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The Orbits about an Oblate Spheroid.

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The subject of the present paper is an analysis of the generalised two-bodies problem, where the central body is an inhomogenous oblate spheroid and the companion can be regarded as a mass point. The general result will be that also in this case the orbits are elliptic (or hyperbolic), but showing a steady advance of periastron. The rate of motion will be found to be a function of the oblateness of the spheroid, of its density condensation, of the ratio of its major semi-axis to the semi-axis of the orbit, of the orbital excentricity, and, for orbits inclined to the equatorial plane, also of the direction of nodes.

The problem which we shall face in the present paper may be stated as follows: In the gravitational field of an inhomogeneous oblate spheroid be moving a body which can be represented as a mass-point. It is to investigate the orbits in which the mass-point will move.

It may be a matter of some surprise that despite of its theoretical as well as practical importance the present problem has not yet been investigated in its general form — except the case of a homogeneous spheroid with very small oblateness, which was dealt with by W. D. Macmillan.[1]) The present paper may be considered as an extension of Macmillan's researches to the case of an inhomogeneous spheroid with considerable oblateness — but, as will be seen, the present method of attacking the problem is essentially different from that of Macmillan. Also the points of view in both investigations differ widely. While the aim of Macmillan's research was to establish the existence of periodic orbits in the considered case, the main bulk of the present paper will be devoted to derivation of the departures of orbits from the classical two-bodies problem[2])

[1]) Cf. F. R. Moulton, Periodic Orbits, Washington 1920. Pp. 99—150.

[2]) Under this term we mean the case where both bodies can be treated as mass-points.

explicitly as functions of the oblateness resp. the density condensation of the spheroid. Anticipating the results we may state that the general property of such orbits is the advance of periastron — a quantity the amount of which can be ascertained by observations. Thus the results of our analysis will yield us a possibility of determining the density condensation of the spheroidal body from observable facts, and the application on celestial bodies may in some cases provide us with certain indications concerning their internal constitution, which untill quite recently has been a subject of purely theoretical research. Finally the fact, that both solutions, Macmillan's and mine, as far as they are consistent, point to exactly the same result, presents a valuable check and leaves no room for doubt that both are generally correct.

The equations of motion for the present problem are:

$$\begin{aligned} \frac{d^2x}{dt^2} &= k^2 \frac{\partial V}{\partial x} \\ \frac{d^2y}{dt^2} &= k^2 \frac{\partial V}{\partial y} \\ \frac{d^2z}{dt^2} &= k^2 \frac{\partial V}{\partial z} \end{aligned} \tag{1}$$

where x, y, z, are Cartesian coordinates (the centre of the spheroid taken as origin), V is the corresponding potential and k the Gaussian constant.

The potential of an inhomogeneous oblate spheroid with respect to a point not too near is[3])

$$V = \frac{\mathfrak{M}}{r} + 4\pi \sqrt{1-e^2} \sum_{i=1}^{\infty} \frac{(-1)^i P_{2i}}{2i+1} \cdot \frac{e^{2i}}{r^{2i+1}} \cdot \int_0^{a_1} \varrho a^{2(i+1)} \, da \tag{2}$$

where $\mathfrak{M}$ is the mass of the spheroid, ϱ its density, e the oblateness

$$e = \sqrt{1 - \frac{b^2}{a^2}}$$

a_1 its major semi-axis, P_{2i} the Legendre polynomials and

$$r = \sqrt{x^2 + y^2 + z^2}.$$

In order to be able to evaluate the integral on the right side of (2) a knowledge of ϱ as function of a is necessary. In the present state of research it might seem most advisable to obtain it from the Emden polytropic equation. But apart from the fact that in the theory of rotating stars the polytropic configurations have

[3]) Cf. Tisserand, Traité de la Mécanique Céleste, II, p. 322. Paris, 1891.

recently rather lost their importance,[4]) the solution of Emden's equation cannot be given in finite number of terms and the convergence of the corresponding series is so slow as to make their application highly impracticable. This is mainly the reason why we prefer for the purposes of the present paper the following relation:

$$\varrho = \varrho_c \left(1 - \frac{a}{a_1}\right)^{\nu-1} \tag{3}$$

where ϱ_c is the central value of the density and ν an arbitrary constant. To get an idea about its physical significance, integrate the above expression for ϱ over the realm $0 < a < a_1$. Denoting ϱ_m the mean density of the configuration, we readily see that

$$\frac{\varrho_c}{\varrho_m} = \frac{\nu(\nu+1)(\nu+2)}{6} = \mu \tag{4}$$

i. e. that ν specifies the density condensation alone. The values $\nu = 1$ and ∞ correspond to the limiting Maclaurin's and Roche's model respectively.

With the aid of (3) the definite integral from (2) can be expressed as follows:

$$\begin{aligned} \int_0^{a_1} \varrho a^{2(i+1)}\, \mathrm{d}a &= \varrho_c a_1^{3+2i} \,.\, B(\nu, 3+2i) = \\ &= \varrho_c a_1^{3+2i} (2i+2)! \frac{\Gamma(\nu)}{\Gamma(3+2i+\nu)} \end{aligned} \tag{5}$$

and by cancellation of Gamma functions[5])

$$\int_0^{a_1} \varrho a^{2(i+1)}\, \mathrm{d}a = \varrho_c a_1^{3+2i} \frac{(2+2i)!}{\nu(\nu+1)\ldots(2+2i+\nu)}$$

Eq. (2) takes now the form:

$$\begin{aligned} V = \frac{\mathfrak{M}}{r} + 4\pi a_1^3 \varrho_c \sqrt{1-e^2} \times \\ \times \sum_{i=1}^{\infty} \frac{(-1)^i (2+2i)!\, P_{2i}}{\nu(\nu+1)\ldots(2+2i+\nu)(2i+1)} \cdot \frac{e^{2i}}{r^{2i+1}} \end{aligned} \tag{6}$$

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[4]) It was found that the polytropic density distribution is not invariant with respect to the rotation. For the discussion of this case the reader is referred to my second paper „On the Internal Constitution of Eclipsing Binaries", M. N. R. A. S. 97 (1937), 646—655.

[5]) Cf. Whittaker and Watson: Modern Analysis, Cambridge 1935; p. 254, where a similar case is left as an exercise to the reader.

which after some minor arrangements can be reduced to

$$V = \frac{\mathfrak{M}}{r}\left\{1 + 3\mu \sum_{i=1}^{\infty} \frac{(-1)^i (2+2i)!\, P_{2i}}{\nu(\nu+1)\ldots(2+2i+\nu)(2i+1)} \cdot \left(\frac{e}{r}\right)^{2i}\right\}. \quad (7)$$

We shall work consistently only up to the fourth order in e, i. e. we consider only such small oblatenesses that the effects arising from e^6 and higher terms can be neglected. Inserting the corresponding expressions for the Legendre polynomials and on going through the algebra, we finally obtain:

$$V = \frac{\mathfrak{M}}{r}\left\{1 + e^2 \cdot \frac{\Phi_2}{3} \frac{x^2 + y^2 - 2z^2}{r^4} + \right.$$
$$\left. + e^4 \frac{\Phi_4}{3} \frac{3x^4 + 3y^4 + 8z^4 + 6x^2y^2 - 24x^2z^2 - 24y^2z^2}{r^8}\right\}. \quad (8)$$

The corresponding derivatives are:

$$\frac{\partial V}{\partial x} = -\frac{\mathfrak{M}}{r^3}\left\{1 + e^2\Phi_2 \frac{x^2 + y^2 + 4z^2}{r^4} + \right.$$
$$\left. + e^4 \Phi_4 \frac{(x^2+y^2)^2 + 8z^4 - 12z^2(x^2+y^2)}{r^8}\right\} x$$

$$\frac{\partial V}{\partial y} = \frac{y}{x}\frac{\partial V}{\partial x} \quad (9)$$

$$\frac{\partial V}{\partial z} = -\frac{\mathfrak{M}}{r^3}\left\{1 + e^2\Phi_2 \frac{3x^2 + 3y^2 - 2z^2}{r^4} + \right.$$
$$\left. + e^4\Phi_4 \frac{15(x^2+y^2)^2 + 8z^4 - 40z^2(x^2+y^2)}{r^8}\right\} z$$

where

$$\Phi_2 = \frac{6a_1^2}{(3+\nu)(4+\nu)}$$
$$\Phi_4 = \frac{27a_1^4}{(3+\nu)(4+\nu)(5+\nu)(6+\nu)}. \quad (10)$$

Combining (1) and (9) we arrived at the fundamental equations of our problem.

As one easily verifies, this set of equations has some integrals with the classical two-bodies problem in common: so the integrals of the centre of the mass, one integral of areas and the vis viva integral. The four remaining integrals cannot, however, be expressed in finite terms.

Seeking for them, we find it advantageous to transform the fundamental equations in the cylindrical coordinates and, for the sake of simplicity, to make use of the canonical form. Owing to that

$$y \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = x \frac{\mathrm{d}^2 y}{\mathrm{d}t^2}. \tag{11}$$

and therefore

$$rv'' + 2r'v' = 0 \tag{12}$$

the fundamental equations take the following forms[6]):

$$r'' = \frac{c_1^2}{r^3} - \frac{r}{(r^2+q^2)^{\frac{3}{2}}} - e^2\Phi_2 \frac{r^3 - 4rq^2}{(r^2+q^2)^{\frac{7}{2}}} - e^4\Phi_4 \frac{r^5 - 12r^3q^2 + 8rq^4}{(r^2+q^2)^{\frac{11}{2}}} \tag{13}$$

$$q'' = -\frac{q}{(r^2+q^2)^{\frac{3}{2}}} - e^2\Phi_2 \frac{3r^2q - 2q^3}{(r^2+q^2)^{\frac{7}{2}}} - e^4\Phi_4 \frac{15r^4q - 40r^2q^3 + 8q^5}{(r^2+q^2)^{\frac{11}{2}}} \tag{14}$$

where

$$r = \sqrt{x^2 + y^2}$$

$$v = \operatorname{arctg} \frac{y}{x}$$

$$z = q$$

which together with the area integral

$$r^2 v' = c_1 \tag{15}$$

specify completely the orbits.

Let us investigate first the orbits lying in the equatoreal plane of the spheroid, i. e. put $q = 0$. Eq. (13) becomes now:

$$r'' = c_1^2 \,.\, r^{-3} - r^{-2} - e^2\Phi_2 \,.\, r^{-4} - e^4\Phi_4 \,.\, r^{-6}. \tag{16}$$

Integrating we obtain

$$(r')^2 = -c_1^2 \,.\, r^{-2} + 2r^{-1} + \tfrac{2}{3}e^2\Phi_2 r^{-3} + \tfrac{2}{5}e^4\Phi_4 \,.\, r^{-5} + c_2. \tag{17}$$

Changing the variable by use of (15), (17) gets the form

$$\left(\frac{\mathrm{d}r}{\mathrm{d}v}\right)^2 = -r^2 + \frac{c_2}{c_1^2} r^4 + \frac{2}{c_1^2} r^3 + \frac{2e^2\Phi_2}{3c_1^2} r + \frac{2e^4\Phi_4}{5c_1^2} r^{-1} \tag{18}$$

which yields

$$v - c_3 = \int \frac{c_1 \, \mathrm{d}r}{\sqrt{r^2 (c_2 r^2 + 2r - c_1^2) + \tfrac{2}{3}e^2\Phi_2 r + \tfrac{2}{5}e^4\Phi_4 r^{-1}}}. \tag{19}$$

The terms multiplied by e^2 resp. e^4 are obviously minor with respect to the first term of the denominator, and we are therefore allowed to expand the integrand in a series of the form:

$$c_1 \sum_{i=0}^{\infty} \binom{-\frac{1}{2}}{i} \frac{(\frac{2}{3}\Phi_2 r + \frac{2}{5}e^2\Phi_4 r^{-1})^i}{(c_2 r^4 + 2r^3 - c_1^2 r^2)^{i+\frac{1}{2}}} \cdot e^{2i}.$$

[6]) The dashes indicate differentiation with respect to the time.

Consistently with our scheme of approximation we take into account only the terms up to the order e^4, and the term-by-term integration gives:

$$v - c_3 = \int \frac{c_1\,\mathrm{d}r}{r\sqrt{c_2 r^2 + 2r - c_1^2}} - $$
$$- e^2 \int \frac{c_1 \Phi_2\,\mathrm{d}r}{3r^2\sqrt{(c_2 r^2 + 2r - c_1^2)^3}} - \tag{20}$$
$$- e^4 \int \frac{c_1 \Phi_4\,\mathrm{d}r}{5r^4\sqrt{(c_2 r^2 + 2r - c_1^2)^3}} + e^4 \int \frac{c_1 \Phi_2^2\,\mathrm{d}r}{6r^3\sqrt{(c_2 \nu^2 + 2r - c_1^2)^5}}.$$

For $e = 0$, (20) should become the usual expression of the classical two-bodies problem. The reader may verify that this is actually the case, and its well-known solution suggests itself.[7]) But it is easy to see that also for $e \neq 0$ the solution may formally be put in the form

$$r = \frac{c_1^2}{1 + \varepsilon \cos (v - \omega')} \tag{21}$$

but ω' ceases to be constant, being a function of the radius vector and therefore (for the non-circular orbits) of the anomaly. Its evaluation consists of expressing the remaining integrals of (20) in terms of v [8]); after performing this long and rather complicated procedure we finally obtain:

$$\omega' = \omega + \{A \,.\, e^2 + B \,.\, e^4 + \ldots\}\, v \tag{22}$$

where

$$A = \Phi_2 \sum_{i=0}^{\infty} (1 + i)\, \varepsilon^{2i} \tag{22,1}$$

$$B = \sum_{i=0}^{\infty} \left\{ (2\Phi_4 + \Phi_2^2) + (4\Phi_4 + \Phi_2^2)\, i + \right.$$
$$\left. + \frac{1}{2}\Phi_2^2 \left[\binom{2+i}{i-1} + 5 \binom{3+i}{i} \right] \right\} \varepsilon^{2i}. \tag{22,2}$$

Thus we arrived at the conclusion that in the considered

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[7]) This enables us to determine the values of the three integration constants. It follows that

$$c_1 = \sqrt{1 - \varepsilon^2}$$
$$c_2 = -1$$
$$c_3 = \omega,$$

ε and ω being the orbital excentricity and the length of periastron respectively.

[8]) As we are concerned only with secular terms, we have to take in account only the terms multiplied by v, as those multiplied by periodic functions of v clearly cannot contribute to the motion of apses.

case the orbits about an inhomogeneous oblate spheroid are again, in general, of the type (21), but showing a continuous advance of periastron, and we carried out the necessary computation up to the fourth order in e; the corresponding coefficients have just been written down as functions of the density condensation of the spheroid and of the orbital excentricity. It results therefrom that the rate of apsidal motion is the higher,

(I) the greater the oblateness of the spheroid,
(II) the greater its major semi-axis,[9])
(III) the smaller the density condensation,
(IV) the greater the orbital excentricity.

The all above said is true, of course, only for orbits lying in the equatoreal plane of the spheroid and it would be desirable now to extend it to inclined orbits. The idea of simultaneous integration of (13) and (14) suggests itself, but the procedure involved would be extremely tedious. The matter may, however, be much simplified by a very reasonable assumption, that also in this case the orbits can be considered as two-dimensional curves lying in a revolving plane.

Let us turn back to (13). Introducing the new variable

$$u = \frac{q}{r} \tag{23}$$

(13) assumes the following form:

$$r'' = \frac{c_1^2}{r^3} - \frac{1}{(1+u^2)^{\frac{3}{2}}} \cdot \frac{1}{r^2} - e^2\Phi_2 \frac{1-u^2}{(1+u^2)^{\frac{7}{2}}} \cdot \frac{1}{r^4} - e^4\Phi_4 \frac{8u^4 - 12u^2 + 1}{(1+u^2)^{\frac{11}{2}}} \cdot \frac{1}{r^6} \tag{21}$$

and, on the other hand, it is clear from the above assumption that

$$u = \operatorname{tg} i \, . \sin(v - \omega'' + \Omega) \tag{22}$$

i being the angle of inclination of the revolving plane to the equator of the spheroid, ω'' the complete expression for the periastron motion for inclined orbits, and Ω the length of node. As $u \lesseqgtr \operatorname{tg} i$, we may, limiting ourselves to small inclinations, develop (21) in a Maclaurin's series

$$r'' = f(0) + f'(0)\, u + \tfrac{1}{2} f''(0)\, u^2 + \ldots \tag{23}$$

where $f(0)$ is obviously the right side of (13). As the right side of (21) contains only the odd powers of u, all derivatives of the even order must vanish when $u \to 0$, and (23) reduces to a power

[9]) Expressed on account of the canonical transformation in terms of the major semi-axis of the orbit as unit.

series in $\text{tg}^2 i$. Supposing that for small values values of i, ω'' comes very near to ω', we may express $(v - \omega'')$ by means of (21) as function of r. Neglecting further the terms of the order $\text{tg}^4 i$ and higher, (23) can be integrated in an analogous way as eq. (16) in the preceding case. The whole procedure is now of course much longer and too complicated to be reproduced here, and except the cases $\Omega = 2n\pi$ and $\Omega = 2(n+1)\pi$, n being an integer, the mathematical difficulties could not yet be mastered completely.

For $\Omega = 2n\pi$, however, il follows[9]):

$$\omega''_n = \omega' + \{[\Phi_2 \Psi_\varepsilon (A_1 + B_1\varepsilon^2 + C_1\varepsilon^4)\,\text{tg}^2 i + \ldots]\, e^2 + \ldots\}\, v \quad (24)$$

where

$$\Psi_\varepsilon = \frac{3}{20} \sum_{i=0}^{\infty} \binom{-4}{i} \varepsilon^{2(i-1)} \quad (24, 1)$$

and

$$A_1 = 5\,(2 + 9\Phi_2);\ B = -(20 - 69\Phi_2);\ C = 2\,(5 + 3\Phi_2). \quad (24, 2)$$

For $\Omega = 2(n+1)\pi$, finally, it results:

$$\omega''_{n+1} = \omega' + \{[\Phi_2 \Psi_\varepsilon (A_2 + B_2\varepsilon^2 + C_2\varepsilon^4 + D_2\varepsilon^6 + E_2\varepsilon^8)\,\text{tg}^2 i + \ldots]\, e^2 + \ldots\}\, v \quad (25)$$

where

$$A_2 = 140\Phi_2 \qquad D_2 = 10\,(1 - 6\Phi_2)$$
$$B_2 = -2\,(5 + 447\Phi_2) \qquad E_2 = -10 \quad (25, 1)$$
$$C_2 = 2\,(5 + 12\Phi_2)$$

It comes out that also for $i \neq 0$, the motion of apsidal line remains always positive, and the greater the inclination, the greater the additional terms. There is also an evident dependance of the rate of motion on the direction of nodes, for in both preceding cases the amount of motion results different, but owing to purely mathematical difficulties this relation could not explicitly be ascertained for all values of Ω.

Numerical Example. As an illustration of the above methods one numerical example may be added. Jupiter and its fifth satellite form a system which comes very closely to the model stated at the outset and discussed in the present paper. The orbit of Jupiter's fifth satellite shows indeed an advance of periastron by the amount 883° a year. As its excentricity as well as its major semi-axis and the oblateness of Jupiter are known, we are enabled with the aid of our results to calculate the density condensation of Jupiter itself.

The following values of elements are adopted:

$$a_1 = 0{,}401$$
$$e = 0{,}362$$
$$\varepsilon = 0{,}006$$

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[9]) Here we have evidently neglected also terms multiplied by $e^4\,\text{tg}^2 i$, i. e. those of the order Φ_4.

and because of the minuteness of inclination of the orbit to Jupiter's equator eq. (22) is directly applicable. The results of computations are:

ν	Φ_2	Φ_4	A	B	$(\omega' - \omega)$ per orbit
1,0	0,0482	0,0008	0,0483	0,0098	2,334°
2,0	0,0322	0,0004	0,0322	0,0045	1,546
3,0	0,0230	0,0002	0,0230	0,0023	1,099
4,0	0,0172	0,0002	0,0172	0,0013	0,819
5,0	0,0134	0,0001	0,0134	0,0008	0,637

As to 883° per year corresponds a value 1,21° per orbit, we obtain by interpolation and by use of (4) for the density condensation of Jupiter the value $\frac{\varrho_c}{\varrho_m} = 7,5$ — a value which comes closely to the density condensation inferred for Saturn.[10])

Concluding I would like to stress that the present paper can contain scarcely more than a mere outline of the whole problem, of the methods employed as well as of the results arrived at. Its full-dressed analysis would exceed widely the scope of the present paper and the detailed treatment will be published elsewhere.

It is also my pleasure to express my sincere thanks to Docent Dr. V. Nechvíle, who has discussed with me the above problems in all stages and to whose stimulating suggestions the present paper owes a great deal.

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O drahách kolem rotačního sferoidu.

(Obsah předešlého článku.)

Tématem práce je zobecnění problému dvou těles pro případ, kdy kolem nehomogenního rotačního sferoidu obíhá hmotný bod, a to jednak pro dráhy v ekvatoreální rovině sferoidu, jednak — v nejstručnějším nárysu — i pro dráhy k rovníkové rovině skloněné. Výsledkem rozboru je, že v obou případech obíhá bod obecně po kuželosečce, ale po dráze s neustále postupující délkou periastra. Velikost posuvu byla pak explicite vyjádřena jako funkce sploštění elipsoidu, jeho vnitřní stavby, velikosti jeho poloosy i výstřednosti dráhy; aplikace výsledku na Jupitera a jeho pátý měsíc, u něhož pohyb periastra je znám z pozorování, vedl k zjištění vnitřního složení Jupitera.

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[10]) See S. Chandrasekhar, M. N. **93** (1933), 559.