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Petr Mariel

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A DYNAMIC MODEL OF ADVERTISING COMPETITION: AN EMPIRICAL ANALYSIS OF FEEDBACK STRATEGIES[1]

PETR MARIEL[2]

This paper provides an empirical framework for estimating the parameters of a differential game of advertising competition taking into account the informative and predatory contents of advertising. The estimated model is a simultaneous equations model consisting of the firms' response functions and the profit maximizing first-order conditions. Hausman's specification test is used to examine the appropriateness of the Nash equilibrium assumption for the German automobile industry market.

1. INTRODUCTION

The present study provides empirical investigation of dynamic advertising competition using a differential game model defined in [12]. This model resembles the Vidale–Wolfe generalizations and Excess advertising models but includes a new effect not considered in the previous models: the informative part of advertising.

There is a large number of studies devoted to differential game models in the field of advertising. [24] provides a very extensive survey on optimal control theory applications to the fields of advertising including, among others, two large families of models: Advertising capital models (Nerlove–Arrow model) and Sales advertising response models (Vidale–Wolfe model). [20] surveys differential games in advertising and sums up conclusions of Vidale–Wolfe, Lanchester-type, 'Leitmann', Excess advertising and Combined Vidale–Wolfe/excess advertising models. Another review of the existing literature can be found in the monograph [9] on dynamic models of advertising competition.

Most of the studies in this field are, however, analytical investigations of a given model and only a few of them offer empirical validation of a proposed model. We could mention here as exceptions studies [2] and [1]. In the first study the authors derive open-loop and closed-loop equilibria for the Lanchester model of combat and estimate the discrete-time analog of the kinematic equations to simulate equilibrium

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advertising policies in the soft-drink industry: Coca–Cola and Pepsi–Cola. They estimate the market share response relationship as one equation separately from closed loop strategies. In the second study, the econometric framework is completely different, as the estimation procedure accounts for the joint endogeneity of market shares and marketing efforts, that is, the market share and the advertising of competitors are assumed to be simultaneously determined. The authors use data from the pharmaceutical, soft-drink, beer and detergent industries to carry out their empirical analysis based on the simultaneous equations system and test model misspecification. The test does not reject no misspecification, so the proposed specification (modification of Case game, see [25]) is appropriate for all four industries.

[10] is another paper which uses the Lanchester model as the basis for empirical investigation in two duopolies: Coca–Cola versus Pepsi–Cola (soft-drink industry) and Anheuser–Bush versus Miller (beer industry). The author considers market shares and advertising levels as a simultaneous system and after estimation of nonlinear relationships of the system, he carries out some tests to detect whether closed-loop equilibrium advertising strategies are used by the competitors rather than open-loop strategies.

2. A DYNAMIC MODEL OF ADVERTISING COMPETITION AS A DIFFERENTIAL GAME

As a basis for the empirical study I use a dynamic model of advertising competition defined in [12]. Consider a duopoly market. The two firms are denoted 1 and 2 and the advertising levels u_1 and u_2 respectively. The following differential equations give the evolution of sales as a function of advertising levels:

$$\begin{aligned} \dot{X}_i(t) &= w_i(u_i(t) - u_j(t)) + z_i(u_i(t) + u_j(t)) + K_i - a_i X_i(t) \\ X_i(0) &= X_{i0} > 0, \quad i, j = 1, 2 \qquad i \neq j. \end{aligned} \tag{1}$$

where (t) denotes time; $a_i X_i(t)$, with $a_i \in (0,1)$, is the decay term and parameters a_i, K_i, w_i, and z_i are assumed to be non-negative and constant over time.

The system dynamics (1) express two different effects of advertising. The first term on the right hand side of equation (1) is the business stealing effect. The second term on the right hand side is the total demand effect. In industries where advertising is mainly informative z_i is expected to be relatively high with respect to w_i, and in industries where it is very competitive z_i should be relatively low with respect to w_i. It is assumed that all the other factors affecting the growth of sales are collected in the constant K_i.

The objective of each firm i is to maximize the discounted sum of its instantaneous profit over an infinite time horizon:

$$J^i = \int_{t_0}^{\infty} e^{-rt}[q_i X_i(t) - u_i^2(t)]\,\mathrm{d}t \tag{2}$$

where q_i, the price-cost margin, is assumed to be constant over time, and r is the discount rate, common to both firms.

Firm i faces the problem of maximizing (2) subject to (1), $u_i(t) \geq 0$ and $X_i(t) \geq 0$, with the initial state $(t_0, X(0)) = (t_0, (X_{10}, X_{20}))$. The appropriate framework for analyzing this problem is a differential game. The competing advertising strategies are developed using two kinds of Nash equilibria which are usually studied in the literature: open-loop, in which advertising is a function only of time, and feedback equilibria, which define advertising as a function not only of time but also the current state of the system.

There is a unique open-loop Nash equilibrium for the differential game defined by (2) and (1). The equilibrium advertising levels are:

$$\overline{u}_i^{\mathrm{OL}} = \frac{(w_i + z_i)}{2}\gamma_i \quad i = 1, 2, \tag{3}$$

where $\gamma_i = \frac{q_i}{r+a_i}$.

In contrast to the open-loop Nash equilibrium, in the case of feedback equilibrium a firm cannot commit itself in advance to any given advertising spending path. The optimal advertising levels change in response to changes in the state variables $X(t) = (X_i(t), X_j(t))$. In an equilibrium in feedback strategies for the game, players are using optimal paths for the control variables u_i, which are also optimal in every subgame.

It is generally more difficult to obtain feedback strategies than open-loop strategies as they usually involve partial differential equations. We can use the value fun tion approach in the above game to derive the feedback equilibria. We get the general form of the optimal strategy for firm i, proposing the value function as a quadratic polynom of the state variables (this restr'cts our strategies to being linear in state variables, but this approach is usually used in the literature: see for example [15], [21] or [22]):

$$\begin{aligned} u_i^{FS} &= \frac{w_i + z_i}{2}c_2^i + \frac{z_j - w_j}{2}c_3^i + \left(\frac{w_i + z_i}{2}c_4^i + \frac{z_j - w_j}{2}c_5^i\right) X_i \\ &\quad + \left(\frac{w_i + z_i}{2}c_5^i + \frac{z_j - w_j}{2}c_6^i\right) X_j, \end{aligned} \tag{4}$$

where c_j^i, $j = 2, 3, \ldots, 6$; $i = 1, 2$, are unknown parameters which can be calculated as solutions of the system of twelve nonlinear algebraic equations obtained by equating coefficients of the state variables in the Hamilton–Jacobi–Bellman equation ([26]).

In general, there are multiple feedback equilibria for the above game. One of them is for example the open-loop equilibrium, because $c_2^i = \gamma_i$, $c_3^i = c_4^i = c_5^i = c_6^i = 0$, $i = 1, 2$ satisfies the Hamilton–Jacobi–Bellman equation. This strategy is, however, a degenerate feedback strategy as in this case u_i^{FS} does not react to changes in X_i nor X_j. The other solutions which define non-degenerate feedback strategies are so complicated that some simplification must be imposed in order to obtain closed-form equilibrium feedback strategies for both firms.

First of all, let us assume symmetry between the two firms. (i.e. $w_1 = w_2 = w$, $z_1 = z_2 = z$, $q_1 = q_2 = q$, $a_1 = a_2 = a$, $K_1 = K_2 = K$). The feedback strategies can then be found for the following three types of markets. The first types of market is a market where advertising is mainly informative and increases demand for both firms (informative advertising; $w = 0$); the second is a market where every unit spent by a firm affects market share and so has a negative impact on the rival's sales (predatory advertising; $z = 0$); and the last is a market where the two effects are of the same size (informative and predatory advertising; $w = z$). Table 1 summarizes optimal strategies for the three different type of markets. See [12] and [11] for more details.

Table 1. Feedback strategies for different type of markets.

Type of Advertising		Feedback strategy
Informative	$(z=0)$	$u_i^{FS_P} = \dfrac{-zq}{2a} + \dfrac{2a+r}{6z}(X_i + X_j)$
Predatory	$(w=0)$	$u_i^{FS_I} = \dfrac{3}{2}\dfrac{qw}{(a+2r)} + \dfrac{2a+r}{6w}(X_i - X_j)$
Predatory and Informative	$(z=w)$	$u_i^{FS_{IP}} = \dfrac{-q(w+z)}{2a} - \dfrac{K(2a+r)}{a} + \dfrac{2a+r}{(w+z)}X_i$

The parameters c_j^i, $j = 2, 3, \ldots, 6$; $(i = 1, 2)$ cannot be obtained for a general asymmetric case ($z_i \neq w_i$, $z_i \neq z_j$, $w_i \neq w_j$, $K_i \neq K_j$ and $z_i, w_i \neq 0$) as a function of z_i, w_i, a_i, K_i, $(i = 1, 2)$ and r as has been done above for the three type of markets. But for given values of z_i, w_i, a_i, K_i the parameters c_j^i can be obtained through numerical methods. How the parameters z_i, w_i, a_i, K_i, $(i = 1, 2)$ can be estimated starting from sales and advertising expenditures data is shown in the next section.

3. ESTIMATION PROCEDURE

Our objective in this section is to see how to estimate the discrete-time analog of (1), where $\dot{X}_i(t) = X_{i,t} - X_{i,t-1}$:

$$X_{it} = w_i(u_{it} - u_{jt}) + z_i(u_{it} + u_{jt}) + K_i + (1 - a_i)X_{i,t-1}, \tag{5}$$

Define $(z_i + w_i) = \alpha_i$ and $(z_i - w_i) = \beta_i$. Then (5) can be rewritten as:

$$X_{it} = \alpha_i u_{it} + \beta_i u_{jt} + K_i + (1 - a_i)X_{i,t-1}. \tag{6}$$

As stated in [10] and [1], when we accept the Nash equilibrium as the appropriate solution concept, the equilibrium levels of advertising efforts derived from the first order necessary conditions for Nash equilibrium are usually functions of sales (or market share) level. For the above game for example, the first order necessary conditions for Nash equilibrium define feedback strategies which are linear in the state variables (see (4)):

$$u_i^{FS} = k_{i1} + k_{i2}X_i + k_{i3}X_j \quad i, j = 1, 2, \; i \neq j. \tag{7}$$

That is why the advertising efforts u_i and u_j cannot be considered exogenous variables. If we do not take into account the endogeneity of u_i and u_j and estimate parameters α_i, β_i, K_i, and $(1 - a_i)$ from the relation (6) directly by OLS, the estimators will be inconsistent.

We define an econometric model as a system of simultaneous equations which consists of response functions (6) as well as the equilibrium conditions (7). This is called the joint estimation approach.

$$\begin{aligned} X_{1t} &= \alpha_1 u_{1t} + \beta_1 u_{2t} + K_1 + (1-a_1)X_{1,(t-1)} + v_{1t} && (8)\\ X_{2t} &= \alpha_2 u_{2t} + \beta_2 u_{1t} + K_2 + (1-a_2)X_{2,(t-1)} + v_{2t} && (9)\\ u_{1t} &= k_{11} + k_{12}X_{1t} + k_{13}X_{2t} + v_{3t} && (10)\\ u_{2t} &= k_{21} + k_{22}X_{1t} + k_{23}X_{2t} + v_{4t}. && (11) \end{aligned}$$

The variables v_{it}, $i = 1, 2, 3, 4$ denote random error terms which are assumed to have a joint distribution with mean zero and covariance matrix Ω. These four equations represent the econometric formulation of the dynamic model we proposed in the previous section.

It is easy to see that parameters α_i, β_i, K_i and $(1 - a_i)$, $i = 1,2$, in the first two equations are not identified and the remaining parameters k_{i1}, k_{i2} and k_{i3}, $i = 1,2$, are just identified. The restrictions needed for identification of the first two equations can of course be of different types and will vary in different industries. Let us suppose $\alpha = \alpha_i = \alpha_j$ and $\beta = \beta_i = \beta_j$, that is, the effects of informative and predatory advertising on sales are equal for both firms. These conditions seem reasonable as both competitors usually use the same media for their advertising. Different combinations of restrictions such as $\alpha_i = \alpha_j = 0$, $\beta_i = \beta_j = 0$, $a_i = a_j$ or $K_i = K_j$, which also lead to identification of the first two equation, seem more restrictive.

The identified system can be rewritten as:

$$\begin{bmatrix} X_{\cdot 1} \\ X_{\cdot 2} \\ u_{\cdot 1} \\ u_{\cdot 2} \end{bmatrix} =$$

$$= \begin{bmatrix} u_{\cdot 1} & u_{\cdot 2} & X_{\cdot 1_{-1}} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ u_{\cdot 2} & u_{\cdot 1} & 0 & 0 & X_{\cdot 2_{-1}} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & X_{\cdot 1} & X_{\cdot 2} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & X_{\cdot 1} & X_{\cdot 2} & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha \\ \beta \\ 1-a_1 \\ K_1 \\ 1-a_2 \\ K_2 \\ k_{12} \\ k_{13} \\ k_{11} \\ k_{22} \\ k_{23} \\ k_{21} \end{bmatrix}$$

$$+\begin{bmatrix} v_{\cdot 1} \\ v_{\cdot 2} \\ v_{\cdot 3} \\ v_{\cdot 4} \end{bmatrix} = \begin{bmatrix} Z_{12} & 0 & 0 \\ 0 & Z_2 & 0 \\ 0 & 0 & Z_3 \end{bmatrix} \begin{bmatrix} \delta_{12} \\ \delta_3 \\ \delta_4 \end{bmatrix} + v = Z^\star \delta + v \tag{12}$$

where $X_{\cdot 1}, X_{\cdot 2}, u_{\cdot 1}, u_{\cdot 2}, v_{\cdot 1}, v_{\cdot 2}, v_{\cdot 3}$ and $v_{\cdot 4}$ are $(T-1) \times 1$ vectors of values $X_{1t}, X_{2t}, u_{1t}, u_{2t}, v_{1t}, v_{2t}, v_{3t}$ and v_{4t} respectively ($t = 2, 3, \ldots, T$). T is the number of observations and the subscript -1 for $X_{1_{-1}}$ and $X_{2_{-1}}$ denotes the one period lagged values of X_1 and X_2. Matrices Z_{12}, Z_3, Z_4 are submatrices of $Z^\star$ defined according to $\delta_{12} = (\alpha, \beta, 1-a_1, K_1, 1-a_2, K_2)'$, $\delta_3 = (k_{12}, k_{13}, k_{11})'$ and $\delta_4 = (k_{22}, k_{23}, k_{21})'$.

We can write system (12) in the usual structural form as:

$$Y\Gamma + XD = V, \tag{13}$$

where Y, X and V are respectively the $((T-1) \times 4)$, $((T-1) \times 3)$ and $((T-1) \times 4)$ matrices of the endogenous variables, predetermined variables (including constant term) and disturbances. The tth rows of Y, X and V are $y_{t\cdot} = (X_{1t}, X_{2t}, u_{1t}, u_{2t})$, $x_{t\cdot} = (X_{1,(t-1)}, X_{2,(t-1)}, 1)$ and $v_{t\cdot} = (v_{1t}, v_{2t}, v_{3t}, v_{4t})$ respectively. The matrices of unknown parameters are defined as:

$$\Gamma = \begin{bmatrix} 1 & 0 & -k_{12} & -k_{22} \\ 0 & 1 & -k_{13} & -k_{23} \\ -\alpha & -\beta & 1 & 0 \\ -\beta & -\alpha & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} -(1-a_1) & 0 & 0 & 0 \\ 0 & -(1-a_2) & 0 & 0 \\ -K_1 & -K_2 & -k_{11} & -k_{21} \end{bmatrix}. \tag{14}$$

The reduced form of the model is:

$$Y = X\Pi + W, \tag{15}$$

where $\Pi = -D\Gamma^{-1}$ and $W = V\Gamma^{-1}$. The properties of the structural disturbances V obviously determine the selection of the estimation procedure. There are only two predetermined (lagged endogenous) variables $X_{1,(t-1)}$ and $X_{2,(t-1)}$ so that the presence of autocorrelation in the disturbances leads to inconsistency of the usual procedures (such as 2SLS or 3SLS) used in empirical applications. The presence of autocorrelation in the structural disturbances U can easily be detected through the reduced form disturbances V. Note that all elements of Γ^{-1} are non-zero, so that autocorrelation in v_{1t}, v_{2t}, v_{3t} or v_{4t} leads to the presence of autocorrelation in all four elements of error terms included in W. I will now describe the estimation procedure for the case of autocorrelation as this is the method which will be used in the next section for estimating model (12) using data from the German automobile industry.

Write equations (13) compactly as.

$$ZA = V, \tag{16}$$

where $Z = (Y, Y_{-1}, 1)$ is $(T-1) \times 9$ matrix with tth row defined as

$$Z_{t\cdot} = \left(X_{1t},\, X_{2t},\, u_{1t},\, u_{2t},\, X_{1,(t-1)},\, X_{2,(t-1)},\, u_{1,(t-1)},\, u_{2,(t-1)},\, 1\right).$$

The corresponding matrix A is:

$$A = \begin{bmatrix} 1 & 0 & -k_{12} & -k_{22} \\ 0 & 1 & -k_{13} & -k_{23} \\ -\alpha & -\beta & 1 & 0 \\ -\beta & -\alpha & 0 & 1 \\ -(1-a_1) & 0 & 0 & 0 \\ 0 & -(1-a_2) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -K_1 & -K_2 & -k_{11} & -k_{21} \end{bmatrix}. \tag{17}$$

About V in (17) we assume that

$$V = V_{-1}R + E, \tag{18}$$

where R is a stable (4×4) matrix of unknown coefficients. I do not impose any a priori restriction on the form of R so that any element of $v_{t\cdot}$ can be correlated not onlv with its own one period lagged value but also with one period lagged values of the other elements of $v_{t\cdot}$. Vector $e_{t\cdot} = (e_{1t}, e_{2t}, e_{3t}, e_{4t})$ is the tth row of the matrix E of order $((T-1) \times 4)$ and the following assumptions about its elements are made:

1. $\mathcal{E}(E) = 0$;
2. $\mathcal{E}(e_{t\cdot}e'_{t\cdot}) = \Sigma \otimes I$, $t = 2, \dots T$;
3. $\mathcal{E}(e_{t\cdot}e'_{s\cdot}) = 0$, $t, s = 2, \dots T$, $t \neq s$.

The ith equation can be rewritten as[1]

$$y_{\cdot i} = Z_i \delta_i + v_{\cdot i}, \quad i = 12, 3, 4 \tag{19}$$

where $y_{\cdot i}$ and $v_{\cdot i}$ are the ith columns, respectively, of Y and V. The matrix Z_i and the vector δ_i are defined as

$$Z_i = (Y_i, Y_{i_{-1}}, 1) \quad \delta_i = (b'_{\cdot i}, c'_{\cdot i}, d'_{\cdot i})', \quad i = 12, 3, 4 \tag{20}$$

with Y_i and $Y_{i_{-1}}$ being submatrices of Y and Y_{-1} respectively, corresponding to the variables included in the ith equation and $b_{\cdot i}$, $c_{\cdot i}$ and $d_{\cdot i}$ are the corresponding non-zero parameters of the ith equation. The complete system

$$y = Z^{\star}\delta + v, \tag{21}$$

[1] The subscript 12 is used for the first two equations as they must be treated together because of the cross-equation identification restriction.

where $y = \text{vec}(Y)$, $v = \text{vec}(V)$, $Z^\star = \text{diag}(Z_{12}, Z_3, Z_4)$ and $\delta = (\delta'_{12}, \delta'_3, \delta'_4)'$ is nothing more than form (12) of the system.

There are different approaches for estimating the model (16) under assumption (18). The maximum likelihood (ML) approach, described in [23], involves many computational difficulties, so that a lot of simpler and relative to the maximum likelihood asymptotically efficient estimators have been proposed in the literature. Some of these estimators are for example the three-stage-least-squares-like estimator, termed the full information dynamic autoregressive (FIDA) estimator, and the converging iterate of FIDA estimator, called CIFIDA. These estimators are described in [7] and [8]. In the first paper the authors describe the FIDA estimator, compare it to the ML estimator and derive the asymptotic distribution of the FIDA estimator. In the second study, the authors extend the previous work, showing that the asymptotic distribution of the CIFIDA and ML estimator are identical and provide a simple two step procedure which is fully efficient as CIFIDA and ML estimators.

The equations defining the FIDA estimates are obtained by minimizing

$$\text{tr}\ \Sigma^{-1}(\tilde{Z}A - Z_{-1}AR)'(\tilde{Z}A - Z_{-1}AR) \tag{22}$$

with respect to the unknown parameters of A subject to prior estimates of Σ and R. The matrix $\tilde{Z}$ in (22) is defined as $\tilde{Z} = (\tilde{Y}, Y_{-1}, 1)$, where

$$\tilde{Y} = Q(QQ)^{-1}Q'Y \tag{23}$$

and the tth row of the $((T-2)\times 9)$ matrix Q is

$$Q_{t\cdot} = \left(X_{1,(t-1)}, X_{2,(t-1)}, u_{1,(t-1)}, u_{2,(t-1)}, X_{1,(t-2)}, X_{2,(t-2)}, u_{1,(t-2)}, u_{2,(t-2)}, 1\right).$$

Note the similarity to the 3SLS estimator, as combining (16) and (18) we get $ZA - V_{-1}R = E$ which is the same as $ZA - Z_{-1}AR = E$ (as $Z_{-1}A = V_{-1}$). This expression is a part of the well known OLS criteria with the slight difference that in (22) the endogenous variables Z are "purged" of their stochastic components, using $\tilde{Z}$ instead of Z. The matrix $\tilde{Z}$ contains the predicted values of the endogenous variables from the reduced form of the model where the lag structure in the error process is reduced. From (13) and (18) we get $Y\Gamma + XD - V_{-1}R = E$ and using $Y_{-1}\Gamma + X_{-1}D = V_{-1}$ the reduced form when the autocorrelation process of V is eliminated is:

$$Y = -XD\Gamma^{-1} + Y_{-1}\Gamma R\Gamma^{-1} + X_{-1}DR\Gamma^{-1} + E\Gamma^{-1}. \tag{24}$$

The right hand side variables of this reduced form are the columns of Q.

The FIDA estimator of δ is defined as:

$$\tilde{\delta}_{FIDA} = (G'\left(\tilde{\Sigma}^{-1} \otimes I_{T-1}\right)G)^{-1}G'\left(\tilde{\Sigma}^{-1} \otimes I_{T-1}\right)g, \tag{25}$$

where

$$G = \tilde{Z}^\star - (\tilde{R} \otimes I_{T-1})Z^\star_{-1} \tag{26}$$

$$g = y - (\tilde{R} \otimes I_{T-1})y_{-1}. \tag{27}$$

The matrix $\tilde{Z}^\star$ denotes the matrix $Z^\star$ (see (12)) except that the current endogenous variables Y are replaced by their predicted values $\tilde{Y}$. To complete the estimation scheme the formulae for estimating R and Σ are given as:

$$\begin{aligned} \tilde{R} &= (\tilde{A}Z'_{-1}Z_{-1}\tilde{A})^{-1}\tilde{A}Z'_{-1}Z\tilde{A} && (28) \\ \tilde{\Sigma} &= \frac{1}{T-1}(Z\tilde{A}-Z_{-1}\tilde{A}\tilde{R})'(Z\tilde{A}-Z_{-1}\tilde{A}\tilde{R}) && (29) \end{aligned}$$

The first formula represents simple OLS regressions of every element of $v_{t.}$ on one period lagged values of all elements of $v_{t.}$, that is on $\left(v_{1,(t-1)}, v_{2,(t-1)}, v_{3,(t-1)}, v_{4,(t-1)}\right)$ and the second formula is the usual estimator of the covariance matrix which uses the residuals from the four previous regressions (28).

The estimation procedure should start with a consistent estimate of A, say $A^{(0)}$ which enables to us to obtain consistent estimates of R and Σ (see (28) and (29)) and subsequently apply (25). This procedure yields a consistent estimator of δ, R and Σ. If we want to get asymptotically efficient estimations we have to iterate back and forth between the estimates of the structural coefficient A and the matrix R using (25) and (28). The result of these iterations would be the CIFIDA estimations. Convergence is, however, not guaranteed from iterating (see [14]). [18] and [8] propose a simple two step procedure which is fully as efficient as CIFIDA and ML estimators and is a natural extension of the result in [6] and [17].

The first step of this simple two step method is the same as in the previous case.

1. Estimate by instrumental variables (16) to get a consistent estimate of A denoted $\tilde{A}^{(0)}$. Then compute $\tilde{R}^{(0)}$ and $\tilde{\Sigma}^{(0)}$ using (28) and (29).

2. Compute the predicted value of the current endogenous variables using (23). Form the matrices G and $\tilde{V} = Z\tilde{A}^{(0)}$ and the vector g using the initial estimation $\tilde{R}^{(0)}$ and $\tilde{\Sigma}^{(0)}$. Then run generalized least squares.

$$\begin{aligned} \phi^{(1)} &= \begin{pmatrix} \tilde{\delta}^{(1)} \\ \mathrm{vec}(\tilde{R}^{(1)}) \end{pmatrix} && (30) \\ &= \left((G, I_4 \otimes \tilde{V}_{-1})'(\tilde{\Sigma}^{(0)} \otimes I_{T-1})^{-1}(G, I_4 \otimes \tilde{V}_{-1})\right)^{-1} \\ &\quad \cdot \left(G, I_4 \otimes \tilde{V}_{-1}\right)' \left(\tilde{\Sigma}^{(0)} \otimes I_{T-1}\right)^{-1} g. && (31) \end{aligned}$$

The final estimation of this two step procedure is:

$$\begin{pmatrix} \tilde{\delta}^{f} \\ \mathrm{vec}(\tilde{R}^{f}) \end{pmatrix} = \begin{pmatrix} \tilde{\delta}^{(1)} \\ \mathrm{vec}(\tilde{R}^{(0)}) + \mathrm{vec}(\tilde{R}^{(1)}) \end{pmatrix}. \qquad (32)$$

Note that the second step takes into account the parts of the terms $V_{-1}R$ which may not have been eliminated by transformations (26) and (27) because of the initial inefficient IV estimation. These transformations are simple multivariate generalizations of the single equation quasi-differences used in very well known Cochrane–Orcutt procedure modified by [17] allowing for the presence of lagged endogenous variables in the model.

As shown in [8] and [18] the asymptotic covariance matrix of the full information maximum likelihood estimator of $(\delta', \operatorname{vec}(R)')'$ denoted D_ϕ^{-1} coincides with the asymptotic covariance matrix of $(\tilde{\delta}^{f'}, \operatorname{vec}(\tilde{R}^{f'})')$. The matrix D_ϕ is defined as:

$$D_\phi = \begin{pmatrix} M & P_1 \\ P_1' & \Sigma^{-1} \otimes \Omega \end{pmatrix}, \tag{33}$$

where

$$\begin{aligned} M &= \operatorname{plim} T^{-1} \ (G' (\Sigma \otimes I_{T-1})^{-1} G) \\ P_1 &= \operatorname{plim} T^{-1} \ (G' (\Sigma \otimes I_{T-1})^{-1} (I_4 \otimes V_{-1})) \\ \Omega &= \mathcal{E}(u_{t.}' u_{t.}) = \sum_{j=0}^{\infty} R' \Sigma R. \end{aligned}$$

The objective of this paper is to test model misspecification, that is: do the observed data confirm simultaneity between sales levels and advertising expenditures? The possible misspecification of the model (8)–(11) can be detected by the [19] specification test. This test rests on a comparison of limited information (LI) to full information (FI) estimators. The null hypothesis is a correct specification of the model. Under the null hypothesis the FI estimator is efficient but yields inconsistent estimates of all equations if one of them is misspecified. On the other hand the LI estimator is not efficient as it does not take into account the correlation among error terms of different equations of the system. However only the misspecified equation is estimated inconsistently, that is, any specification error is not propagated throughout the model.

Denote the LI estimator by $\hat{\delta}_{\mathrm{LI}}$ and the FI estimator by $\hat{\delta}_{\mathrm{FI}}$. The Hausman specification test can be then defined by the following statistic

$$m = (\hat{\delta}_{\mathrm{LI}} - \hat{\delta}_{\mathrm{FI}})' \hat{V}^{-1} (\hat{\delta}_{\mathrm{LI}} - \hat{\delta}_{\mathrm{FI}}) \,. \tag{34}$$

The matrix $\hat{V}$ is a consistent estimator of the asymptotic covariance matrix of $(\hat{\delta}_{\mathrm{LI}} - \hat{\delta}_{\mathrm{FI}})$. Under the null hypothesis of no misspecification the m statistic is distributed asymptotically χ^2 with K degrees of freedom, where K is the dimension of the vector $(\hat{\delta}_{\mathrm{LI}} - \hat{\delta}_{\mathrm{FI}})$. In our case the LI estimator will be the initial equation by equation IV estimation and the FI estimator will be the asymptotically efficient two step estimator $\tilde{\delta}^f$.

4. EMPIRICAL ANALYSIS

The objective of this section is to estimate the parameters of the model proposed in Section 2 and to test the model misspecification using data from two major competitors in the German automobile industry (Volkswagen–Opel GM). The analyzed market is the medium car market in Germany for the period from January 1993 to July 1995. Two models, VW–Golf and Opel Astra, accounted for a large part of industry sales (more than 60 % for the period under study) whereas other competitors in this market, such as the BMW 316–325 with 12 % and the Ford Escort with 9 %, were left behind. Hence, the assumption of a duopoly market can be accepted. Sales are monthly data obtained from the Report of New Car Registration in Germany[1] which is a detailed report by car models and manufacturers. The first series (X_1) VW Golf includes all Golf III, Kombi and Variant models and the second one (X_2) all Opel Astra Models. The advertising expenditures data[2] are the observed number of pages (newspapers, journals) minutes (TV, Radio) and posters, devoted to the advertising of the models under study, multiplied by the official (for us unknown) price of one page, minute or poster. Therefore, both series u_1 (advertising expenditures for Golf) and u_2 (advertising expenditures for Astra) are values in DM and represent advertising efforts of the competitors.

Figures 1 and 2 plot sales and advertising data[3] deflated by consumer price index[4].

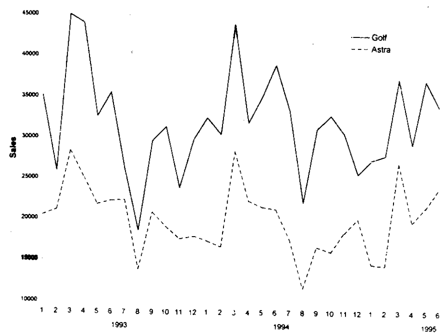

Fig. 1. Sales for VW Golf and Opel Astra.

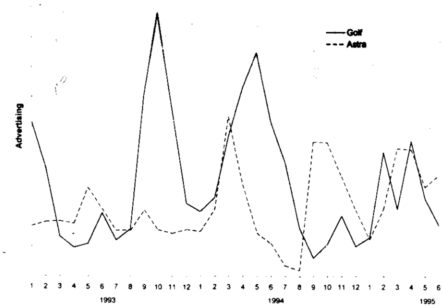

Fig. 2. Advertising for VW Golf and Opel Astra.

First of all the four equations of the reduced form of the system (15) were estimated by OLS to test for the presence of autocorrelation in the error terms W. Because

[1] Source: Zulassungen von fabrikneuen Personenkraftwagen in Deutschland nach Herstellern und Typgruppen, published by Kraftfahrt-Bundesamt.

[2] Source: Nielsen Werbeforschung S+P GmbH, Hamburg

[3] I am not authorized to publish the advertising expenditure data, so the scale of the vertical axis cannot be given.

[4] Source: CD-ROM International Financial Statistics, International Monetary Fund.

of the presence of the lagged endogenous variables in these regressions the following (well known Durbin's) approach was used for doing it. I regressed residuals from the above regressions on their own lagged values and all variables included in the reduced form (15), that is 1, $X_{1,(t-1)}$ and $X_{2,(t-1)}$. Then I tested the joint significance of the coefficients on the lagged residuals with the standard F test. In the third and fourth equation the null hypothesis of no autocorrelation was rejected. The presence of autocorrelation in the reduced form implies presence of autocorrelation in the structural form (as $W = V\Gamma^{-1}$). In such a situation the application of usual methods (2SLS, 3SLS) leads to inconsistent estimations, so I used the procedure described in the previous section.

All estimators based on the transformations (26) and (27) such as the CIFIDA or the full information two step estimator described in the previous section need a consistent initial estimation of the structural parameters. There are many ways in which this can be done. The technique usually used is to treat the lagged endogenous variables as endogenous, and estimate each equation of the system by IV ignoring the autoregressive structure of the disturbances. This yields a consistent estimation of the structural parameters as only exogenous and lagged exogenous variables are used as instruments. There are of course other techniques which can be used in special cases. If the matrix R is diagonal a simple grid method can be used, as the elements of R are between minus one and one and the equations can be treated separately ([13]).

Note however that apart from the constant terms there are only endogenous and lagged endogenous variables in the system (8) – (11). Obviously, some new information must be used for the initial IV estimation. This information cannot be included in the structural form of the system as the relations (8) – (11) are assumed to be well specified and the inclusion of new exogenous variables would make it impossible to compare the empirical results with the theoretical conclusions (see Table 1). The solution adopted here is the use of three deterministic variables (constant, trend and dummy variables for July and August) to get valid instruments needed for the initial estimation. Sales and advertising activities usually decrease in the vacational period so that the selection of this variable is quite reasonable. To get valid instruments $\hat{X}_{1t}$, $\hat{X}_{2t}$, $\hat{u}_{1t}$ and $\hat{u}_{2t}$ we just regress the endogenous variables X_{1t}, X_{2t}, u_{1t} and u_{2t} on constant, trend and dummy variables for the summer months and use these estimations for computing the predicted values of the endogenous variables. Let $\hat{Z}$ denote the matrix Z except for the replacement of the current and lagged endogenous variables by their predicted values. The initial estimation $\hat{\delta}$ is defined as $\hat{\delta}_{IV} = (\hat{Z}'Z)^{-1}\hat{Z}'y$. The residuals from this regression are used for the estimation of Ω. The asymptotic covariance matrix of $\hat{\delta}_{IV}$ is computed as $\widehat{\text{acov}}(\hat{\delta}_{IV}) = (\hat{Z}'Z)^{-1}(\hat{Z}'\hat{\Omega}\hat{Z})(Z'\hat{Z})^{-1}$. The initial IV estimations of the system (8) – (11) is (standard deviation in parenthesis):

$$X_{1t} = \underset{(0.85)}{0.82}\ u_{1t} + \underset{(1.07)}{0.41}\ u_{2t} - \underset{(12248.54)}{3278.36} + \underset{(0.39)}{0.90}\ X_{1,(t-1)} + \hat{v}_{1t} \quad (35)$$

$$X_{2t} = \underset{(0.85)}{0.82}\ u_{2t} + \underset{(1.07)}{0.41}\ u_{1t} - \underset{(8679.87)}{6361.76} + \underset{(0.49)}{0.95}\ X_{2,(t-1)} + \hat{v}_{2t} \tag{36}$$

$$u_{1t} = \underset{(6884.72)}{-2120.66} + \underset{(1.03)}{0.44}\ X_{1t} - \underset{(1.67)}{0.27}\ X_{2t} + \hat{v}_{3t} \tag{37}$$

$$u_{2t} = \underset{(3474.13)}{404.96} + \underset{(0.52)}{1.67}\ X_{1t} - \underset{(0.84)}{2.47}\ X_{2t} + \hat{v}_{4t}. \tag{38}$$

The estimations (35)–(38) imply that $z = 0.62$ and $w = 0.20$, that is both effects of advertising (informative and competitive) are present but the informative effect is higher. The estimations of decay terms are $a_1 = 0.10$ and $a_2 = 0.05$, that is, both stocks of goodwill depreciate very slowly.

In [12] equilibrium feedback strategies were obtained for some extreme cases such as informative ($w_i = w_j = 0$) and predatory ($z_i = z_j = 0$) advertising for symmetric firms. The equilibrium feedback strategies for the general asymmetric case ($z_i \neq w_i$, $z_i \neq z_j$, $w_i \neq w_j$ and $z_i, w_i \neq 0$) were not given because it is impossible to obtain the parameters c_j^i, $j = 2, 3, \ldots, 6$, $i = 1, 2$ in (4) analytically in such situation. The strategies defined in Table 1 show that the presence of competitive advertising implies negative reaction to the rival's sales. This is the explanation for the negative coefficient of X_{2t} in (37). The negative reaction of firm 2 to its own sales has no relation to Table 1. The explanation however could be that firm 2 increases its advertising effort when sales decreases and vice versa. This case does not appear in the Table 1 but is clearly possible if $z_2, w_2 \neq 0$ and $z_2 \neq w_2$ (see (4)).

In general, the precision of the estimation increases considerably in the following two step estimation:

$$X_{1t} = \underset{(0.24)}{0.36}\ u_{1t} + \underset{(0.32)}{0.34}\ u_{2t} + \underset{(6296.76)}{12304.23} + \underset{(0.18)}{0.44}\ X_{1,(t-1)} + \hat{e}_{1t} \tag{39}$$

$$X_{2t} = \underset{(0.24)}{0.36}\ u_{2t} + \underset{(0.32)}{0.34}\ u_{1t} + \underset{(3781.16)}{5163.01} + \underset{(0.21)}{0.51}\ X_{2,(t-1)} + \hat{e}_{2t} \tag{40}$$

$$u_{1t} = \underset{(10259.09)}{-7161.76} + \underset{(0.23)}{0.20}\ X_{1t} + \underset{(0.53)}{0.38}\ X_{2t} + \hat{e}_{3t} \tag{41}$$

$$u_{2t} = \underset{(17446.78)}{-11520.69} + \underset{(0.46)}{1.47}\ X_{1t} - \underset{(0.98)}{1.46}\ X_{2t} + \hat{e}_{4t}. \tag{42}$$

The corresponding estimation of the matrix R is:

$$R = \begin{pmatrix} -0.28 & -0.08 & -0.07 & -0.11 \\ -0.24 & -0.12 & -0.19 & -0.17 \\ 0.46 & -0.35 & 0.67 & 0.32 \\ -0.23 & 0.23 & -0.29 & 0.06 \end{pmatrix}.$$

The highest element r_{33} says that the strongest dependence on the one period lagged value corresponds to v_{3t}. The reason is probably the two huge peaks in the advertising expenditures series of the Golf model in the last months of 1993 and spring 1994 .

As expected, the standard deviations of these FI estimations decrease. The estimations of the coefficients of the informative and predatory effect of advertising are $z = 0.35$ and $w = 0.01$ respectively. The surprisingly high informative effect of advertising in this very competitive market can be explained by inclusion of exhibitions of new (by the consumers unknown) elements in car advertising such as ABS, side bars or airbag which explain what these elements are useful for, and thus, also inform the rival's customers as the car characteristics are very similar.

The estimations of the decay term $a_1 = 0.56$ and $a_2 = 0.49$ are higher than in the LI estimation and indicate that both stocks of goodwill depreciate at approximately the same rate. The estimated feedback strategies (41) – (42) show a slightly different behavior than in LI estimation. Firm 1 reacts positively to both own and rival's sales. This reaction resembles the optimal feedback strategy in the extreme case of informative advertising, where the positive reaction to the rival's sales works as a promise of cooperation (see Table 1). The explanation of the signs in (42) does not change with respect to the previous LI case.

It is important to highlight that the selection of the additional deterministic variables used for the initial estimation by IV has very little impact on the final results. I estimated the above model using the CIFIDA estimator starting with different initial values $A^{(0)}$ and, when convergence was reached, the results were very similar to the presented in (39) – (42). The common results in all converged estimations for this market were: the informative effect of advertising is higher than the predatory effect, the decay terms are similar for both firms (about 0.5) and the first firm reacts positively to its own and rival's sales. The initial values $A^{(0)}$ used for this CIFIDA iterations were of different natures. First I tried different "reasonable" expected values set completely ad hoc. Then I used grid method for the elements of R estimating equation by equation. This is a generalization of the procedure proposed in [13]. Simply transform the original variables according to (26) and (27) forming the "multivariate" quasi-differences and estimate every equation for different values of r_{ij} involved in every equation. Then choose those values of r_{ij} and the corresponding estimates of δ_i which yield the smallest sum of squared residuals. The first two equations involved 8 unknown elements of R (the first two rows of R) and the third and fourth equation only 4 elements (the third and fourth row of R). Note that the grid for the first two equations was not very dense as 10 possible values of every r_{ij}, $(i = 1, 2;\ j = 1, 2, 3, 4)$ implied 10^8 iterations. Nevertheless, this grid procedure yielded a good initial values for the CIFIDA iterations.

I will turn now to the testing of misspecification in the estimated model. The LI and FI estimators $\hat{\delta}_{\mathrm{LI}}$ and $\hat{\delta}_{\mathrm{FI}}$ are defined by $\hat{\delta}_{IV}$ and $\tilde{\delta}^f$. The estimator of the asymptotic covariance matrix of $(\hat{\delta}_{IV} - \tilde{\delta}^f)$ is $V = \widehat{\mathrm{acov}}(\hat{\delta}_{IV}) - \widehat{\mathrm{acov}}(\hat{\delta}^f)$, where $\widehat{\mathrm{acov}}(\hat{\delta}^f)$ is computed as $(G'\ (\tilde{\Sigma}^{-1} \otimes I_{T-1})\ G)^{-1}$ (see (33)). The value of the Hausman specification statistic m is 1.84. This means that the null hypothesis of no misspecification cannot be rejected at 5%, as the critical value for 12 degrees of freedom is 21.0. That is, the observed data from the German car industry confirm not only the correct definition of the kinematic equations of the model (1) but also the correct specification of the linear feedback strategies (7). The fact that the data are in accordance with the linear form of the equilibrium feedback strategies is important, since the functional form of the equations (7) is determined by the value functions, which appear as unknown arguments in the Hamilton–Jacobi–Bellman equation. The only way to solve this equation for the above model is by proposing the functional form of the value functions as a quadratic polynom of the state variables (X_1, X_2). The empirical evidence then supports this proposal.

5. CONCLUSION

The main result of this paper is that the Nash equilibrium assumption is consistent with the data used in the estimation and provides empirical support for the theoretical model defined in the form of the sales advertising response model. Other implication of the results is that advertising expenditures must be considered endogenous in a similar type of data as ignoring simultaneity between response functions and equilibrium conditions would lead to inconsistency in the parameter estimates. This means that in situations in which the frequency of the data coincides with the decision-making horizon (as it is in our case) a simultaneous equation approach is needed, since the advertising decision rules are based on expected sales, creating simultaneity in advertising and sales (see [1]).

The parameter estimations for the German automobile industry indicate a surprisingly high informative effect of advertising not expected in such a competitive market and the majority of the estimated signs in the feedback strategies coincide with the analytically derived optimal strategies. The proposed approach can easily be used for data from other industries and surely helps to understand advertising-sales relationships a little better.

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Petr Mariel, Departamento de Econometría y Estadística, Universidad del País Vasco – Euskal Herriko Unibertsitatea, Avenida Lehendakari Aguirre 83, E 48015 Bilbao. Spain.