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The Minimum Value of $\sum_{n=0}^{\infty}[n\, e(n)]^2$ as the Quality Control Criterion

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The paper deals with the quadratic criterion of control process quality, which is given as the minimum value of the sum of the squares of error discrete values multiplied by the linear weighting function. First the basic relations are introduced. The main part of the paper deals with the minimization of the criterion with respect to system parameters.

INTROUCTION

There exists a number of control process quality criteria. Among them the quadratic one are most important and frequently used. Their advantages are quite clear when they are compared with the linear criteria. But quadratic criteria emphasize higher values at the beginning of the response while they suppress the low values as the time increases. The result is an overshoot which may be undesirable in some cases. We can remove this disadvantage by introducing a weighting function by which the discrete values of error will be multiplied.

In this paper I use the notation introduced in the book by Prof. Strejc [1] and also some standard results proved therein.

BASIC RELATIONS

Before analysing the criterion I shall introduce some basic relations. Let us presuppose the Z-transform of an error in the form of a rational function:

$$E(z) = \frac{\sum_{i=0}^{l} b_i z^i}{\sum_{i=0}^{l} a_i z^i}\,. \tag{1}$$

According to [1] the sum I of squares of the discrete values is given as the ratio of two determinants:

$$I = \frac{\Delta_b}{\Delta_a}, \tag{2}$$

where

$$\Delta_a = \begin{vmatrix} a_0 & a_1 & a_2 & \dots & a_{l-2} & a_{l-1} & a_l \\ a_1 & a_0 + a_2 & a_3 & \dots & a_{l-1} & a_l & 0 \\ a_2 & a_1 + a_3 & a_0 + a_4 & \dots & a_l & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_l & a_{l-1} & a_{l-2} & \dots & a_2 & a_1 & a_0 \end{vmatrix}. \tag{3}$$

The determinant Δ_b is similar to (3), the difference being in the first column only. The elements a_k are substituted here by the elements γ_k, which depend on the coefficients b_i. Let us introduce the auxiliary function

$$\beta(z) = \frac{\left(\sum_{i=0}^{l} b_i z^i\right)\left(\sum_{i=0}^{l} b_i z^{l-i}\right)}{\sum_{i=0}^{l} a_i z^{l-i}}. \tag{4}$$

Then the elements of the first column of determinant Δ_b are defined by

$$\gamma_h = \frac{1}{h!} \lim_{z \to 0} \frac{d^h}{dz^h} \beta(z) \tag{5}$$

and

$$\Delta_b = \begin{vmatrix} \gamma_0 & a_1 & a_2 & \dots & a_{l-2} & a_{l-1} & a_l \\ \gamma_1 & a_0 + a_2 & a_3 & \dots & a_{l-1} & a_l & 0 \\ \gamma_2 & a_1 + a_3 & a_0 + a_4 & \dots & a_l & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \gamma_l & a_{l-1} & a_{l-2} & \dots & a_2 & a_1 & a_0 \end{vmatrix}. \tag{6}$$

The detailed derivation of the above relations as well as the discussion of simple cases up to $l = 3$ can be found in the book [1].

THE SUM OF SQUARES OF DISCRETE VALUES MULTIPLIED BY LINEAR WEIGHTING FUNCTION

The weighting function may have different forms. In the present paper I use the linear weighting function. The criterion in question is defined as

$$\mathrm{I} = \sum_{n=0}^{\infty} [n\, e(n)]^2, \tag{7}$$

where n is the linear weighting function, $e(n)$ are the values of the error in the sampling instants $n = 0, 1, \dots$.

The relationship (7) can be computed in two ways. Let us suppose that the Z-transform of (7) is a rational function. In this case the response is given as the sum of exponential functions. The first method is simple – it is based on the sum of exponential functions but it requires the knowledge of the roots of characteristic polynomial, which cannot be expressed in general form for polynomials of higher order. Considering the fact that the described criterion should make the calculation of the studied parameters possible, it would be necessary to obtain the general expression for the roots and their dependence on the control system parameters. For this reason the above mentioned method is disadvantageous.

The second method is based on the Z-transform of the error. This makes it possible to express the studied parameters. When using this method it is not necessary to compute the roots of the characteristic polynomial. This is the most important advantage of the described method. Another advantage of it consists in the possibility of using generally expressed coefficients containing the studied parameters in general form. These are the reasons why I shall follow the second method.

We introduce a new function in the expression (7)

$$e_1(n) = n\, e(n)\,. \tag{8}$$

Differentiating the basic relation of the Z-transform

$$E(z) = \sum_{n=0}^{\infty} e(n)\, z^{-n} \tag{9}$$

with respect to z, the following relation is obtained in [1]

$$\frac{\mathrm{d}}{\mathrm{d}z} E(z) = -\sum_{n=0}^{\infty} n\, e(n)\, z^{-(n+1)}\,. \tag{10}$$

Therefore,

$$E_1(z) = -z \frac{\mathrm{d}}{\mathrm{d}z} E(z)\,. \tag{11}$$

We shall substitute the Z-transform of error (1) into (11). The first derivative of the error Z-transform has the form

$$\frac{\mathrm{d}}{\mathrm{d}z} E(z) = \frac{\left(\sum_{i=1}^{l} i b_i z^{i-1}\right)\left(\sum_{i=0}^{l} a_i z^i\right) - \left(\sum_{i=0}^{l} b_i z^i\right)\left(\sum_{i=1}^{l} i a_i z^{i-1}\right)}{\left(\sum_{i=0}^{l} a_i z^i\right)^2} \tag{12}$$

It is necessary to arrange the terms of numerator and denominator by the powers of z. After substitution into (11) we get

$$E_1(z) = \frac{\sum_{k=0}^{2l}\left(\sum_{j=0}^{k} j a_j b_{k-j} - j b_j a_{k-j}\right) z^k}{\sum_{k=0}^{2l}\left(\sum_{j=0}^{k} a_j a_{k-j}\right) z^k}\,. \tag{13}$$

When expressing in numbers it is necessary to respect the fact that the coefficients a_j and b_j for $j > l$ are equal to zero. Similarly the coefficients a_{k-j} and b_{k-j} for $k - j > l$ are equal to zero.

Let us introduce new coefficients in expression (13) by

$$ {}^1a_k = \sum_{j=0}^{k} a_j a_{k-j} , \tag{14} $$

$$ {}^1b_k = \sum_{j=0}^{k} j(a_j b_{k-j} - a_{k-j} b_j) . \tag{15} $$

After substitution, the expression (13) will be simplified to the form

$$ E_1(z) = \frac{\sum_{k=0}^{2l} {}^1b_k z^k}{\sum_{k=0}^{2l} {}^1a_k z^k} , \tag{16} $$

which is similar to expression (1). Consequently, from the coefficients of Z-transform (16), we can form the determinant similar to (3)

$$ {}^1\Delta_a = \begin{vmatrix} {}^1a_0 & {}^1a_1 & {}^1a_2 & \dots & {}^1a_{2l-2} & {}^1a_{2l-1} & {}^1a_{2l} \\ {}^1a_1 & {}^1a_0 + {}^1a_2 & {}^1a_3 & \dots & {}^1a_{2l-1} & {}^1a_{2l} & 0 \\ {}^1a_2 & {}^1a_1 + {}^1a_3 & {}^1a_0 + {}^1a_4 & \dots & {}^1a_{2l} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ {}^1a_{2l} & {}^1a_{2l-1} & {}^1a_{2l-2} & \dots & {}^1a_2 & {}^1a_1 & {}^1a_0 \end{vmatrix} \tag{17} $$

and the determinant similar to (6)

$$ {}^1\Delta_b = \begin{vmatrix} {}^1\gamma_0 & {}^1a_1 & {}^1a_2 & \dots & {}^1a_{2l-2} & {}^1a_{2l-1} & {}^1a_{2l} \\ {}^1\gamma_1 & {}^1a_0 + {}^1a_2 & {}^1a_3 & \dots & {}^1a_{2l-1} & {}^1a_{2l} & 0 \\ {}^1\gamma_2 & {}^1a_1 + {}^1a_3 & {}^1a_0 + {}^1a_4 & \dots & {}^1a_{2l} & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ {}^1\gamma_{2l} & {}^1a_{2l-1} & {}^1a_{2l-2} & \dots & {}^1a_2 & {}^1a_1 & {}^1a_0 \end{vmatrix} \tag{18} $$

The ratio of both determinants, i.e. (17) and (18), gives the final expression

$$ I = \frac{{}^1\Delta_b}{{}^1\Delta_a} \tag{19} $$

for the sum of the squares of discrete values multiplied by the linear weighting function.

It is possible to obtain the minimum value of expression (19) by equating its first derivative to zero. Let us differentiate expression (19) with respect to the studied parameters. We shall designate the derivative by a prime. Consequently, the first derivative will be

$$I' = \frac{{}^1\Delta_a {}^1\Delta'_b - {}^1\Delta'_a \Delta_b}{{}^1\Delta_a^2}. \tag{20}$$

If investigating a larger number of parameters, we must differentiate expression (19) with respect to all parameters under investigation and the result equate to zero. In this way we get a system of equation from which we can determine the minimizing parameters.

Expression (20) is equal to zero if and only if its numerator is zero.

$${}^1\Delta_a {}^1\Delta'_b - {}^1\Delta'_a {}^1\Delta_b = 0. \tag{21}$$

The derivative of determinants appears twice in expression (21). The evaluation of the determinants and their differentiation as well as multiplication is difficult. Therefore, I shall give a method which will make the differentiation of determinants possible without actually computing their values.

Let $\Delta_m = |a_{ij}|$, $i, j = 1, 2, \ldots, m$ denote an m-th order determinant with elements a_{ij} and let A_{ij} denote the cofactor of a_{ij}. Further write

$${}^k\Delta_m = \begin{vmatrix} a_{11} & \ldots & a'_{1k} & \ldots & a_{1m} \\ a_{21} & \ldots & a'_{2k} & \ldots & a_{2m} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{m1} & \ldots & a'_{mk} & \ldots & a_{mm} \end{vmatrix}$$

for the m-th order determinant with the same elements as Δ_m except for the k-th column, which contains the derivatives a'_{ik} of a_{ik}.

Then we can write

$$\Delta'_m = \sum_{k=1}^{m} {}^k\Delta_m. \tag{22}$$

To prove (22) we shall proceed by induction. The expression evidently holds for $m = 2$ because

$$\Delta_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} + a_{21}(-a_{12})$$

and

$$\Delta'_2 = a'_{11}a_{22} + a_{11}a'_{22} + a'_{21}(-a_{12}) + a_{21}(-a'_{12}) = \\ = \begin{vmatrix} a'_{11} & a_{12} \\ a'_{21} & a_{22} \end{vmatrix} + \begin{vmatrix} a_{11} & a'_{12} \\ a_{21} & a'_{22} \end{vmatrix} = {}^1\Delta_2 + {}^2\Delta_2.$$

Assuming that (22) holds for $m - 1$ we are to prove that it also holds for m. We just write

$$\Delta_m = a_{11}A_{11} + a_{21}A_{21} + \ldots + a_{m1}A_{m1}$$

and

$$\Delta'_m = a'_{11}A_{11} + a_{11}A'_{11} + a'_{21}A_{21} + a_{21}A'_{21} + \ldots + a'_{m1}A_{m1} + a_{m1}A'_{m1} \,. \tag{23}$$

But A_{i1} itself is a determinant of order $m - 1$ and hence, applying (22) for $m - 1$, we obtain

$$A'_{i1} = {}^1\Delta_{i1,m-1} + {}^2\Delta_{i1,m-1} + \ldots + {}^{m-1}\Delta_{i1,m-1}$$

for $i = 1, 2, \ldots, m$. It can be seen that

$$a'_{11}A_{11} + a'_{21}A_{21} + \ldots + a'_{m1}A_{m1} = {}^1\Delta_m \tag{24}$$

and

$$\begin{aligned} a_{11}{}^1\Delta_{11,m-1} + a_{21}{}^1\Delta_{21,m-1} + \ldots + a_{m1}{}^1\Delta_{m1,m-1} &= {}^2\Delta_m \,, \\ a_{11}{}^2\Delta_{11,m-1} + a_{21}{}^2\Delta_{21,m-1} + \ldots + a_{m1}{}^2\Delta_{m1,m-1} &= {}^3\Delta_m \,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{11}{}^{m-1}\Delta_{11,m-1} + a_{21}{}^{m-1}\Delta_{21,m-1} + \ldots + a_{m1}{}^{m-1}\Delta_{m1,m-1} &= {}^m\Delta_m \,. \end{aligned} \tag{25}$$

Substituting (24) and (25) into (23) we obtain (22). Thus expression (22) is proved for any m.

In words, we can say that the derivative of an m-th degree determinant is given as the sum of m determinants ${}^i\Delta_m$ of m-th degree, which contain the derivative of original elements in the i-th column.

If we differentiate with respect to the studied parameters the determinants will become simpler because the derivatives of the elements not containing the studied parameters are equal to zero. We substitute the resultant expression into equation (21) and calculate the minimizing parameter. If the number of studied parameters is higher than one, it is necessary to differentiate with respect to all of them. In this way we get a set of equations that determines the required parameters.

CONCLUSION

If we compare the classical quadratic criterion with that described in the presented paper, we can see that they differ in the degrees of the determinants. The determinant for the calculation of the $\sum_{n=0}^{\infty} e^2(n)$ is of l-th degree, while that for the $\sum_{n=0}^{\infty} [n\, e(n)]^2$ is of $2l$-th degree. The weighting function can be in general $n^{\varkappa}$. But in this case the degree of determinant increases $2^{\varkappa}$ times.

It is not possible to compare the achieved results within the scope of the presented paper. They will be discussed in a special paper allong with the criteria already known. This will be of great importance from the objectivity point of view. We can say, however, that the above described criterion smaller overshoot than classical criteria.

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