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Palaniappan Kannappan; Pushpa Narayan Rathie

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Generalization of Sum Representation Functional Equations II Generalized Directed Divergance*)

PL. KANNAPPAN, P. N. RATHIE

The measurable solution of a functional equation in three variables useful in the axiomatic characterization of generalized directed-divergence is given in this paper.

1. INTRODUCTION

Let

$$K =]0, 1[\times]0, 1[\times]0, 1[\cup \{(0, y, z)\} \cup \{(1, u, w)\}$$

with $y, z \in [0, 1[$ and $u, w \in]0, 1]$.

Let $F_{ij}, G_i, H_j : K \to R$ (reals) $(i = 1, 2; j = 1, 2, 3)$ be functions which are measurable in each variable satisfying the functional equation

$$\sum_{i=1}^{2} \sum_{j=1}^{3} F_{ij}(x_i p_j, y_i q_j, z_i r_j) = \sum_{i=1}^{2} G_i(x_i, y_i, z_i) + \sum_{j=1}^{3} H_j(p_j, q_j, r_j) \tag{1.1}$$

where $P, Q, R \in \Delta_3$ and $X, Y, Z \in \Delta_2$ with

$$\Delta_n = \{P = (p_1, p_2, \ldots, p_n) : p_i \geqq 0, \quad \sum_{i=1}^{n} p_i = 1\}$$

for all $n = 1, 2, \ldots$, the set of all complete finite n-ary probability distributions.

The equation (1.1) in a particular case, is useful to characterize the following measure of information known as the generalized directed-divergence [6]

$$I_n(p_1, \ldots, p_n : q_1, \ldots, q_n : r_1, \ldots, r_n) = \sum_{i=1}^{n} p_i \log (q_i/r_i), \tag{1.2}$$

where $P, Q, R \in \Delta_n$, with the convention that whenever a q_i or a r_i is zero then the

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corresponding p_i is also zero, and $0 \,.\, \log 0 = 0$. The measure generalized directed divergence given by (1.2) satisfies many algebraic properties. It was shown by C.T. Ng that any measure I_n satisfying the properties of symmetry, expansibility and branching can be represented in the sum form $I_n(P|\,|Q|\,R) = \sum_{i=1}^{n} F(p_i, q_i, r_i)$ where F is an arbitrary function satisfying $F(0, 0, 0) = 0$. In addition if I_n is additive, then F satisfies (2.27), that is (1.1) in which all functions appearing are the same. For details refer to [2].

The object of this paper is to find the measurable solutions of the equation (1.1).

2. THE MAIN THEOREM

The following theorem will be proved through a series of lemmas.

Theorem. Let $F_{ij}, G_i, H_j : K \to R$ $(i = 1, 2; j = 1, 2, 3)$ be functions measurable in each variable satisfying the functional equation (1.1). Then the solutions of (1.1) are given by

$$
\begin{aligned}
(2.1)\quad F_{11}(p, q, r) &= \Delta(p, q, r) + (d_1 - c_1)\,p + (d_2 - c_2)\,q + (d_3 - c_3)\,r + e_2\,,\\
F_{12}(p, q, r) &= \Delta(p, q, r) + (d_1' - c_1)\,p + (d_2' - c_2)\,q + (d_3' - c_3)\,r + e_5\,,\\
F_{13}(p, q, r) &= \Delta(p, q, r) + (d_1'' - c_1)\,p + (d_2'' - c_2)\,q + (d_3'' - c_3)\,r + e_8\,,\\
F_{21}(p, q, r) &= \Delta(p, q, r) + (d_1 - c_4)\,p + (d_2 - c_5)\,q + (d_3 - c_6)\,r + e_3\,,\\
F_{22}(p, q, r) &= \Delta(p, q, r) + (d_1' - c_4)\,p + (d_2' - c_5)\,q + (d_3' - c_6)\,r + e_6\,,\\
F_{23}(p, q, r) &= \Delta(p, q, r) + (d_1'' - c_4)\,p + (d_2'' - c_5)\,q + (d_3'' - c_6)\,r + e_9\,,\\
G_2(p, q, r) &= g(p, q, r) \quad \text{which is arbitrary}\,,\\
G_1(p, q, r) &= -g(p, q, r) - L_{p,q,r}(1, 1, 1) + (c_4 - c_1)\,p + (c_5 - c_2)\,q +\\
&\quad + (c_6 - c_3)\,r - c_4 - c_5 - c_6 - e_1 + e_2 + e_3 - e_4 + e_5 +\\
&\quad + e_6 - e_7 + e_8 + e_9\,,\\
H_1(p, q, r) &= \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + e_1\,,\\
H_2(p, q, r) &= \Delta(p, q, r) + d_1' p + d_2' q + d_3' r + e_4\,,\\
H_3(p, q, r) &= \Delta(p, q, r) + d_1'' p + d_2'' q + d_3'' r + e_7\,,
\end{aligned}
$$

where $c_1, c_2, c_3, c_4, c_5, c_6, d_1, d_2, d_3, d_1', d_2', d_3', d_1'', d_2'', d_3'', e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9$ are arbitrary constants,

$$(2.2)\qquad \Delta(p, q, r) = p[a \log p + b \log q + c \log r] + dq \log q + er \log r, \qquad ((p, q, r) \in K)$$

(2.3) $L_{x,y,z}(p, q, r) = p[a\, I(x, x) + b\, I(x, y) + c\, I(x, z)] + dq\, I(y, y) + er\, I(z, z)$,

$$((p, q, r), (x, y, z) \in K)$$

with

(2.4) $$I(x, y) = -x \log y - (1 - x) \log (1 - y), \quad x, y \in [0, 1].$$

The theorem will be established with the help of four lemmas which will be proved one by one in what follows:

Lemma 1. Let $F : K \to R$ be measurable in each variable and satisfy

(2.5) $$F(p, q, r) - F(px, qy, rz) - F((1 - x)\, p, (1 - y)\, q, (1 - z)\, r) = A_{x,y,z}(p, q, r),$$

where $A_{x,y,z} : K \to R$, for fixed $(x, y, z) \in K$, satisfies the equation

(2.6) $$A_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2) = A_{x,y,z}(p_1, q_1, r_1) + A_{x,y,z}(p_2, q_2, r_2).$$

Then the solutions of (2.5) are given by

(2.7) $$F(p, q, r) = \Delta(p, q, r) + d_1 p + d_2 q + d_3 r, \quad A_{x,y,z}(p, q, r) = L_{x,y,z}(p, q, r),$$

where the functions Δ and L are given by (2.2) and (2.3) respectively and d_1, d_2, d_3 are arbitrary constants.

Proof of Lemma 1. In order to prove Lemma 1, first we derive the measurable solution of (2.6).

For $p_1 = 0 = p_2$, (2.6) reduces to

$$A_{x,y,z}(0, q_1 + q_2, r_1 + r_2) = A_{x,y,z}(0, q_1, r_1) + A_{x,y,z}(0, q_2, r_2)$$

which is the same as discussed in [3]. Hence

(2.8) $$A_{x,y,z}(0, q, r) = B_2(x, y, z)\, q + B_3(x, y, z)\, r$$

where B_2 and B_3 are arbitrary functions of x, y, z.

For $p_1 = 0$, (2.6) gives

$$A_{x,y,z}(p_2, q_1, + q_2, r_1 + r_2) = A_{x,y,z}(0, q_1, r_1) + A_{x,y,z}(p_2, q_2, r_2) =$$
$$= B_2(x, y, z)\, q_1 + B_3(x, y, z)\, r_1 + A_{x,y,z}(p_2, q_2, r_2)$$

(by using (2.8))

$$= B_2(x, y, z)\, q_2 + B_3(x, y, z)\, r_2 + A_{x,y,z}(p_2, q_1, r_1)$$

(by interchanging q_1 and q_2 and r_1 and r_2 respectively).

This implies that

$$A_{x,y,z}(p_2, q_2, r_2) - B_2(x, y, z)\, q_2 - B_3(x, y, z)\, r_2 =$$
$$= A_{x,y,z}(p_2, q_1, r_1) - B_2(x, y, z)\, q_1 - B_3(x, y, z)\, r_1 =$$
$$= \text{Independent of } q_i \text{ and } r_i \ (i = 1, 2) =$$
$$= \alpha_{x,y,z}(p_2), \quad \text{say.}$$

Then

(2.9) $$A_{x,y,z}(p_2, q_2, r_2) = \alpha_{x,y,z}(p_2) + B_2(x, y, z)\, q_2 + B_3(x, y, z)\, r_2 .$$

Hence (2.6) and (2.9) yield

(2.10) $$\alpha_{x,y,z}(p_1 + p_2) = \alpha_{x,y,z}(p_1) + \alpha_{x,y,z}(p_2), \quad p_1, p_2, p_1 + p_2 \in [0, 1] .$$

Clearly, the measurable solution of (2.10) is given by

(2.11) $$\alpha_{x,y,z}(p) = B_1(x, y, z)\, p, \quad p \in [0, 1] ,$$

where B_1 is an arbitrary function of x, y, z.

Thus (2.9) and (2.11) give

(2.12) $$A_{x,y,z}(p, q, r) = B_1(x, y, z)\, p + B_2(x, y, z)\, q + B_3(x, y, z)\, r ,$$
$$(p, q, r),\ (x, y, z) \in K .$$

Hence (2.5), on using (2.12), becomes

(2.13) $$F(p, q, r) - F(px, qy, rz) - F(p(1 - x), q(1 - y), r(1 - z)) =$$
$$= B_1(x, y, z)\, p + B_2(x, y, z)\, q + B_3(x, y, z)\, r .$$

For fixed r and z, (2.13) takes the following form

(2.14) $$g_1(p, q) - g_2(px, qy) - g_3(p(1 - x), q(1 - y)) = A'_{x,y}(p, q) + B(x, y) ,$$

where $A'_{x,y}(p, q) = B_1(x, y, z)\, p + B_2(x, y, z)\, q$ is additive in both p and q. The measurable solution of (2.14) are given by [4].

(2.15) $$g_1(p, q) = ap \log p + bp \log q + cq \log q + dp + eq + d_1 ,$$
$$g_2(p, q) = ap \log p + bp \log q + cq \log q + (d - c_1)\, p + (e - c_2)\, q + d_2 ,$$
$$g_3(p, q) = ap \log p + bp \log q + cq \log q + (d - c_3)\, p + (e - c_4)\, q + d_3 ,$$

$$A'_{x,y}(p, q) = p[a\, I(x, x) + b\, I(x, y)] + q\, I(y, y) + (c_1 - c_3)\, px + $$
$$+ (c_2 - c_4)\, qy + c_3 p + c_4 q\,,$$
$$B(x, y) \quad = d_1 - d_2 - d_3\,,$$

where $a, b, c, d, e, d_1, d_2, d_3, c_1, c_2, c_3, c_4$ are arbitrary constants (here functions of r and z). Thus

$$F(p, q, r) = a(r, z)\, p \log p + b(r, z)\, p \log q + c(r, z)\, q \log q + \tag{2.16}$$
$$+ d(r, z)\, p + e(r, z)\, q + d_1(r, z)\,, \quad \text{etc;}$$

that is,

$$F(p, q, r) = a(r)\, p \log p + b(r)\, p \log q + c(r)\, q \log q + \tag{2.17}$$
$$+ d(r)\, p + e(r)\, q + d_1(r)\,.$$

Now (2.17) in (2.13) yields

$$a(r) = \text{constant} = a\,, \tag{2.18}$$
$$b(r) = \text{constant} = b\,,$$
$$c(r) = \text{constant} = d\,.$$

Hence (2.17) and (2.18) give

$$F(p, q, r) = ap \log p + bp \log q + dq \log q + d(r)\, p + e(r)\, q + d_1(r)\,. \tag{2.19}$$

Substituting (2.19) in (2.13) and equating similar terms on both sides, we get

$$B_1(x, y, z) = a\, I(x, x) + b\, I(x, y) + d(r) - x\, d(rz) - (1 - x)\, d(r(1 - z))\,, \tag{2.20}$$

$$B_2(x, y, z) = c\, I(y, y) + e(r) - y\, e(rz) - (1 - y)\, e(r(1 - z)) \tag{2.21}$$

and

$$B_3(x, y, z)\, r = d_1(r) - d_1(rz) - d_1(r(1 - z))\,. \tag{2.22}$$

Equation (2.21) for $y = 0$, $z = 1$ yields

$$e(r) = \text{constant} = d_2\,, \quad \text{say}\,. \tag{2.23}$$

The measurable solution of equation (2.22) is given by ([2], [5])

$$d_1(r) = er \log r + d_3 r\,, \tag{2.24}$$

where e and d_3 are arbitrary constants.

The equation (2.20) can be rewritten as follows:

$$d(r) - d(r(1-z)) = B_1(x, y, z) - a\,I(x, x) - b\,I(x, y) + x[d(rz) - d(r(1-z))]$$

$$= \text{independent of } r \text{ (by letting } x = 0)$$

$$= h(z), \quad \text{say}.$$

This is a Pexider equation with measurable solution given by

$$d(r) = c \log r + d_1, \tag{2.25}$$

where c and d_1 are arbitrary constants.

Thus (2.19), (2.23), (2.24) and (2.25) yield

$$F(p, q, r) = ap \log p + bp \log q + dq \log q + (c \log r + d_1)\, p + d_2 q + er \log r + d_3 r \tag{2.26}$$

which is the same as (2.7), proving Lemma 1.

Remark 1. The Lemma 1 leads to the measurable solution of the functional equation

$$\sum_{i=1}^{2} \sum_{j=1}^{3} F(x_i p_j, y_i q_j, z_i r_j) = \sum_{i=1}^{2} F(x_i, y_i, z_i) + \sum_{j=1}^{3} F(p_j, q_j, r_j) \tag{2.27}$$

with $\sum x_i = 1 = \sum y_i = \sum z_i = \sum p_j = \sum q_j = \sum r_j$, as

$$F(p, q, r) = \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + d_1 + d_2 + d_3, \tag{2.28}$$

connected with the generalized directed divergence (1.2), mentioned in Section 1 [2].

Lemma 2. Let $F : K \to R$ be measurable in each variable and satisfy

$$F(p, q, r) - F(px, qy, rz) - F(p(1-x), q(1-y), r(1-z)) = A_{x,y,z}(p, q, r) + B(x, y, z), \tag{2.29}$$

where $A_{x,y,z}$ satisfies (2.6) and $B(.,.,.)$ is a function of x, y, z. Then the solutions of (2.29) are given by

$$F(p, q, r) = \Delta(p, q, r) + d_1 p + d_2 q + d_3 r - l, \tag{2.30}$$

$$B(x, y, z) = l,$$

$$A_{x,y,z}(p, q, r) = L_{x,y,z}(p, q, r).$$

Proof of Lemma 2. For $p = q = r = 0$, (2.29) yields

$$B(x, y, z) = -F(0, 0, 0) = l, \quad \text{say}. \tag{2.31}$$

Then $G(p, q, r) = F(p, q, r) + l$ satisfies the functional equation (2.5). Hence (2.7) and (2.31) give (2.30), proving the Lemma 2.

Lemma 3. Let the functions $F_i : K \to R$ $(i = 1, 2, 3)$ be measurable, in each variable and satisfy

$$F_1(p, q, r) - F_2(px, qy, rz) - F_3(p(1 - x), q(1 - y), r(1 - z)) = \\ = A_{x,y,z}(p, q, r) + B(x, y, z), \tag{2.32}$$

where $A_{x,y,z}$ satisfies (2.6). Then the solutions of (2.32) are given by

(2.33)

$$\begin{aligned} F_1(p, q, r) &= \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + a_1, \\ F_2(p, q, r) &= \Delta(p, q, r) + (d_1 - c_1)\, p + (d_2 - c_2)\, q + (d_3 - c_3)\, r + a_2, \\ F_3(p, q, r) &= \Delta(p, q, r) + (d_1 - c_4)\, p + (d_2 - c_5)\, q + (d_3 - c_6)\, r + a_3, \\ A_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r) + (c_1 - c_4)\, px + (c_2 - c_5)\, qy + (c_3 - c_6)\, rz + \\ &\quad + c_4 p + c_5 q + c_6 r, \\ B(x, y, z) &= a_1 - a_2 - a_3. \end{aligned}$$

Proof of Lemma 3. The equation (2.32) with $p = q = r = 0$ yields

$$B(x, y, z) = a_1 - a_2 - a_3, \tag{2.34}$$

where $a_i = F_i(0, 0, 0)$ $(i = 1, 2, 3)$.

Substituting $x = y = z = 1$ and $x = y = z = 0$ respectively in (2.32), and using (2.34), we get

$$F_2(p, q, r) = F_1(p, q, r) - A_{1,1,1}(p, q, r) + a_2 - a_1, \tag{2.35}$$

and

$$F_3(p, q, r) = F_1(p, q, r) - A_{0,0,0}(p, q, r) + a_3 - a_1. \tag{2.36}$$

Hence (2.32) with the help of (2.34), (2.35) and (2.36) gives

$$F_1(p, q, r) - F_1(px, qy, rz) - F_1(p(1 - x), q(1 - y), r(1 - z)) = \\ = A^1_{x,y,z}(p, q, r) - a_1, \tag{2.37}$$

where

$$(2.38)\qquad A^1_{x,y,z}(p, q, r) = A_{x,y,z}(p, q, r) - A_{1,1,1}(px, qy, rz) - \\ - A_{0,0,0}(p(1-x), q(1-y), r(1-z)).$$

It is easy to see that $A^1_{x,y,z}(p, q, r)$ also satisfies (2.6). Hence, applying the Lemma 2 to (2.37), we have

$$(2.39)\qquad F_1(p, q, r) = \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + a_1,$$

$$(2.40)\qquad A^1_{x,y,z}(p, q, r) = L_{x,y,z}(p, q, r).$$

Also from (2.12) it follows that

$$(2.41)\qquad A_{1,1,1}(p, q, r) = c_1 p + c_2 q + c_3 r$$

and

$$(2.42)\qquad A_{0,0,0}(p, q, r) = c_4 p + c_5 q + c_6 r,$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants.

Hence (2.35), (2.39), (2.41); (2.36), (2.39), (2.42); (2.38), (2.40), (2.41), (2.42) respectively yield

$$(2.43)\quad F_2(p, q, r) = \Delta(p, q, r) + (d_1 - c_1)\, p + (d_2 - c_2)\, q + (d_3 - c_3)\, r + a_2,$$

$$(2.44)\quad F_3(p, q, r) = \Delta(p, q, r) + (d_1 - c_4)\, p + (d_2 - c_5)\, q + (d_3 - c_6)\, r + a_3$$

and

$$(2.45)\quad A_{x,y,z}(p, q, r) = L_{x,y,z}(p, q, r) + (c_1 - c_4)\, px + (c_2 - c_5)\, qy + \\ + (c_3 - c_6)\, rz + c_4 p + c_5 q + c_6 r.$$

Thus (2.39), (2.43), (2.44), (2.45) and (2.34) prove Lemma 3.

Lemma 4. Let $F_i : K \to R$ $(i = 1, 2, \ldots, 6)$ be measurable in each variable and satisfy

$$(2.46)\quad [F_1(p_1, q_1, r_1) - F_2(p_1 x, q_1 y, r_1 z) - F_3(p_1(1-x), q_1(1-y), r_1(1-z))] + \\ + [F_4(p_2, q, r_2) - F_5(p_2 x, q_2 y, r_2 z) - F_6(p_2(1-x), q_2(1-y), r_2(1-z))] = \\ = A_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2).$$

Then the solutions of (2.46) are given by

$$(2.47)\; F_1(p, q, r) = \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + e_1,$$

$$F_2(p, q, r) = \Delta(p, q, r) + (d_1 - c_1)\, p + (d_2 - c_2)\, q + (d_3 - c_3)\, r + e_2,$$

$$F_3(p,q,r) = \Delta(p,q,r) + (d_1 - c_4)\,p + (d_2 - c_5)\,q + (d_3 - c_6)\,r + e_3\,,$$

$$F_4(p,q,r) = \Delta(p,q,r) + d'_1 p + d'_2 q + d'_3 r + e_4\,,$$

$$F_5(p,q,r) = \Delta(p,q,r) + (d'_1 - c_1)\,p + (d'_2 - c_2)\,q + (d'_3 - c_3)\,r + e_5\,,$$

$$F_6(p,q,r) = \Delta(p,q,r) + (d'_1 - c_4)\,p + (d'_2 - c_5)\,q + (d'_3 - c_6)\,r + e_6\,,$$

$$A_{x,y,z}(p,q,r) = L_{x,y,z}(p,q,r) + (c_1 - c_4)\,px + (c_2 - c_5)\,qy + (c_3 - c_6)\,rz + c_4 p + c_5 q + c_6 r + e_1 - e_2 - e_3 + e_4 - e_5 - e_6\,,$$

where c_i $(i = 1, \ldots, 6)$, $d_1, d_2, d_3, d'_1, d'_2, d'_3$, are constants.

Proof of Lemma 4. The equation (2.46) for $p_2 = q_2 = r_2 = 0$ and $p_1 = q_1 = r_1 = 0$ respectively gives

$$(2.48)\quad F_1(p_1,q_1,r_1) - F_2(p_1x, q_1y, r_1z) - F_3(p_1(1-x), q_1(1-y), r_1(1-z)) = A_{x,y,z}(p_1,q_1,r_1) + e_5 + e_6 - e_4\,,$$

and

$$(2.49)\quad F_4(p_2,q_2,r_2) - F_5(p_2x, q_2y, r_2z) - F_6(p_2(1-x), q_2(1-y), r_2(1-z)) = A_{x,y,z}(p_2,q_2,r_2) + e_2 + e_3 - e_1\,,$$

where $e_i = F_i(0,0,0)$ $(i = 1, 2, \ldots, 6)$.

Adding (2.48) and (2.49) and using (2.46), we get

$$(2.50)\quad A_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2) = A_{x,y,z}(p_1,q_1,r_1) + A_{x,y,z}(p_2,q_2,r_2) + e_2 + e_3 + e_5 + e_6 - e_1 - e_4\,.$$

Let

$$(2.51)\quad A^2_{x,y,z}(p,q,r) = A_{x,y,z}(p,q,r) - e_1 + e_2 + e_3 - e_4 + e_5 + e_6\,.$$

Then (2.50) becomes

$$(2.52)\quad A^2_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2) = A^2_{x,y,z}(p_1,q_1,r_1) + A^2_{x,y,z}(p_2,q_2,r_2)\,.$$

Hence $A^2_{x,y,z}(p,q,r)$ satisfies (2.6).

The equations (2.48) and (2.49) due to (2.51) take the following forms

$$(2.53)\quad F_1(p_1,q_1,r_1) - F_2(p_1x, q_1y, r_1z) - F_3(p_1(1-x), q_1(1-y), r_1(1-z)) = A^2_{x,y,z}(p_1,q_1,r_1) + e_1 - e_2 - e_3$$

and

$$(2.54)\quad F_4(p_2,q_2,r_2) - F_5(p_2x, q_2y, r_2z) - F_6(p_2(1-x), q_2(1-y), r_2(1-z)) = A^2_{x,y,z}(p_2,q_2,r_2) + e_4 - e_5 - e_6\,.$$

Hence applying Lemma 3 to (2.53) and (2.54) respectively, we have

$$(2.55)\quad \begin{aligned} F_1(p, q, r) &= \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + e_1 , \\ F_2(p, q, r) &= \Delta(p, q, r) + (d_1 - c_1)\, p + (d_2 - c_2)\, q + (d_3 - c_3)\, r + e_2 , \\ F_3(p, q, r) &= \Delta(p, q, r) + (d_1 - c_4)\, p + (d_2 - c_5)\, q + (d_3 - c_6)\, r + e_3 , \\ A_{x,y,z}(p, q, r) &= L_{x,y,z}(p, q, r) + (c_1 - c_4)\, px + (c_2 - c_5)\, qy + (c_3 - c_6)\, rz + \\ &\quad + c_4 p + c_5 q + c_6 r , \end{aligned}$$

$$(2.56)\quad \begin{aligned} F_4(p, q, r) &= \Delta'(p, q, r) + d'_1 p + d'_2 q + d'_3 r + e_4 , \\ F_5(p, q, r) &= \Delta'(p, q, r) + (d'_1 - c'_1)\, p + (d'_2 - c'_2)\, q + (d'_3 - c'_3)\, r + e_5 , \\ F_6(p, q, r) &= \Delta'(p, q, r) + (d'_1 - c'_4)\, p + (d'_2 - c'_5)\, q + (d'_3 - c'_6)\, r + e_6 , \\ A^2_{x,y,z}(p, q, r) &= L'_{x,y,z}(p, q, r) + (c'_1 - c'_4)\, px + (c'_2 - c'_5)\, qy + (c'_3 - c'_6)\, rz + \\ &\quad + c'_4 p + c'_5 q + c'_6 r , \end{aligned}$$

where Δ' and L' are given by (2.2) and (2.3) with a, b, c, d, e changed to a', b', c', d', e' respectively.

Comparing the expressions for $A^2_{x,y,z}(p, q, r)$ in (2.55) and (2.56), we get

$$a = a',\ b = b',\ c = c',\ d = d',\ e = e',\ c_4 = c'_4,\ c_5 = c'_5,\ c_6 = c'_6 ,$$
$$c_1 = c'_1,\ c_2 = c'_2,\ c_3 = c'_3 .$$

This proves Lemma 4.

Now we are in a position to prove the main theorem.

Proof of the theorem. Substituting $x_1 = x, x_2 = 1 - x, y_1 = y, y_2 = 1 - y, z_1 = z, z_2 = 1 - z$ in (1.1), we have

(2.57)

$$\begin{aligned} &[H_1(p_1, q_1, r_1) - F_{11}(p_1 x, q_1 y, r_1 z) - F_{21}(p_1(1 - x), q_1(1 - y), r_1(1 - z))] + \\ &+ [H_2(p_2, q_2, r_2) - F_{12}(p_2 x, q_2 y, r_2 z) - F_{22}(p_2(1 - x), q_2(1 - y), r_2(1 - z))] = \\ &= F_{13}(p_3 x, q_3 y, r_3 z) + F_{23}(p_3(1 - x), q_3(1 - y), r_3(1 - z)) - H_3(p_3, q_3, r_3) - \\ &\quad - G_1(x, y, z) - G_2(1 - x, 1 - y, 1 - z) = \\ &\quad = A^3_{x,y,z}(1 - p_3, 1 - q_3, 1 - r_3), \quad \text{say} \\ &\quad = A^3_{x,y,z}(p_1 + p_2, q_1 + q_2, r_1 + r_2) . \end{aligned}$$

Hence applying Lemma 4 to (2.57), we get

(2.58)
$$H_1(p, q, r) = \Delta(p, q, r) + d_1 p + d_2 q + d_3 r + e_1\,,$$
$$F_{11}(p, q, r) = \Delta(p, q, r) + (d_1 - c_1)\,p + (d_2 - c_2)\,q + (d_3 - c_3)\,r + e_2\,,$$
$$F_{21}(p, q, r) = \Delta(p, q, r) + (d_1 - c_4)\,p + (d_2 - c_5)\,q + (d_3 - c_6)\,r + e_3\,,$$
$$H_2(p, q, r) = \Delta(p, q, r) + d'_1 p + d'_2 q + d'_3 r + e_4\,,$$
$$F_{12}(p, q, r) = \Delta(p, q, r) + (d'_1 - c_1)\,p + (d'_2 - c_2)\,q + (d'_3 - c_3)\,r + e_5\,,$$
$$F_{22}(p, q, r) = \Delta(p, q, r) + (d'_1 - c_4)\,p + (d'_2 - c_5)\,q + (d'_3 - c_6)\,r + e_6\,,$$
$$A^3_{x,y,z}(p, q, r) = L_{x,y,z}(p, q, r) + (c_1 - c_4)\,px + (c_2 - c_5)\,qy + (c_3 - c_6)\,rz + c_4 p + c_5 q + c_6 r + e_1 - e_2 - e_3 + e_4 - e_5 - e_6\,.$$

From (2.57) and (2.58), we get

(2.59)
$$H_3(p_3, q_3, r_3) - F_{13}(p_3 x, q_3 y, r_3 z) - F_{23}(p_3(1 - x), q_3(1 - y), r_3(1 - z)) =$$
$$= -A^3_{x,y,z}(1 - p_3, 1 - q_3, 1 - r_3) - G_1(x, y, z) - G_2(1 - x, 1 - y, 1 - z) =$$
$$= [p_3\{a\,I(x, x) + b\,I(x, y) + c\,I(x, z)\} + q_3 d\,I(y, y) + r_3 e\,I(z, z) -$$
$$- (c_4 - c_1)\,p_3 x - (c_5 - c_2)\,q_3 y - (c_6 - c_3)\,r_3 z + c_4 p_3 + c_5 q_3 + c_6 r_3] +$$
$$+ [-\{a\,I(x, x) + b\,I(x, y) + c\,I(x, z)\} - d\,I(y, y) - e\,I(z, z) +$$
$$+ (c_4 - c_1)\,x + (c_5 - c_2)\,y + (c_6 - c_3)\,z - c_4 - c_5 - c_6 - e_1 + e_2 + e_3 -$$
$$- e_4 + e_5 + e_6 - G_1(x, y, z) - G_2(1 - x, 1 - y, 1 - z)] =$$
$$= A_{x,y,z}(p_3, q_3, r_3) + B^*(x, y, z)\,, \text{ say}$$

where A is the same as given in (2.32) and $B^*(x, y, z)$ is the remaining expression on the right hand side of (2.59).

Applying Lemma 3 to (2.59), we get

(2.60)
$$H_3(p, q, r) = \Delta(p, q, r) + d''_1 p + d''_2 q + d''_3 r + e_7\,,$$
$$F_{13}(p, q, r) = \Delta(p, q, r) + (d''_1 - c_1)\,p + (d''_2 - c_2)\,q + (d''_3 - c_3)\,r + e_8\,,$$
$$F_{23}(p, q, r) = \Delta(p, q, r) + (d''_1 - c_4)\,p + (d''_2 - c_5)\,q + (d''_3 - c_6)\,r + e_9\,,$$

$$B^*(x, y, z) = e_7 - e_8 - e_9,$$

where $e_7 = H_3(0, 0, 0)$, $e_8 = F_{13}(0, 0, 0)$, $e_9 = F_{23}(0, 0, 0)$.

The last equation of (2.60) yields

$$(2.61) \quad G_1(x, y, z) = -G_2(1-x, 1-y, 1-z) - L_{x,y,z}(1, 1, 1) + (c_4 - c_1)x + (c_5 - c_2)y + (c_6 - c_3)z - c_4 - c_5 - c_6 - e_1 + e_2 + e_3 - e_4 + e_5 + e_6 - e_7 + e_8 + e_9.$$

The equation (2.61) expresses $G_1(x, y, z)$ in terms of $G_2(1-x, 1-y, 1-z)$ which is arbitrary.

Thus (2.58), (2.60) and (2.61) prove the theorem. It is easy to verify that the eleven functions thus obtained do satisfy (1.1).

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Prof. Pl. Kannappan, Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario. Canada N2L 3G1

Prof. P. N. Rathie, Department of Statistics, State University of Campinas, Caixa Postal 1170, 13100 Campinas. S. P., Brazil.