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# Basic Equations for Source Coding with Side Information at the Decoder and Encoder

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Basic equations for source coding when fidelity criterion is considered over the channel have been obtained by Berger. Currently there is interest in problems allowing use of side information as is evidenced by the work of Wyner, Wyner and Ziv and Slepian and Wolf.

In this paper we examine the form of basic equation when use of side channel is made in the following two ways:

1) The side channel is available both to the decoder and encoder and a fidelity criterion is applied to the main channel only.
2) The side channel is available both to the decoder and encoder and fidelity criteria are applied to the main as well as to the side channel.

Examples of binary symmetric channels have been discussed in both the cases. Convexity of the rate distortion function in the first case has been established.

## 1. INTRODUCTION

Basic equations for source coding when fidelity criterion is considered over the channel have been obtained by Berger [1]. Currently there is interest in problems allowing use of side information, refer to the work of Wyner [5], Wyner and Ziv [6] and Slepian and Wolf [4] in this direction.

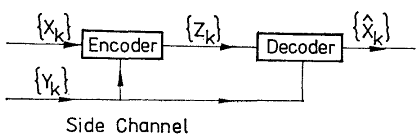


**Fig. 1.**

In the present paper we examine the form of basic equations when a side channel is available both to the decoder and the encoder and there is fidelity criterion over the main channel. Also we consider the situation in which there are different fidelity criteria over the main and side channels. The Fig. 1 schematically explains the problem.

Let there be two random variables $X$ and $Y$ taking values in the finite sets $\mathscr{X}$ and $\mathscr{Y}$ and suppose that the sequence $\{X_k, Y_k\}_{k=1}^{\infty}$ represents independent copies of a pair of dependent random variables $(X, Y)$. The encoder presents binary sequences of length $n$ at a rate $R$ bits per input symbol and the decoder decodes the received message into a sequence $\{\hat{X}_k\}$ of length $n$ which takes values in the finite reproduction alphabet $\hat{\mathscr{X}}$. Also let the fidelity criterion be the expectation of

$$\frac{1}{n}\sum_{k=1}^{n} D(X_k, \hat{X}_k),$$

where $D(x, \hat{x}) \geqq 0$, $x \in \mathscr{X}$, $\hat{x} \in \hat{\mathscr{X}}$ is a given distortion function.

When we confine to the case of there being fidelity criterion on main channel only, refer to Wyner and Ziv [6], we define $R_{X/Y}(d)$ as the minimum rate for which the system can operate when $n$ is large and the average distortion is equal to $d$ i.e.

$$\mathsf{E}\left[\frac{1}{n}\sum_{k=1}^{n} D(X_k, \hat{X}_k)\right] = d,$$

In other words, for $d \geqq 0$, let $M_0(d)$ denote the set of probability distributions $\{p(x, y, \hat{x})\}$, $x \in \mathscr{X}$, $y \in \mathscr{Y}$ and $\hat{x} \in \hat{\mathscr{X}}$ such that

$$p(x, y) = \sum p(x, y, \hat{x}),$$

where the sum is taken over all $\hat{x} \in \hat{\mathscr{X}}$, and

$$\sum_{x,y,\hat{x}} D(x, \hat{x})\, p(x, y, \hat{x}) = d$$

then

$$R_{X/Y}(d) = \min_{p \in M_0(d)} I(X, \hat{X}/Y), \tag{1}$$

where $I(.)$ denotes the ordinary Shannon's Mutual Information.

A further modification in the definition of $R_{X/Y}(d)$ is required if we consider an additional fidelity criterion on the side channel also. The $R_{X/Y}(d)$ now depends on a pair of distortion limits in place of a single $d$. The problem and its mathematical formulation will be found in Section 3.

## 2. BASIC EQUATIONS WHEN FIDELITY CRITERION IS OVER THE MAIN CHANNEL

In what follows the source alphabet will be represented by index $i$, the encoder output by index $j$, decoder output by index $k$ and the side information by index $l$. Also the following notations will be used:

$$P(i, k, l) = p_{ikl}\,; \quad Q(k/i, l) = Q_{k/i,l}\,,$$
$$D(i, k) = \varrho_{ik}\,; \quad Q(k/l) = Q_{k/l} \text{ etc}\,.$$

Thus our problem is to minimize the average mutual information

$$I(Q) = \sum_{i,k,l} p_{il} Q_{k/il} \log \frac{Q_{k/il}}{Q_{k/l}}\,, \tag{2}$$

subject to the constraints:

$$Q_{k/il} \geqq 0\,, \tag{3}$$

$$\sum_k Q_{k/il} = 1\,, \tag{4}$$

$$\sum_{i,k,l} \varrho_{ik} p_{ikl} = d \quad \text{i.e.} \quad \sum_{i,k,l} \varrho_{ik} p_{il} Q_{k/il} = d\,. \tag{5}$$

We will solve this problem by using the method of Lagrange Multipliers ignoring the constraints (3). For this purpose consider the augmented function

$$J(Q) = I(Q) - \sum_{i,l} \mu_{il} \sum_k Q_{k/il} - s \sum_{i,k,l} \varrho_{ik} p_{il} Q_{k/il}\,, \tag{6}$$

where the parameters $\mu_{il}$ and $s$ are Lagrange Multipliers. Taking $\log \lambda_{il} = \mu_{il}/p_{il}$ and using (2) in (6), we may write

$$J(Q) = \sum_{i,k,l} p_{il} Q_{k/il} \left[ \log \frac{Q_{k/il}}{\lambda_{il} Q_{k/l}} - s\varrho_{ik} \right].$$

Now for stationary point, we have

$$\frac{\mathrm{d}J}{\mathrm{d}Q_{k/il}} = p_{il} \left[ \log \frac{Q_{k/il}}{\lambda_{il} Q_{k/l}} - s\varrho_{ik} \right] + p_{il} - p_i p_l = 0$$

i.e.

$$Q_{k/il} = \lambda_{il} Q_{k/l}\, \mathrm{e}^{s\varrho_{ik} - 1 + p_i p_l / p_{il}}\,.$$

Summing over $k$ and using (4), we get

$$\lambda_{il} = [\sum_k Q_{k/l}\, \mathrm{e}^{s\varrho_{ik} - 1 + p_i p_l / p_{il}}]^{-1}\,. \tag{7}$$

Thus the expression for $Q_{k/il}$ can be rewritten as

$$Q_{k/il} = \frac{Q_{k/l}\, e^{s\varrho_{ik} - 1 + p_l p_l / p_{il}}}{\sum_t Q_{t/l}\, e^{s\varrho_{it} - 1 + p_i p_l / p_{il}}} . \tag{8}$$

We solve it for $Q_{k/l}$ since $Q_{k/il}$ is expressed in terms of the conditional probabilities $Q_{k/l}$ of the reproducing letters. Multiplying (8) by $p_{il}$, summing over $i$ and dividing by $Q_{k/l}$ (provided $Q_{k/l} > 0$), we have

$$\sum_i \frac{p_{il}\, e^{s\varrho_{ik} - 1 + p_i p_l / p_{il}}}{\sum_t Q_{t/l}\, e^{s\varrho_{it} - 1 + p_i p_l / p_{il}}} = \sum_i p_{il} . \tag{9}$$

We suppose temporarily that $Q_{k/l} > 0$ for all $k$. Then for a fixed value of $s$, if all the $Q_{k/l}$ obtained by solving the simultaneous non linear equations (9) are positive then equation (8) yields that $Q_{k/il}$ are also positive. Thus a point on the $R_{X/Y}(d)$ curve can be obtained parametrically by expressing both $d(Q)$ and $I(Q)$ in terms of $Q_{k/l}$ and $s$. Thus we have from equations (5) and (2)

$$d = \sum_{i,k,l} p_{il} \lambda_{il} Q_{k/l}\, e^{s\varrho_{ik} - 1 + p_i p_l / p_{il}}\, \varrho_{ik} \tag{10}$$

and

$$I(Q) = \sum_{i,k,l} p_{il} \lambda_{il} Q_{k/l}\, e^{s\varrho_{ik} - 1 + p_i p_l / p_{il}} \left[ \log \lambda_{il} + s\varrho_{ik} - 1 + \frac{p_i p_l}{p_{il}} \right] =$$

$$= sd + \sum_{i,l} p_{il} \log \lambda_{il} .$$

Thus $R_{X/Y}$ which is the minimum of $I(Q)$ is such that

$$R_{X/Y} = sd + \sum_{i,l} p_{il} \log \lambda_{il} , \tag{11}$$

where $\lambda_{il}$ is given by (7).

Now if for a particular value of $s$, the unconstrained solution procedure yields one or more $Q_{k/l} \leqq 0$ then the results can be formulated as in Berger [1], Lemma 1, p. 32.

**Remark.** When side information about the source is not provided at the encoder and decoder, then the equations (10) and (11) reduce to (cf. Berger [1], p. 32)

$$d = \sum_{i,k} p_i \lambda_i Q_k\, e^{s\varrho_{ik}} \varrho_{ik}$$

and

$$R_X = sd + \sum_i p_i \log \lambda_i ,$$

where

$$\lambda_i = [\sum_k Q_k\, e^{s\varrho_{ik}}]^{-1} .$$

**Example.** We consider an example in which the main as well as the side channels are both binary symmetric. The source is memoryless with alphabet $X_k = \{1, 2\}$ and side information alphabet $Y_k = \{1, 2\}$. Also let the received set be $\hat{X}_k = \{1, 2\}$. Further let the distortion be given by the probability of error distortion, i.e.

$$\varrho_{ik} = 1 - \delta_{ik} \quad \text{where} \quad \delta_{ik} = \begin{cases} 1 & \text{for} \quad i = k\,, \\ 0 & \text{for} \quad i \neq k \end{cases}$$

so that

$$\varrho_{ik} = 0 \quad \text{for} \quad i = k\,, \qquad = 1 \quad \text{for} \quad i \neq k\,, \quad i, k = 1, 2\,. \tag{12}$$

Now from equations (9), we have

$$\sum_{i=1}^{2} p_{il}\lambda_{il}\, e^{s\varrho_{ik} - 1 + p_i p_l / p_{il}} = \sum_{i=1}^{2} p_{il}\,; \quad k, l = 1, 2\,.$$

Solving these simultaneous equations, we get

$$\lambda_{il} = \frac{\sum_{j=1}^{2} p_{jl}}{p_{il}(e^s + 1)\exp\left\{\dfrac{\sum_{j=1}^{2} p_{jl} \sum_{j=1}^{2} p_{ij}}{p_{il}} - 1\right\}}\,. \tag{13}$$

From equation (7), we have

$$\frac{1}{\lambda_{il}} = \sum_{k=1}^{2} Q_{k/l}\, e^{s\varrho_{ik} - 1 + p_i p_l / p_{il}}\,.$$

Solving these equations for $Q_{k/l}$ we have

$$Q_{k/l} = \frac{p_{jl}\, e^s - p_{kl}}{(e^s - 1)\sum_{i=1}^{2} p_{il}}\,; \quad j \neq k\,, \quad j = 1, 2\,. \tag{14}$$

On using equations (13) and (14) equations (10) and (11) yield

$$d = \frac{e^s}{e^s + 1}\,, \tag{15}$$

$$R_{X/Y} = sd + (p_{11} + p_{21})\, H\left(\frac{p_{11}}{p_{11} + p_{21}}, \frac{p_{21}}{p_{11} + p_{21}}\right) + (p_{12} + p_{22})\, H\left(\frac{p_{12}}{p_{12} + p_{22}}, \frac{p_{22}}{p_{12} + p_{22}}\right) - \log(e^s + 1)\,. \tag{16}$$

Thus the equations (15) and (16) determine the distortion and rate for the example considered. 

## 3. CONVEXITY OF THE FUNCTION $R_{X/Y}(d)$

We shall prove that $R_{X/Y}(d)$ is a convex $\cup$ function of $d$. Thus for any pair of distortion values $d'$ and $d''$ and any number $\lambda \in [0, 1]$ we shall show that

$$R_{X/Y}(\lambda d' + (1 - \lambda)\, d'') \leqq \lambda R_{X/Y}(d') + (1 - \lambda)\, R_{X/Y}(d'') . \tag{17}$$

Let $q'(\hat{x}/x, y)$ and $q''(\hat{x}/x, y)$ achieve the points $(d', R_{X/Y}(d'))$ and $(d'', R_{X/Y}(d''))$ respectively and let

$$q^*(\hat{x}/x, y) = \lambda q'(\hat{x}/x, y) + (1 - \lambda)\, q''(\hat{x}/x, y) . \tag{18}$$

Then it is easy to see that $\{q^*(\hat{x}/x, y)\}$ is a bonafide conditional probability distribution. Now by definition

$$d(q) = \sum_{\hat{x},x,y} D(x, \hat{x})\, p(x, y)\, q(\hat{x}/x, y)$$

and in particular

$$\begin{aligned} d(q^*(\hat{x}/x, y)) &= d(\lambda q'(\hat{x}/x, y) + (1 - \lambda)\, q''(\hat{x}/x, y)) , \\ &= \lambda d(q'(\hat{x}/x, y)) + (1 - \lambda)\, d(q''(\hat{x}/x, y)) , \\ &= \lambda d' + (1 - \lambda)\, d'' . \end{aligned}$$

This shows that $d(q^*)$ is a linear function of $d$ and that if $q_D$ is defined as

$$q_D = \{q(\hat{x}/x, y) : d(q) \leqq D\}$$

then

$$q^*(\hat{x}/x, y) \in q_{\lambda d' + (1-\lambda)d''} .$$

Next we have

$$R_{X/Y}(\lambda d' + (1 - \lambda)\, d'') \leqq I(q^*(\hat{x}/x, y)) , \tag{19}$$

where

$$I(q^*(\hat{x}/x, y)) = \sum_{x,y,\hat{x}} p(x, y)\, q^*(\hat{x}/x, y) \log \frac{q^*(\hat{x}/x, y)}{q^*(\hat{x}/y)} = \tag{20}$$

$$= \sum_{x,y,\hat{x}} p(x, y)\, [\lambda q'(\hat{x}/x, y) + (1 - \lambda)\, q''(\hat{x}/x, y)] \log \frac{\lambda q'(\hat{x}/x, y) + (1 - \lambda)\, q''(\hat{x}/x, y)}{\lambda q'(\hat{x}/y) + (1 - \lambda)\, q''(\hat{x}/y)} .$$

 Then for $a > 0$, $b \geqq 0$, we have the inequality

$$\log \frac{a+b}{a} \leqq \frac{a+b}{a} - 1$$

i.e.

$$\log (a + b) \leqq \log a + \frac{b}{a} \tag{21}$$

with equality iff $b = 0$.

We shall use the above inequality for two sets of values $a_1$, $b_1$ and $a_2$, $b_2$ given by

$$a_1 = \frac{q'(\hat{x}/x, y)}{q'(\hat{x}/y)}, \quad a_2 = \frac{q''(\hat{x}/x, y)}{q''(\hat{x}/y)}, \tag{22}$$

$$b_1 = \frac{(1-\lambda)\,[q'(\hat{x}/y)\,q''(\hat{x}/x, y) - q''(\hat{x}/y)\,q'(\hat{x}/x, y)]}{q'(\hat{x}/y)\,[\lambda q'(\hat{x}/y) + (1-\lambda)\,q''(\hat{x}/y)]},$$

$$b_2 = \frac{\lambda[q''(\hat{x}/y)\,q'(\hat{x}/x, y) - q'(\hat{x}/y)\,q''(\hat{x}/x, y)]}{q''(\hat{x}/y)\,[\lambda q'(\hat{x}/y) + (1-\lambda)\,q''(\hat{x}/y)]}.$$

Thus we have from equations (20), (21) and (22)

$$I(q^*(\hat{x}/x, y)) \leqq \lambda \sum_{x,y,\hat{x}} p(x, y)\, q'(\hat{x}/x, y) \left[ \log \frac{q'(\hat{x}/x, y)}{q'(\hat{x}/y)} + \right.$$

$$\left. + \frac{(1-\lambda)\,[q''(\hat{x}/x, y)\,q'(\hat{x}/y) - q''(\hat{x}/y)\,q'(\hat{x}/x, y)]}{q'(\hat{x}/x, y)\,[\lambda q'(\hat{x}/y) + (1-\lambda)\,q''(\hat{x}/y)} \right] +$$

$$+ (1-\lambda) \sum_{x,y,\hat{x}} p(x, y)\, q''(\hat{x}/x, y) \left[ \log \frac{q''(\hat{x}/x, y)}{q''(\hat{x}/y)} + \right.$$

$$\left. + \frac{\lambda[q''(\hat{x}/y)\,q'(\hat{x}/x, y) - q'(\hat{x}/y)\,q''(\hat{x}/x, y)]}{q''(\hat{x}/x, y)\,[\lambda q'(\hat{x}/y) + (1-\lambda)\,q''(\hat{x}/y)]} \right] = \lambda I(q'(\hat{x}/x, y)) + (1-\lambda)\, I(q''(\hat{x}/x, y))$$

i.e.

$$I(q^*(\hat{x}/x, y)) \leqq \lambda I(q'(\hat{x}/x, y)) + (1-\lambda)\, I(q''(\hat{x}/x, y)) = \tag{23}$$

$$= \lambda R_{X/Y}(d') + (1-\lambda)\, R_{X/Y}(d'').$$

Combining (19) and (23) we get

$$R_{X/Y}(\lambda d' + (1-\lambda)\, d'') \leqq \lambda R_{X/Y}(d') + (1-\lambda)\, R_{X/Y}(d'').$$

Hence $R_{X/Y}(d)$ is a convex $\cup$ function of $d$.

## 4. BASIC EQUATIONS WHEN FIDELITY CRITERIA ARE OVER MAIN AS WELL AS SIDE CHANNEL

In this section we consider the problem of source encoding with fidelity criteria when the decoder as well as the encoder are provided with side information about the source having distortions $d_1$ and $d_2$ along the main channel and side channel respectively.

Let for $d_1 \geqq 0$, $d_2 \geqq 0$, $M_0(d_1, d_2)$ be the set of probability distributions $\{p(x, y, \hat{x})\}$, $x \in \mathscr{X}$, $y \in \mathscr{Y}$, $\hat{x} \in \hat{\mathscr{X}}$ such that

$$p(x, y) = \sum_{\hat{x}} p(x, y, \hat{x}),$$

$$\sum_{x,y,\hat{x}} D(x, \hat{x})\, p(x, y, \hat{x}) = d_1,$$

$$\sum_{x,y,\hat{x}} D(y, \hat{x})\, p(x, y, \hat{x}) = d_2.$$

Then define rate distortion function with fidelity criteria on the main and side channel as

$$R_{X/Y}(d_1, d_2) = \min_{p \in M_0(d_1, d_2)} I(X, \hat{X}/Y), \tag{24}$$

where $I(.)$ as before denotes the ordinary Shannon's Mutual Information.

Thus the problem of determining $R_{X/Y}(d_1, d_2)$ is of minimizing the average mutual information (24) subject to the constraints (3) and (4) together with

$$\sum_{i,k,l} \varrho_{ik} p_{ikl} = d_1 \quad \text{i.e.} \quad \sum_{i,k,l} p_{il} Q_{k/il} \varrho_{ik} = d_1 \tag{25}$$

and

$$\sum_{i,k,l} \varrho_{kl} p_{ikl} = d_2 \quad \text{i.e.} \quad \sum_{i,k,l} p_{il} Q_{k/il} \varrho_{kl} = d_2. \tag{26}$$

As before for this purpose we consider the augmented function

$$J(Q) = I(Q) - \sum_{i,l} v_{il} \sum_k Q_{k/il} - S_1 \sum_{i,k,l} \varrho_{ik} p_{ikl} - S_2 \sum_{i,k,l} \varrho_{kl} p_{ikl} \tag{27}$$

where the parameters $v_{il}$, $S_1$ and $S_2$ are Lagrange Multipliers. Taking $\log \lambda_{il} = v_{il}/p_{il}$ we may write (27) as

$$J(Q) = \sum_{i,l} p_{il} \sum_k Q_{k/il} \left[ \log \frac{Q_{k/il}}{\lambda_{il} Q_{k/l}} - S_1 \varrho_{ik} - S_2 \varrho_{kl} \right]. \tag{28}$$

For stationary point, we have

$$\frac{\mathrm{d}J}{\mathrm{d}Q_{k/il}} = p_{il} \left[ \log \frac{Q_{k/il}}{\lambda_{il} Q_{k/l}} - S_1 \varrho_{ik} - S_2 \varrho_{kl} \right] + p_{il} - p_i p_l = 0$$

i.e.

$$Q_{k/il} = \lambda_{il} Q_{k/l}\, e^{S_1\varrho_{ik} + S_2\varrho_{kl} + (p_i p_l/p_{il}) - 1} . \tag{29}$$

Summing (29) over $k$ and using (4), we get

$$\lambda_{il} = [\sum_k Q_{k/l}\, e^{S_1\varrho_{ik} + S_2\varrho_{kl} + (p_i p_l/p_{il}) - 1}]^{-1} . \tag{30}$$

Multiplying (29) by $p_{il}$ summing over $i$ and dividing by $Q_{k/l}$ (provided $Q_{k/l} > 0$) we have

$$\sum_i \frac{p_{il}\, e^{S_1\varrho_{ik} + S_2\varrho_{kl} + (p_i p_l/p_{il}) - 1}}{\sum_{k'} Q_{k'/l}\, e^{S_1\varrho_{ik'} + S_2\varrho_{k'l} + (p_i p_l/p_{il}) - 1}} = \sum_i p_{il} . \tag{31}$$

Thus equations (25), (26) and (2) give

$$d_1 = \sum_{i,k,l} p_{ik} p_{il} \lambda_{il} Q_{k/l}\, e^{S_1\varrho_{ik} + S_2\varrho_{kl} + (p_i p_l/p_{il}) - 1} , \tag{32}$$

$$d_2 = \sum_{i,k,l} p_{kl} p_{il} \lambda_{il} Q_{k/l}\, e^{S_1\varrho_{ik} + S_2\varrho_{kl} + (p_i p_l/p_{il}) - 1} \tag{33}$$

and

$$I(Q) = \sum_{i,l} p_{il} \log \lambda_{il} + S_1 d_1 + S_2 d_2 .$$

Thus $R_{X/Y}$ which is the minimum of $I(Q)$ is given by

$$R_{X/Y} = \sum_{i,l} p_{il} \log \lambda_{il} + S_1 d_1 + S_2 d_2 . \tag{34}$$

Equations (32), (33) and (34) are the basic equations when the side information about the source is provided to the encoder as well as to the decoder involving distortions $d_1$ along the main channel and $d_2$ along the side channel.

Now if for a particular value of $s$, one or more $Q_{k/l} \leqq 0$ then as before the results can be formulated as in Berger [1], Lemma 1, p. 32.

**Example.** We give here an example of binary symmetric channels by which rate distortion function can be determined analytically by the method discussed above:

Consider the binary source $X_k = \{1, 2\}$; side information $Y_k = \{1, 2\}$ and the received set $\hat{X}_k = \{1, 2\}$.

Further let the distortion be given by the probability of error distortion, i.e.

$$\varrho_{ik} = 1 - \delta_{ik} \quad \text{where} \quad \delta_{ik} = 1 \quad \text{for} \quad i = k , \qquad = 0 \quad \text{for} \quad i \neq k . \tag{35}$$

From equation (31), we have

$$\sum_{i=1}^{2} p_{il} \lambda_{il}\, e^{S_1\varrho_{ik} + S_2\varrho_{kl} + (p_i p_l/p_{il}) - 1} = \sum_{i=1}^{2} p_{il} ; \quad k = 1, 2 ; \quad l = 1, 2 .$$

Solving these simultaneous equations, we get 

$$(36)\qquad \lambda_{il} = \left(e^{S_1} - e^{S_2}\right) A \quad \text{for} \quad i = l\,,$$
$$= \left(e^{S_1+S_2} - 1\right) A \quad \text{for} \quad i \neq l\,, \quad i, l = 1, 2\,,$$

where

$$A = \frac{\sum_{k=1}^{2} p_{kl}}{p_{il}(e^{2S_1} - 1)\, e^{S_2 - 1} \exp\left[\left(\sum_{k=1}^{2} p_{ik} \sum_{k=1}^{2} p_{kl}\right)/p_{il}\right]}\,.$$

Now from equation (30), we have

$$\frac{1}{\lambda_{il}} = \sum_{k=1}^{2} Q_{k/l}\, e^{S_1 \varrho_{ik} + S_2 \varrho_{kl} + (p_i p_l/p_{il}) - 1}\,.$$

Solving these equations, for $Q_{k/l}$, we get

$$(37)\quad Q_{k/l} = \frac{1}{\sum_{i=1}^{2} p_{il}} \left[ p_{jl} - \left\{ \frac{p_{kl}\, e^{S_2}}{e^{S_1} - e^{S_2}} - \frac{p_{jl}}{e^{S_1+S_2} - 1} \right\} \right] \quad \text{for} \quad k = l\,, \quad j \neq k\,,$$
$$= \frac{1}{\sum_{i=1}^{2} p_{il}} \left[ \frac{p_{jl}\, e^{S_1}}{e^{S_1} - e^{S_2}} - \frac{p_{kl}}{e^{S_1+S_2} - 1} \right] \quad \text{for} \quad k \neq l\,, \quad j = l\,, \quad i, j, k, l = 1, 2\,.$$

On using equations (36) and (37), equations (32), (33) and (34) give

$$(38)\quad d_1 = \frac{e^{2S_1}}{e^{2S_1} - 1} - \frac{e^{S_1}}{e^{2S_1} - 1} \left[ \frac{e^{S_1} - e^{S_2}}{e^{S_1+S_2} - 1} (p_{12} + p_{21}) + \frac{e^{S_1+S_2} - 1}{e^{S_1} - e^{S_2}} (p_{11} + p_{22}) \right],$$

$$(39)\qquad d_2 = \frac{e^{S_1}(p_{11} + p_{22})}{e^{S_1} - e^{S_2}} - \frac{p_{12} + p_{21}}{e^{S_1+S_2} - 1}$$

and

$$(40)\qquad R_{X/Y} = S_1 d_1 + S_2(d_2 - 1) + (p_{11} + p_{22}) \log\left(e^{S_1} - e^{S_2}\right) +$$
$$+ (p_{12} + p_{21}) \log\left(e^{S_1+S_2} - 1\right) + (p_{11} + p_{21})\, H\left(\frac{p_{11}}{p_{11} + p_{21}}, \frac{p_{21}}{p_{11} + p_{21}}\right) +$$
$$+ (p_{12} + p_{22})\, H\left(\frac{p_{12}}{p_{12} + p_{22}}, \frac{p_{22}}{p_{12} + p_{22}}\right) - \log\left(e^{2S_1} - 1\right).$$

These equations determine the distortions and rate for the example considered.

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