recourse*.

The aim of the paper is to present optimality conditions of the just introduced two-stage stochastic programming problems.

Remarks.

1. It can generally happen that some symbols in (2), (3) and (4) are not meaningful. This situation cannot appear under the assumptions which will be respected in this paper.
2. The problem given by (2), (3) and (4) is a special case of the two-stage stochastic nonlinear programming problems introduced in [2], [4].

It will be useful to substitute

$$(5)\qquad y_i = h_i(u, z_2),\ i = 1, 2, \ldots, l\,, \quad y = (y_1, \ldots, y_l) \in E_l$$

and to formulate the problem given by (2), (3) as the following parametric optimization problem:

Find

$$(6)\qquad \bar{\varphi}(z_1, y) = \sup\{f_0(x, z_1): x \in \bar{K}(z_1, y)\}\,,$$

where

$$\bar{K}(z_1, y) = \{x \in X: f_i(x, z_1) \leqq y_i\ i = 1, 2, \ldots, l\}\,,$$

$z_1 \in Z_1$, $y \in E_l$.

It is easy to see that

$$\begin{aligned} K(u, z_1, z_2) &= \bar{K}[z_1, (h_1(u, z_2), \ldots, h_l(u, z_2))]\,, \\ \varphi(u, z_1, z_2) &= \bar{\varphi}[z_1, (h_1(u, z_2), \ldots, h_l(u, z_2))]\,. \end{aligned}$$

Further we denote by $Y \subset E_l$ the set for which

$$(7) \qquad [(h_1(u, z_2), \ldots, h_l(u, z_2)] \subset \operatorname{int} Y \quad \text{for} \quad u \in U\,, \quad z_2 \in Z_2$$

and by $h(u, z_2)$, $f(x, z_1)$ and $F(u)$ the following vector functions

$$(8) \qquad \begin{aligned} h(u, z_2) &= [h_1(u, z_2), \ldots, h_l(u, z_2)]\,, \\ f(x, z_2) &= [f_1(x, z_1), \ldots, f_l(x, z_1)]\,, \\ F(u) &= [F_1(u), \ldots, F_m(u)]\,. \end{aligned}$$

for $u \in E_r$, $x \in E_n$, $z_1 \in E_{s_1}$, $z_2 \in E_{s_2}$.

The problem given by (6) is a parametric optimization problem with parameters $z_1 \in Z_1$, $y \in Y$. However, we can also consider this problem separately for every $z_1 \in Z_1$, $y \in Y$ and define the Lagrangian function $L(x, v/z_1, y)$ and the Kuhn-Tucker vector $v(z, y) = v$. So let $z_1 \in Z_1$, $y \in Y$ be arbitrary given points, then

$$(9) \qquad L(x, v/z_1, y) = f_0(x, z_1) + \sum_{i=1}^{l} v_i[y_i - f_i(x, z_1)]\,,$$

$$x = (x_1, \ldots, x_n) \in E_n\,, \quad v = (v_1, \ldots, v_l) \in E_l\,.$$

Definition 1. A vector $v = v(z_1, y)$, $v \geqq 0$, $v \in E_l$ is the Kuhn-Tucker vector of the problem (6) if

$$\bar{\varphi}(z_1, y) = \sup \{L(x, v/z_1, y) : x \in X\}$$

$$(v = (v_1, \ldots, v_l) \geqq 0 \text{ denotes that } v_i \geqq 0 \text{ for all } i = 1, 2, \ldots, l)\,.$$

Further, the Hausdorff distance between two subsets in E_n can be defined in the following way.

Definition 2. If X', $X'' \subset E_n$, $n \geqq 1$ are two non-empty sets then the Hausdorff distance of these sets $\Delta_n(X', X'')$ is defined by

$$\Delta_n(X', X'') = \max [\delta_n(X', X''), \delta_n(X'', X')]\,,$$

$$\delta_n(X', X'') = \sup_{x' \in X'} \inf_{x'' \in X''} \varrho_n(x', x'')\,,$$

where ϱ_n denotes the Euclidean metric in E_n. (We usually omit the subscripts in the symbols Δ_n, ϱ_n, δ_n.)

If we denote by $P_{\cdot|\cdot}$ and $\mathsf{E}_{\cdot|\cdot}$ the conditional probability measure and the conditional mathematical expectation respectively, and if we denote by $X(\varepsilon)$, $\varepsilon > 0$ the sets

$$X(\varepsilon) = X + B(\varepsilon) = \{x = x_1 + x_2 : x_1 \in X,\ x_2 \in B(\varepsilon)\}\,,$$

where $B(\varepsilon)$ is the ε-surroundings of $0 \in E_n$, then we can recall Theorem of [5]. This can be here formulated in the following way.

Theorem 1. Let X, U, Y be convex sets, and let $h_i(u, z_2)$, $i = 1, 2, \ldots, l$ be differentiable functions on U for every $z_2 \in Z_2$. If relations (1) are fulfilled and if

1) $h(u, \xi(\omega))$ is for every $u \in U$ a random vector such that the conditional probability measure $P_{h|\eta}$ is absolutely continuous with respect to the Lebesgue measure in E_l,

2) there exists a constant $g \in E_1$ such that the condition

$$\varrho[h(u, z_2), h(u', z_2)] \leqq g\,\varrho(u, u')$$

is fulfilled for every $u', u \in U$, $z_2 \in Z_2$,

3) $y \in Y$, $z_1 \in Z_1$ implies $\bar{K}(z_1, y) \neq \emptyset$ and the fulfillment of at least one of the following two conditions

a) $\bar{K}(z_1, y)$ is a compact set,

b) $f_0(x, z_1)$ is a bounded function on $\bar{K}(z_1, y)$,

4) $f_0(x, z_1)$ is for every $z_1 \in Z_1$ a Lipschitz function on E_n with the Lipschitz constant c_1 not depending on $z_1 \in Z_1$,

5) there exists a constant c_2 such that

$$\Delta[\bar{K}(z_1, y), \bar{K}(z_1, y')] \leqq c_2\,\varrho(y, y')$$

for every $z_1 \in Z_1$, $y, y' \in Y$,

6) $f_0(x, z_1)$ and $f_i(x, z_1)$, $i = 1, 2, \ldots, l$, are respectively concave and convex functions on E_n for every $z_1 \in Z_1$,

7) there exists finite $\mathsf{E}\bar{\varphi}(\eta(\omega), h(u, \xi(\omega))$ for every $u \in \operatorname{int} U$,

8) there exists finite, differentiable $\mathsf{E}h_0(u, \xi(\omega))$ for every $u \in \operatorname{int} U$,

then there exists the vector of the partial derivatives of the function $\mathsf{E}\psi(u, \eta(\omega), \xi(\omega))$ for $u \in \operatorname{int} U$, and moreover

$$(10) \qquad \frac{\partial \mathsf{E}\psi(u, \eta(\omega), \xi(\omega))}{\partial u_j} = \frac{\partial \mathsf{E}h_0(u, \xi(\omega))}{\partial u_j} + $$

$$+ \mathsf{E}\sum_{i=1}^{l} v_i(\eta(\omega), h(u, \xi(\omega))\frac{\partial h_i(u, \xi(\omega))}{\partial u_j}, \quad j = 1, 2, \ldots, l$$

where $v = v(z_1, y) = [v_1(z_1, y), \ldots, v_l(z_1, y)]$ is a Kuhn-Tucker vector of the parametric optimization problem to find

$$\sup \{f_0(x, z_1): x \in X, f_i(x, z_1) \leqq y_i, i = 1, 2, \ldots, l\}.$$

Since the problem given by (2), (3) and (4) is a problem of the two-stage decision, we can consider its solution (of course, if it exists) as a vector $[u^*, x^* = x^*(z_1, h(u^*, z_2))]$ for which

$$(11) \qquad f_0[x^*(z_1, h(u, z_2)), z_1] = \sup \{f_0(x, z_1): x \in \bar{K}(z_1, h(u, z_2))\}$$

and

$$(12) \qquad \mathsf{E}\{h_0(u^*, \xi(\omega)) + f_0(x^*(\eta(\omega), h(u^*, \xi(\omega))), \eta(\omega))\} = $$

$$= \sup \mathsf{E}\{h_0(u, \xi(\omega)) + \varphi(u, \eta(\omega), \xi(\omega)): u \in U'\}.$$

Remark. The conditions under which the problem given by (2), (3), (4) is equivalent to the problem to find

$$\sup \mathsf{E}\{f_0(x, \eta(\omega)) + h_0(u, \xi(\omega))\}$$

under the conditions

$$u \in U'$$

$$x = x(\eta(\omega), \xi(\omega)) \in K(u, \eta(\omega), \xi(\omega)) \quad \text{a.s.}$$

$$x(\eta(\omega), \xi(\omega)) \in L_1$$

are given in [10].

2. MAIN RESULTS

To get the optimality conditions we restrict the introduced problem given by (2), (3) and (4) to that in which the function $\varphi(u, z_1, z_2)$ is concave on E_r^+ ($E_r^+ = \{u \in E_r: u = (u_1, \dots, u_r), u_i \geqq 0, i = 1, 2, \dots, r\}$) only.

We can formulate the main result of this paper as the following theorem.

Theorem 2. Let the assumptions of Theorem 1 be fulfilled for $U = E_r^+$, $X = E_n^+$. If

(i) $h_i(u, z_2)$, $i = 0, 1, 2, \dots, l$ for every $z_2 \in Z_2$ and $F_j(u)$, $j = 1, 2, \dots, m$, are concave functions on E_r,

(ii) there exists $u \in E_r^+$ such that

$$F_j(u) > 0\,, \quad j = 1, 2, \dots, m\,,$$

(iii) for every $u \in U$, $z_1 \in Z_1$, $z_2 \in Z_2$ there exists $x = x(u, z_1, z_2) \in X$ such that

$$f_i(x, z_1) < h_i(u, z_2)\,, \quad i = 1, 2, \dots, l\,,$$

(iv) there exist continuous partial derivatives of the function $F_j(u)$, $j = 1, 2, \dots, m$ and $f_i(x, z_1)$, $i = 0, 1, 2, \dots, l$, respectively, with respect to the components of the vector u and the vector x (on $E_r \times E_n$),

then u^*, $x^* = x^*(z_1, h(u^*, z_2))$ is the solution of the problem given by (11), (12) (of course, also by (2), (3), (4)) if and only if there exist vectors $v^* = v^*(z_1, h(u^*, z_2)) \in E_l^+$, $t^* \in E_m^+$, $v^* = (v_1^*, \dots, v_l^*)$, $v_i^* = v_i^*(z_1, h(u^*, z_2))$, $i = 1, 2, \dots, l$, $t^* = (t_1^*, \dots, t_m^*)$ such that the following conditions are fulfilled.

$$\nabla_x f_0(x^*, z_1) - \sum_{i=1}^{l} v_i^*(z_1, h(u^*, z_2)) \nabla_x f_i(x^*, z_1) \leqq 0 \tag{13}$$

$$\langle x^*, \nabla_x f_0(x^*, z_1) - \sum_{i=1}^{l} v_i^*(z_1, h(u^*, z_2)) \nabla_x f_i(x^*, z_1)\rangle = 0$$

$$h(u^*, z_2) - f(x^*, z_1) \geqq 0$$

$$\langle v^*(z_1, h(u^*, z_2)), h(u^*, z_2) - f(x^*, z_1)\rangle = 0$$

for all $z_1 \in Z_1$, $z_2 \in Z_2$,

$$\nabla_u \mathsf{E} h_0(u^*, \xi(\omega)) + \mathsf{E} \sum_{i=1}^{l} v_i^*(\eta(\omega), h(u^*, \xi(\omega)) \nabla_u h_i(u^*, \xi(\omega)) + \tag{14}$$

$$+ \sum_{j=1}^{m} t_j^* F_j(u^*) \leqq 0$$

$$\langle u^*, \nabla_u \mathsf{E} h_0(u^*, \xi(\omega)) + \mathsf{E} \sum_{i=1}^{l} v_i^*(\eta(\omega), h(u^* \xi(\omega)) \nabla_u h_i(u^*, \xi(\omega)) +$$

$$+ \sum_{j=1}^{m} t_j^* \nabla_u F_j(u^*)\rangle = 0$$

$$F(u^*) \geqq 0\,, \quad \langle t^*, F(u^*)\rangle = 0$$

where $v_i^*(z_1, y)$ is a Kuhn-Tucker vector of the parametric optimization problem to find

$$\sup \{f_0(x, z_1): x \in \bar{K}(z_1, y)\}\,.$$

$\langle \cdot, \cdot \rangle$ denotes the scalar product in E_n, $\nabla_x f_i(\cdot, \cdot)$ the vector of the partial derivatives with respect to the components of the vector x.

Proof. The problem given by (6) is (under the assumptions of Theorem 2) a problem of the concave programming for every $z_1 \in Z_1$, $y \in Y$. Moreover, since it is possible to generalize Lemma 1 of [2] to an arbitrary set $Z = Z_1 \times Z_2$, we can easily see that $\mathsf{E}\varphi(u, \eta(\omega), \xi(\omega))$ is a concave function too.

The proof of Theorem 2 will be divided into several parts. Let $[u^*, x^* = x^*(u^*, z_1, z_2)]$ be a solution of the problem given by (11), (12). First, we prove necessity of the system (13). It follows from the assumptions that we can utilize (for every $z_1 \in Z_1$, $u \in U$, $z_2 \in Z_2$, $y = h(u, z_2)$ separately) the assertion of Theorem 7.1.1 in [6]. According to this we obtain the existence of the vector $v^* = v^*(z_1, y)$ such that

$$L(x, v^*/z_1, y) \leqq L(x^*, v^*/z_1. y) \leqq L(x^*, v/z_1, y)$$

for all $x \in E_n^+$, $v \geqq 0$, $v \in E_l$ and $\langle v^*, y - f(x^*, z_1)\rangle = 0$. However, since the function $L(x, v/z_1, y)$ also fulfils the assumptions of Theorem 7.1.3. in [6], we can see that the following relations hold

$$\nabla_x f_0(x^*, z_1) - \sum_{i=1}^{l} v_i^*(z_1, y) \nabla_x f_i(x^*, z_1) \leqq 0\,,$$

$$\langle x^*, \nabla_x f_0(x^*, z_1) - \sum_{i=1}^{l} v_i^*(z_1, y) \nabla_x f_i(x^*, z_1)\rangle = 0\,,$$

$$y - f(x^*, z_1) \geqq 0\,,$$

$$\langle v^*, y - f(x^*, z_1)\rangle = 0\,.$$

Using the substitution $y = h(u, z_2)$ we get immediately the validity of relations (13) in the parametric point u^*, $z_1 \in Z_1$, $z_2 \in Z_2$. According to Lemma 5 of [5] the vector v^* is a Kuhn-Tucker vector of the problem given by (6).

Further, we shall deal with the problem given by (12). First, in this case, using Theorem 1 we obtain the form of the partial derivatives of the optimalized function $\mathsf{E}\psi(u, \eta(\omega), \xi(\omega))$. Since $\mathsf{E}\psi(u, \eta(\omega), \xi(\omega))$ is a concave function it follows from Theorem 24.1 in [9] that these partial derivatives are continuous. According to this fact and to the assumptions of Theorem 2 we can utilize Theorems 7.1.1 and 7.1.3 in [6] again. We obtain the validity of the system

$$
\begin{aligned}
&(15) \qquad && \nabla_u L_1(u^*, t^*) \leqq 0 \\
& && \langle u^*, \nabla_u L_1(u^*, t^*)\rangle = 0 \\
& && \nabla_t L_1(u^*, t^*) \geqq 0 \\
&(16) && \langle t^*, \nabla_t L_1(u^*, t^*)\rangle = 0 \quad \text{for a } t^* \in E_m^+ \,.
\end{aligned}
$$

$$L_1(u, t) = \mathsf{E}\psi(u, \eta(\omega), \xi(\omega)) + \sum_{j=1}^{m} t_j F_j(u)$$

is the Lagrangian function of the problem given by (12).

It is easy to see that necessity of the system (14) follows from (10) and (15) immediately.

It remains to prove that conditions (13), (14) are sufficient too. To this end, let $x^* = x^*(z_1, h(u^* z_2))$, $v^* = (z_1, h(u^*, z_2))$, u^*, t^* fulfil (13), (14). Since the problem given by (12) is a concave programming problem, Theorem 1 and Theorems 7.1.1, 7.1.3 of [6] imply that conditions (14) are sufficient for the problem given by (12). Further, as u^* is an optimal solution of problem (12) it is enough to prove that the system (13) is sufficient in the parametric points u^*, $z_1 \in Z_1$, $z_2 \in Z_2$ only. However, we get this in a similar way using Theorems 7.1.1 and 7.1.3 of [6] again. This completes the proof. □

It is easy to see that for the stochastic programming problem given by (2) and (4), we can formulate a similar assertion (cf. the following Remark).

Remark. If the assumptions of Theorem 2 are fulfilled then u^* is a solution of the problem given by (12) if and only if there exists a vector t^* such that the system (14) holds under the assumption that $x^*(z_1, h(u, z_2))$ and $v^*(z_1, h(u, z_2))$ respectively are a solution and a Kuhn-Tucker vector of the parametric optimization problem given by (11).

3. SPECIAL CASES

In the previous part of the paper we have presented the necessary and sufficient optimality conditions for quite general case of two-stage stochastic programming problems. Now we shall apply these results to some special cases. Namely we consider the case, when the problem (6) is a linear or quadratic parametric problem.

a) Linear Case

First, we consider the problem in which $f_i(x, z_1)$, $i = 0, 1, 2, \ldots, l$ for every $z_1 \in Z_1$ are linear functions of x. For this case let $A(\omega)$ and $q(\omega)$ be $(l \times n)$ and $(n \times 1)$ random matrices defined on $(\Omega, \mathscr{S}, P)$ respectively. If we denote by $\mathscr{A}$, $\mathscr{Q}$

the supports of the random matrices $A(\omega)$, $q(\omega)$ then

$$Z_1 = \mathscr{A} \times \mathscr{Q}\,, \quad \eta(\omega) = [A(\omega), q(\omega)]\,, \quad s_1 = ln + n\,.$$

In this special case the problem given by (12) is the problem to find

$$\sup \{\mathsf{E}\psi_1(u, A(\omega), q(\omega), \xi(\omega)) \colon u \in U'\} \tag{17}$$

where

$$\begin{aligned} U' &= \{u \in E_r^+ \colon F_j(u) \geqq 0, j = 1, 2, \ldots, m\}\,, \\ \psi_1(u, A, q, z_2) &= h_0(u, z_2) + \varphi_1(u, A, q, z_2)\,, \\ \varphi_1(u, A, q, z_2) &= \sup \{q'x \colon x \in K_1(u, A, z_2)\}\,, \\ K_1(u, A, z_2) &= \{x \in E_n^+ \colon Ax \leqq h(u, z_2)\}\,, \end{aligned} \tag{18}$$

for $A \in \mathscr{A}$, $q \in \mathscr{Q}$, $z_2 \in Z_2$, $u \in U$. (q' denotes the transposition of the matrix q.) It is easy to see that the inner problem given by (18) is the problem of the linear parametric programming. If we apply substitution (5) then it is the problem to find

$$\sup q'x \tag{19}$$

under the constraints $Ax \leqq y$, $x \geqq 0$. The corresponding dual problem is to find

$$\inf y'v \tag{20}$$

under the constraints $A'v \geqq q$, $v \geqq 0$.

If we denote x^* and v^* the solutions of the problem (19) and the problem (20) respectively then, according to Theorem 2 (under some additional assumptions), we get that u^* is a solution of the problem (17) if and only if there exists $t^* \in E_m^+$ such that the following system holds

$$\begin{aligned} &\nabla_u \mathsf{E} h_0(u^*, \xi(\omega)) + \mathsf{E} \sum_{i=1}^{l} v_i^*(A(\omega), q(\omega), h(u^*, \xi(\omega))\, \nabla_u h_i(u^*, \xi(\omega)) + \\ &\qquad + \sum_{j=1}^{m} t_j^*\, \nabla F_j(u^*) \leqq 0 \\ &\langle u^*, \nabla_u \mathsf{E} h_0(u^*, \xi(\omega)) + \mathsf{E} \sum_{i=1}^{l} v_i^*(A(\omega), q(\omega), h(u^*, \xi(\omega))\, \nabla_u h_i(u^*, \xi(\omega) + \\ &\qquad + \sum_{j=1}^{m} t_j^*\, \nabla F_j(u^*)\rangle = 0 \\ &F(u^*) \geqq 0\,, \quad \langle t^*, F(u^*)\rangle = 0 \end{aligned} \tag{21}$$

We can formulate this consideration as the following corollary.

Corollary 1. Let X, U, Y be convex sets. Let, further, relation (1), assumptions 1, 2, 8 of Theorem 1 and assumptions (i), (ii), (iv) of Theorem 2 be fulfilled. If

(i') A is a deterministic matrix,

(ii') $\{x \in E_n^+ \colon Ax \leqq h(u, z_2)\} \subset K$ and simultaneously $\{x \in E_n^+ \colon Ax < h(u, z_2)\} \neq \emptyset$ for every $u \in U$, $z_2 \in Z_2$ and a compact set K,

(iii') $\mathscr{Q}$ is a compact set,

(iv′) there exists $\mathsf{E}\{\sup [q'(\omega) x \mid x: Ax \leqq h(u, \xi(\omega)]\}$ for every $u \in U$

(v′) there exist the continuous partial derivatives of the functions $h_i(u, z_2)$, $i = 1, 2, \ldots, l$, $F_j(u), j = 1, 2, \ldots, m$ with respect to the components of the vector u on $U \times Z_2$,

then u^* is a solution of the problem given by (17) if and only if there exists a vector $t^* \in E_m$ such that the system (21) holds for t^*, u^* and v^* given by (20).

Proof. It is easy to see that to prove Corollary 1 it is sufficient to verify assumption (5) of Theorem 1. However it follows from assumptions (ii′) and a simply provable generalization of Lemma 1 in [4] for an arbitrary set Y. □

Remark. It follows, for example, from [1] (Theorem 12) that assumption (iv′) of Corollary 1 is fulfilled if the following conditions are satisfied.

a) the probability measure of the random vector $h(u, \xi(\omega))$ is, for every u, absolutely continuous with respect to the Lebesgue measure in E_l,

b) the components of the random vectors $q(\omega)$, $h(u, \xi(\omega))$ are square integrable for every $u \in U$,

c) A is a deterministic matrix.

b) Quadratic Case

In the last part of the paper we apply the assertion of Theorem 2 to the stochastic programming problem in which $f_i(x, z_1)$, $i = 1, \ldots, l$ and $f_0(x, z_1)$ are respectively linear and quadratic functions of the vector x. This means that the parametric problem given by (6) is a problem of the quadratic programming.

Let $A(\omega)$, $C(\omega)$ and $d(\omega)$ be respectively random matrices of the types $(l \times n)$, $(m \times n)$ and $(n \times 1)$ defined on $(\Omega, \mathscr{S}, P)$. We denote by $\mathscr{A}$, $\mathscr{C}$ and $\mathscr{D}$ the supports of $A(\omega)$, $C(\omega)$ and $d(\omega)$. We can define the stochastic programming problem given by (2), (4) in this special case as the problem to find

$$\sup \{\mathsf{E}\psi_2(u, A(\omega), C(\omega), d(\omega), \xi(\omega)) : u \in U'\} \tag{22}$$

where

$$U' = \{u \in E_r^+ : F_j(u) \geqq 0, j = 1, 2, \ldots, m\} \tag{23}$$

$$\psi_2(u, A, C, d, z_2) = h_0(u, z_2) + \varphi_2(u, A, C, d, z_2)$$

$$\varphi_2(u, A, C, d, z_2) = \sup \{x'Cx + d'x : x \in K_2(u, A, z_2)\}$$

$$K_2(u, A, z_2) = \{x \in E_n^+ : Ax \leqq h(u, z_2)\}$$

for $A \in \mathscr{A}$, $C \in \mathscr{C}$ $d \in \mathscr{D}$, $z_2 \in Z_2$ and $u \in U$. It is easy to see that in this case $Z_1 = \mathscr{A} \times \mathscr{C} \times \mathscr{D}$, $\eta(\omega) = [A(\omega), C(\omega), d(\omega)]$, $s_1 = l \,.\, n + n \,.\, n + n$. Further, the inner problem given by (23) is a problem of the quadratic parametric programming. Moreover, if we recall the substitution (5) we obtain its in the form to find

$$\sup \{x'Cx + d'x\} \tag{24}$$

under the constrains $Ax \leqq y$, $x \geqq 0$.

Corollary 2. Let X, U, Y be convex sets. Let, further, relation (1), assumptions 1, 2, 8 of Theorem 1 and assumptions (i), (ii), (iv) of Theorem 2 hold. If

(i″) $\{x \in E_n^+ : Ax \leqq h(u, z_2)\} \subset K$ and simultaneously $\{x \in E_n^+ : Ax < h(u, z_2)\} \neq \emptyset$, for every $u \in U$, $z_2 \in Z_2$ and some compact set $K \subset E_n$,

(ii″) $C \in \mathscr{C}$ implies C is a symmetric, negative definite matrix,

(iii″) A is a deterministic matrix, $\mathscr{C}$, $\mathscr{D}$ are compact sets,

(iv″) there exist the continuous partial derivatives of the functions $h_i(u, z_2)$, $i = 1, 2, \ldots, l$, $F_j(u)$, $j = 1, 2, \ldots, m$ with respect to the components of the vector u on $U \times Z_2$,

(v″) there exists $\mathsf{E}\{\sup x'Cx + d'x \mid x'Ax \leqq h(u, \xi(\omega)\}$ for every $u \in U$

then u^* is a solution of the problem given by (22) if and only if there exists a vector t^* such that the following system holds

$$\nabla_u \mathsf{E} h_0(u^*, \xi(\omega)) + \mathsf{E} \sum_{i=1}^{l} v_i^*(A, C(\omega), d(\omega), h(u^*, \xi(\omega)) \nabla_u h_i(u^*, \xi(\omega)) + \\ + \sum_{j=1}^{m} t_j^* \nabla_u F_j(u^*) \geqq 0$$

$$\langle u^*, \nabla_u \mathsf{E} h_0(u^*, \xi(\omega)) + \\ + \mathsf{E} \sum_{i=1}^{l} v_i^*(A, C(\omega)\, d(\omega), h(u^*, \xi(\omega)) \nabla_u h_i(u^*, \xi(\omega)) + \sum_{j=1}^{m} t_j^* \nabla_j F_j(u^*)\rangle = 0\,, \tag{25}$$

$$F(u^*) \geqq 0\,, \quad \langle t^*, F(u^*)\rangle = 0\,,$$

where the vector $v^* = (v_1^*, \ldots, v_l^*)$, $v_i^* = v_i^*(A, C, d, y)$ and the vector $x^* = x^*(A, C, d, y)$ fulfil the conditions

$$2Cx^*(A, C, d, y) - d - A'v^*(A, C, d, y) \leqq 0\,, \tag{26}$$

$$\langle x^*(A, C, d, y), 2Cx^*(A, C, d, y) - d - A'v^*(A, C, d, y)\rangle = 0\,,$$

$$y - Ax^*(A, C, d, y) \geqq 0$$

$$\langle v^*(A, C, d, y), y - Ax^*(A, C, d, y)\rangle = 0$$

for every $C \in \mathscr{C}$, $d \in \mathscr{D}$, $y \in Y$, $y = h(u, z_2)$, $z_2 \in Z_2$.

Proof. It is easy to see that the systems (25), (26) are equivalent to systems (13), (14) in the special case considered in Corollary 2. According to this fact the assertion will be proved if we verify the assumptions of Theorem 2. To this end it is necessary to deal with assumptions 3, 4, 5, 6, 7 of Theorem 1. Assumption 3 follows immediately from assumptions i″). Since, according the assumptions (ii″), the function $x'Cx$ is concave for every $C \in \mathscr{C}$ and, since $\mathscr{C}$, $\mathscr{D}$ are compact sets, assumption 4 follows from Corollary of [3]. However, by this approach we have verified also assumption 6. Finally, assumption 5 follows from the generalized (for arbitrary set Y) Lemma 1 of paper [4]. So we have completely finished the proof of Corollary 2. □

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