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Sets which satisfy certain avoidability conditions.*)

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In a recent paper[1]) I have made use of certain avoidability conditions in order to define a type of generalized closed manifold. These conditions are repeated in Definitions I and II below, and together with other types of avoidability introduced in the subsequent definitions, are employed in the present paper to obtain further results concerning the relations of closed sets to their complements in euclidean n-space.

We precede the applications by a determination of the logical relationships between the various definitions in certain special types of closed sets. This is done not only with a view to settling these relations once and for all for the sake of subsequent developments and abbreviation of proofs, but because many results may be seen, later on, to hold for alternative choices of the types of avoidability used in the hypotheses of theorems. Where the latter is the case, we have sometimes explicitly pointed out the fact; where we have not done so, it is left to the reader to observe that such is the case.

In the following definitions, M denotes a metric space, and P a point of M.

Definition I. M is *completely i-avoidable at* P if for every $\varepsilon > 0$ there exist δ and η such that $\varepsilon > \delta > \eta > 0$ and every i-cycle of $F(P, \delta)$ bounds on $S(P, \varepsilon) - S(P, \eta)$.

Definition II. M is *locally i-avoidable at* P if for every $\varepsilon > 0$ there exist δ and η such that $\varepsilon > \delta > \eta > 0$ and every i-cycle of $F(P, \delta)$ bounds on $M - S(P, \eta)$.

Definition III. M is *i-avoidable at* P if for every $\varepsilon > 0$ there

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[1]) *Generalized closed manifolds in n-space*, Annals of Math. 35 (1934), pp. 876—903; to be referred to hereafter as G. C. M. (For other definitions of generalized manifolds the reader is referred to recent works of Čech, Lefschetz, Alexandroff and Pontrjagin.)

exists a $\delta > 0$ such that $\varepsilon > \delta$ and every i-cycle of $F(P, \varepsilon)$ bounds on $M - S(P, \delta)$.

Definition IV.[2]) P is a *non-i-cut-point* of M if every i-cycle of $M - P$ bounds on $M - P$.

Definition V. P is a *local non-i-cut-point* of M if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that every i-cycle of $S(P, \delta) - P$ bounds on $S(P, \varepsilon) - P$.

In establishing the relations between these definitions, we shall use the following conventions concerning symbols: The symbol $\supset$ will mean „implies";[3]) the symbol non $\supset$ means „does not imply". Thus, I $\supset$ II will mean that the property (of a space M under consideration) of being completely i-avoidable (for any i) at P implies that M is locally i-avoidable at P. For the sake of brevity, we shall also use the symbol $>$ as indicated in the following example: I $>$ II means that I $\supset$ II and that II non $\supset$ I. Finally, I $\equiv$ II means I $\supset$ II and II $\supset$ I. If no implication relates two or more definitions, we state simply that they are independent.

Lemma A. *In a compact (or any more general) metric space,*

(a) I $>$ II; III $>$ II; I *and* III *are independent.*

(b) IV *is independent of* I, II, III *and* V

(c) V *is independent of* I, II, III *and* IV.

Proof of (a). That I $\supset$ II and III $\supset$ II is obvious. That II non $\supset$ I is shown by the

Example α_1: The euclidean n-sphere with $i = n - 1$. That II non $\supset$ III is shown by the

Example α_2: The set of points (x, y, z) of cartesian 3-space such that $x^2 + y^2 + z^2 \leqq 1$, and the set such that $x^2 + y^2 = 4$, $z = 0$, with $P = (0, 0, 0)$, $\varepsilon = 2$ and $i = 1$.

That I non $\supset$ III is shown by Example α_2, and that III non $\supset$ I is shown by Example α_1.

Proof of (b). That IV non $\supset$ II is shown by the

Example α_3: The set of points (ϱ, Θ) of the polar coordinate plane such that (1) $\Theta = \pi/4^n$, $n = 1, 2, 3, \ldots, 0 \leqq \varrho \leqq 1$; (2) $\Theta = 0$, $0 \leqq \varrho \leqq 1$; (3) $\varrho = \tan \Theta$, $0 \leqq \Theta \leqq \pi/4$; (4) any arc joining the point $P = (0, 0)$ to $(1, \pi/4)$, but otherwise not containing any of the points defined in (1)—(3); with $i = 0$.

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[2]) In this definition we do not restrict the notion to simply i-connected spaces. Although for $i = 0$ this restriction has customarily been made in the theory of sets of points, we deem it inadvisable in the case $i > 0$. (See Example α_4 below for instance.)

[3]) That $\supset$ and $>$ are also used in another sense below (the former as a set-theoretic symbol and the latter as a symbol for numerical magnitude) should occasion no difficulty, the meanings of symbols between which these binary relations are used being sufficient to indicate the correct meaning in each case.

Consequently, by (a), IV non $\supset$ I, IV non $\supset$ III. That IV non $\supset$ $\supset$ V follows from Example α_1.

That I non $\supset$ IV is shown by Example α_2. That III non $\supset$ IV follows from the trivial fact that III may be satisfied for every $\varepsilon > 0$, and yet $M - P$ contain a cycle non-bounding on $M - P$ which lies on no $F(P, \varepsilon)$; thus, with the usual euclidean metric, in the

Example α_4: The set of points (x, y) of the cartesian plane such that $x^2 + y^2 = 1$, and the points $1 < x \leqq 2$, $y = 0$, with $P = (2, 0)$ and $i = 1$.

That II non $\supset$ IV now follows from (a). Finally, V non $\supset$ IV; for instance, consider a space which is the sum of two closed mutually exclusive subsets A and B, where A satisfies V at a point P, and B contains a cycle which is unbounding in the space.

Proof of (c). The independence of IV and V is already shown in (b). That V non $\supset$ II is shown by the set of points (ϱ, Θ) which satisfy conditions (1) and (2) in Example α_3, as well as the points for which $\varrho = 1/2^n$, $\pi/4^n < \Theta < 2\pi$; let $i = 0$, $P = (0, 0)$. Consequently, by (a), V non $\supset$ I and V non $\supset$ III.

To show that I non $\supset$ V and III non $\supset$ V, and hence by (a) that II non $\supset$ V, consider, with $i = 1$,

Example α_5: The set of points (x, y) of the cartesian plane lying on circles whose respective diameters are the portions of the x-axis from $(1/n, 0)$ to $1/n + 1, 0)$ for all natural numbers n, together with the origin.

Lemma B. *In a semi-i-connected,*[4]) *compact metric space,*

(a) I $>$ II; III $>$ II; I *and* III *are independent,*

(b) IV *is independent of* I, II, III *and* V.

(c) I $>$ V; V *is independent of* II, III *and* IV.

The proofs of (a) and (b) are as in Lemma A, (a) and (b) respectively.

Proof of (c). That IV and V are independent is proved as in Lemma A (c). That V non $\supset$ II, and hence, by (a), V non $\supset$ I and V non $\supset$ III, is shown as in Lemma A (c). To show that III non $\supset$ V, consider

Example α_6: The set of points (x, y, z) in cartesian 3-space whose x- and y-coordinates satisfy the equations of the circles defined in Example α_5, and such that $0 \leqq z \leqq 1$; the set of points for which $z = 1$ and whose projections on the xy-plane lie interior

[4]) A metric space M is called semi-i-connected if, given a point P of M, there exists an $\varepsilon > 0$ such that all i-cycles of $S(P, \varepsilon)$ bound on M; for a previous use of this notion, see my paper *On locally connected spaces,* Duke Math. Journ. 1 (1935), 543—555 (to be referred to hereafter as L. C.) If all i-cycles bound on M, we call M simply i-connected.

to the circles of Example α_5; and the set of points $(0, 0, z)$ such that $0 \leqq z \leqq 1$ with $P = (0, 0, 0)$ and $i = 1$.

Consequently, by (a), II non $\supset$ V. It remains to show that I $\supset$ V.

Let M be a semi-i-connected space which satisfies I at a certain point P, and consider an arbitrary $\varepsilon > 0$. Then there exist δ and η such that any i-cycle of $F(P, \delta)$ bounds on $S(P, \varepsilon) - S(P, \eta)$. Since M is semi-i-connected, we may assume η to be so small that i-cycles of $S(P, \eta)$ bound on M.

Consider any cycle γ^i of $S(P, \eta) - P$. Select ε' so that $|\gamma^i| \cdot \overline{S(P, \varepsilon')} = 0$.[5] Since I holds at P, there exist δ' and η' such that any cycle Γ^i of $F(P, \delta')$ bounds on $S(P, \varepsilon') - S(P, \eta')$. Consider any chain $K^{i+1} \to \gamma^i$ on M. By infinitesimal alterations of K^{i+1} and harmonizing[6]) of chains, we can say that the portion of K^{i+1} exterior to $S(P, \delta)$ is a chain F^{i+1} whose boundary is a cycle Γ^i of $F(P, \delta)$, and we let $H^{i+1} \to \Gamma^i$ be a chain of $S(P, \varepsilon) - S(P, \eta)$. Similarly, we may regard the portion of K^{i+1} interior to $S(P, \delta')$ as a chain F_1^{i+1} bounded by a Γ_1^i of $F(P, \delta')$, and we let $H_1^{i+1} \to \Gamma_1^i$ be a chain of $S(P, \varepsilon') - S(P, \eta')$. The chain $K^{i+1} - (F^{i+1} + F_1^{i+1}) + (H^{i+1} + H_1^{i+1}) \to \gamma^i$ lies in $S(P, \varepsilon) - P$. Thus, any i-cycle of $S(P, \eta) - P$ bounds on $S(P, \varepsilon) - P$, and M satisfies V at P.

Lemma C. *In a simply i-connected compact metric space,*

(a) I $>$ II $\equiv$ III $>$ IV,

(b) I $>$ V $>$ IV; II *and* V *are independent.*

Proof of (a). By Lemma A, I $\supset$ II and III $\supset$ II. That I $>$ II is shown as in Lemma A. We can show that II $\supset$ III as follows: Consider any P and $\varepsilon > 0$. By II, there exist δ and η such that any cycle γ^i of $F(P, \delta)$ bounds on $M - S(P, \eta)$. Consider a Γ^i of $F(P, \varepsilon)$, and let $K^{i+1} \to \Gamma^i$ be a chain of M. As in the proof that I $\supset$ V in Lemma B, we may consider the portion of K^{i+1} interior to $S(P, \delta)$ as a chain F^{i+1} bounded by a γ^i of $F(P, \delta)$. There exists an $H^{i+1} \to \gamma^i$ on $M - S(P, \eta)$. Then $K^{i+1} - F^{i+1} + H^{i+1} \to \gamma^i$ is a chain of $M - S(P, \eta)$.

We next show that III $\supset$ IV. Consider any P and let γ^i be a cycle of $M - P$. Select an $\varepsilon > 0$ such that $|\gamma^i| \cdot \overline{S(P, \varepsilon)} = 0$. By III, there is a $\delta > 0$ such that any i-cycle of $F(P, \varepsilon)$ bounds on $M - S(P, \delta)$. Let $K^{i+1} \to \gamma^i$ be a chain of M. As in the preceding paragraph, the chain K^{i+1} may be converted into a chain of $M - S(P, \delta)$.

That IV non $\supset$ III is shown by the following example, with $i = 0$. The set of points (x, y) of the cartesian plane (1) lying on the

[5]) By $|\gamma^i|$ we mean the closure of the set of all points on the cycle γ^i.

[6]) See L. C.

curve $y = \sin 1/x; 0 < x \leqq 1/\pi$; (2) all points $x = 0, -1 \leqq y \leqq 1$; (3) an arc joining $(1/\pi, 0)$ to $(0, 1)$, but otherwise not containing any points defined in (1) and (2). Let P be the point $(0, -1)$.

Proof of (b). That I $>$ V follows as in Lemma B. To show that V $\supset$ IV, consider any cycle γ^i of $M - P$, and let $\varepsilon > 0$ be such that $|\gamma^i| \,.\, \overline{S(P, \varepsilon)} = 0$. By V, there exists a $\delta > 0$ such that any i-cycle of $S(P, \delta) - P$ bounds on $S(P, \varepsilon) - P$. Consider any positive number $\eta < \delta$, and let K^{i+1} be a chain of M bounded by γ^i. If $|K^{i+1}| \supset P$, the portion of K^{i+1} in $S(P, \eta)$ is a chain F^{i+1} bounded by a cycle Γ^i of $F(P, \eta)$. Let $H^{i+1} \to \Gamma^i$ be a chain of $S(P, \varepsilon) - P$. Then $K^{i+1} - F^{i+1} + H^{i+1} \to \gamma^i$ is on $M - P$. That IV non $\supset$ V follows from Example α_1.

That V non $\supset$ III follows as in Lemma A (c), and that III non $\supset$ V follows from example α_6.

Lemma D. *In a compact* J^k,[7]) where $i \leqq k$,

(a) I $>$ II; III $>$ II; I *and* III *are independent.*

(b) IV $>$ III; IV *is independent of* I *and* V.

(c) I $\equiv$ V.

Proof of (a). As in Lemma A (a).

Proof of (b). That IV is independent of V, as well as that III non $\supset$ IV is shown as in Lemma A (b). We have to show that IV $\supset$ III. Consider any P and $\varepsilon > 0$. Let δ and η be arbitrary, except that $\varepsilon > \delta > \eta > 0$. As our space is a J^k, there exist[8]) cycles γ^i_m, $m = 1, 2, \ldots, s$, of $F(P, \delta)$ which form a basis for homologies in $S(P, \varepsilon) - \overline{S(P, \eta)}$. As P is a non-i-cut-point, there exist chains $K_m^{i+1} \to \gamma^i_m$ on $M - P$. Let $\delta' > 0$ be such that for each m, $|K_m^{i+1}| \,.\, \overline{S(P, \delta')} = 0$. Let γ^i be a cycle of $F(P, \varepsilon)$. As P is a non-i-cut-point, there is a $K^{i+1} \to \gamma^i$ on $M - P$. The portion of K^{i+1} in $S(P, \delta)$ can be considered as a chain F^{i+1} bounded by a Γ^i of $F(P, \delta)$. Since Γ^i is related to the cycles γ^i_m by a homology in $S(P, \varepsilon) - S(P, \eta)$, and the γ^i_m's in turn bound exterior to $S(P, \delta')$, the chain K^{i+1} can be replaced by one not meeting $S(P, \delta')$.

Proof of (c). By Lemma B (c), I $\supset$ V. We have to show that V $\supset$ I. Consider any P and $\varepsilon > 0$, and take $\delta < \varepsilon$ such that every cycle of $F(P, \delta)$ bounds on $S(P, \varepsilon) - P$. Let η be any positive number less than δ. Since our space is a J^k, there is a finite basis of cycles γ^i_m of $F(P, \delta)$ for homologies in $S(P, \varepsilon) - \overline{S(P, \eta)}$. For each m, there is a $K_m^{i+1} \to \gamma^i_m$ on $S(P, \varepsilon) - P$. Let η' be a positive number such that $|K^{i+1}| \,.\, S(P, \eta') = 0$ for each m. The remainder of the proof should be obvious from the methods used above.

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[7]) We use the symbol J^k to denote a metric space that is locally i-connected for $i = 0, 1, \ldots, k$ (See L. C.).

[8]) See Theorem 2 of L. C.

Lemma E. *In a simply i-connected compact J^k,*

$$\mathrm{I} \equiv \mathrm{V} > \mathrm{II} \equiv \mathrm{III} \equiv \mathrm{IV}.$$

Proof. By Lemma D (c), I $\equiv$ V. By Lemma C(a), II $\equiv$ III. By Lemma C (a), III $\supset$ IV, and by Lemma D (b), IV $\supset$ III; accordingly III $\equiv$ IV. That V $\supset$ II follows from Lemma D (a), (c). That II non $\supset$ V is shown by Example α_1.

We now turn to the study of some of the relations of closed point sets, that satisfy various avoidability conditions, to their complements in euclidean spaces. In the theorems (this does not include the lemmas) that follow we assume that the sets considered lie in the euclidean n-space, E_n, $n \geqq 2$. (In case $n = 2$, and a condition is stated in a hypothesis for $i = 0, 1, \ldots, n - 3$, it is to be understood that this condition is deleted.)

Theorem 1. *In E_n, let M be a closed point set and r a non-negative integer such that the complementary domains of M have (1) diameters that form a null sequence,*[9] *(2) boundaries that are locally r-connected, and (3) boundaries all but a finite number of which are simply r-connected. Then M is locally r-connected.*

Proof. Consider any point P of M and $\varepsilon > 0$. We may assume ε so small that any complementary domain of M that lies wholly in $S(P, \varepsilon)$ has a simply r-connected boundary. As the diameters of the complementary domains form a null sequence, there is an $\varepsilon' < \varepsilon$ such that if a complementary domain meets both $F(P, \varepsilon)$ and $F(P, \varepsilon')$, it has P on its boundary.

Denote the domains that meet both $F(P, \varepsilon)$ and $F(P, \varepsilon')$ by D_m, $m = 1, 2, \ldots, s$. There exists, by (2), a $\delta < \varepsilon'$ such that any r-cycle of $B_m \,.\, S(P, \delta)$, where B_m is the boundary of D_m, bounds a chain of $B_m \,.\, S(P, \varepsilon)$.

Consider any cycle γ^r of $M \,.\, S(P, \delta)$, and suppose it fails to bound on $M \,.\, \overline{S(P, \varepsilon)}$. Then in the complement of the latter set there exists a cycle Γ^{n-r-1} that is linked with γ^r. However, consider any chain $K^{r+1} \rightarrow \gamma^r$ in $S(P, \delta)$. The intersections of K^{r+1} and Γ^{n-r-1} must lie in a finite number of domains complementary to M. These intersections may be removed as follows: The portion of K^{r+1} in a domain D complementary to M is a chain H^{r+1} bounded by a cycle on the boundary B of D. If D is a D_m, H^{r+1} may be replaced by a chain of $B_m \,.\, S(P, \varepsilon)$. If D is not a D_m, then it must lie wholly in $S(P, \varepsilon)$, is therefore a domain with simply r-connected boundary, and H^{r+1} may be replaced by a chain of this boundary. The total effect of these replacements is the replacement of K^{r+1} by a new chain $L^{r+1} \rightarrow \gamma^r$ in $S(P, \varepsilon)$ and not meeting Γ^{n-r-1},

[9]) We call a sequence of numbers ε_k a *null sequence* if $\lim\limits_{k\to\infty} \varepsilon_k = 0$.

contradicting the fact that γ^r and Γ^{n-r-1} are linked. Thus γ^r must bound on $M \,.\, \overline{S(P, \varepsilon)}$ and M is locally r-connected.

Theorem 2. *In E_n, let M be a closed point set whose complementary domains have (1) diameters that form a null sequence, and (2) boundaries that are g. c. $(n-1)$-m.'s*[10]) *all but a finite number of which are simply i-connected for $i = 1, 2, \ldots, n-2$. Then M is a J^{n-2}.*

Since the g. c. $(n-1)$-m.'s are locally i-connected for $i = 0, 1, \ldots, n-2$, Theorem 2 follows from Theorem 1.

We digress at this point to prove some lemmas needed in the sequel.

Lemma F. *If M is both r-avoidable and completely r-avoidable at P, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any positive number $\eta < \delta$, there exists an $\eta' < \eta$ such that if γ^r is any cycle on $F(P, \Theta)$, where $\delta \geqq \Theta \geqq \eta$, then γ^r bounds on $M \,.\, [S(P, \varepsilon) - S(P, \eta')]$.*[11])

Proof. We first select δ and η satisfying the definition of complete r-avoidability, and let η' be a positive number $< \eta$ such that any r-cycle of $M \,.\, F(P, \delta)$ or $M \,.\, F(P, \eta)$ bounds on $M - M \,.\, S(P, \eta')$. Consider any number Θ such that $\delta \geqq \Theta \geqq \eta$, and let γ^r be a cycle of $F(P, \Theta)$. Since M is r-avoidable at P, γ^r bounds a chain K^{r+1} on $M - P$. If this chain meets $S(P, \eta')$, the portion of it in $S(P, \eta)$ is a chain H^{r+1} bounded by a cycle Z^r on $F(P, \eta)$, and we may replace H^{r+1} by a chain L^{r+1} on $M - M \,.\, S(P, \eta')$. Then the chain $F^{r+1} = K^{r+1} - H^{r+1} + L^{r+1} \rightarrow \gamma^r$ lies on $M - M \,.\, S(P, \eta')$. If F^{r+1} meets $F(P, \varepsilon)$, the portion of it on $M - M \,.\, S(P, \delta)$ is a chain h^{r+1} bounded by a cycle z^r of $F(P, \delta)$ which may be replaced by a chain $\bar{h}^{r+1}$ of $M \,.\, [S(P, \varepsilon) - S(P, \eta)]$. We then have $F^{r+1} - h^{r+1} + \bar{h}^{r+1} \rightarrow \gamma^r$ on $M \,.\, [S(P, \varepsilon) - S(P, \eta')]$. We observe, finally, that any number $\eta < \delta$, greater than the η obtained above from the definition of complete r-avoidability, may be used with the same η' as determined above. Also, that if any η less than that used above is assigned, a new η' may be obtained and the conclusion holds as before.

Lemma F'. *If certain sets M_m, $m = 1, 2, \ldots, s$, finite in number, have a point P in common, and if for each m, M_m is both r-avoidable and completely r-avoidable at P, then for any $\varepsilon > 0$ there exists a $\delta > 0$ such that for any $\eta < \delta$, there exists an $\eta' < \eta$ such that if γ^r is a cycle on $M_m \,.\, F(P, \Theta)$, $(m = 1, 2, \ldots, s)$, where $\delta \geqq \Theta \geqq \eta$, then γ^r bounds on $M_m \,.\, [S(P, \varepsilon) - S(P, \eta')]$.*

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[10]) G. c. n-m. = generalized closed n-manifold as defined in G. C. M.

[11]) Compare this lemma with Axiom H^k of Čech, Annals of Math. 34 (1933), p. 667.

Proof. We select δ_m as provided relative to ε for each M_m by Lemma F. Let δ be a positive number less than the minimum δ_m, and let $\eta < \delta$ be arbitrary. For each m, by Lemma F, there exists an $\eta'_m < \eta$ as provided by Lemma F. Let η' be the smallest η'_m. Now if γ^r is a cycle of any $M_m . F(P, \Theta)$ for $\delta \geqq \Theta \geqq \eta$, then since $\delta_m > \delta \geqq \Theta \geqq \eta \geqq \eta_m$, γ^r bounds a chain on $M_m . [S(P, \delta_m) — - S(P, \eta'_m)]$ and hence on $M_m . [S(P, \varepsilon) — S(P, \eta')]$.

Lemma G. *If M is completely r-avoidable at P, and for some neighborhood U of P all r-cycles of $U — P$ bound on M, then the conclusion of Lemma F holds.*

Proof. For $\varepsilon > 0$ arbitrary, subject to the condition that all r-cycles of $M . S(P, \varepsilon) — P$ bound on M, we determine δ and η as in the definition of complete r-avoidability. Obviously any smaller number than η may be selected. We then determine δ_1 and η_1 such that any γ^r on $M . F(P, \delta_1)$ bounds on $M . [S(P, \eta) — S(P, \eta_1)]$. Let Θ be such that $\delta \geqq \Theta \geqq \eta$. Then if γ^r is on $M . F(P, \Theta)$, it bounds a chain K^{r+1} on M. If H^{r+1} meets $S(P, \eta_1)$, the portion of it in $S(P, \delta_1)$ is a chain H^{r+1} bounded by a cycle Z^r of $F(P, \delta_1)$. This may be replaced by a chain L^{r+1} on $M . [S(P, \eta) — S(P, \eta_1)]$. Similarly a portion exterior to $S(P, \varepsilon)$ may be replaced by a chain on $M . [S(P, \varepsilon) — S(P, \eta)]$. We observe, finally, that η may be replaced by any number greater than η and less than δ, by retaining δ_1 and η_1 as already determined above.

The following lemma now follows from Lemma G just as Lemma F′ follows from Lemma F:

Lemma G′. *If certain sets M_m, finite in number, have a point P in common, and if for each m, M_m is completely r-avoidable at P and for some neighborhood U_m of P all r-cycles of $U_m — P$ bound on M_m, then the conclusion of Lemma F′ holds.*

Theorem 3. *Let M be a closed point set and r a non-negative integer $\leqq n — 2$ such that the complementary domains of M have (1) diameters that form a null sequence, (2) boundaries that satisfy at all points the hypothesis of either Lemma F or Lemma G, and (3) boundaries all but a finite number of which are simply r-connected. Then M is completely r-avoidable at all its points.*

Proof. We proceed as in the first paragraph of the proof of Theorem 1, and define the domains D_m (with boundaries B_m) as in the second paragraph of that proof. By Lemma F′ or Lemma G′, there exist δ and η such that $\varepsilon' > \delta > \eta > 0$ and such that any r-cycle of $B_m . F(P, \delta)$ bounds on $B_m . [S(P, \varepsilon) — S(P, \eta)]$. Of the domains complementary to M that do not meet $F(P, \varepsilon)$ but do meet $F(P, \delta)$, only a finite number have P on their boundaries, and there exists an η' such that $\eta > \eta' > 0$ and such that of these domains only the latter have points in $S(P, \eta')$. Denote those domains that meet $F(P, \delta)$, have P on their boundaries, and do not meet $S(P, \varepsilon)$, by G_k, $k = 1, 2, \ldots, t$. Then there exist δ_1 and η_1 such that $\eta' > \delta_1 > \eta_1 > 0$, and such that any r-cycle of F_k, boundary of G_k, on $F(P, \delta_1)$ bounds on $F_k . [S(P, \eta') - S(P, \eta_1)]$.

The numbers ε, δ and η_1 satisfy the complete avoidability definition. For consider a cycle γ^r on $M . F(P, \delta)$. As $r \leqq n - 2$, there exists on $F(P, \delta)$ a chain $K^{r+1} \rightarrow \gamma^r$. Suppose γ^r does not bound on $H = M . [\overline{S(P, \varepsilon)} - S(P, \eta_1)]$. Then there exists a cycle Γ^{n-r-1} of $E_n - H$ that is linked with γ^r. The intersections of Γ^{n-r-1} and K^{r+1} lie in a finite number of the complementary domains of M, and these intersections may be removed as follows: If D is a domain containing such an intersection, then the portion of K^{r+1} in D is a chain H^{r+1} bounded by a cycle Z^r on the boundary B of D. Now if D is a D_m, H^{r+1} may be replaced by a chain $\bar{H}^{r+1} \rightarrow Z^r$ on $B_m . [S(P, \varepsilon) - S(P, \eta)] \subset H$. If D is not a D_m and does not have P on its boundary, then its boundary B is simply r-connected and lies in H, and hence the chain $\bar{H}^{r+1}$ may be chosen on $B \subset H$. The only remaining possibility is for D to be a domain G_k. In this case we first let L^{r+1} be any chain of F_k bounded by Z^r. If L^{r+1} lies on H, we denote it by $\bar{H}^{r+1}$; otherwise, the portion of it in $S(P, \eta_1)$ is a chain N^{r+1} bounded by a cycle z^r on $F(P, \eta_1)$. But as we have chosen δ_1 and η_1, there is a chain $h^{r+1} \rightarrow z^r$ on $F_k . [(S(P, \eta') - - S(P, \eta_1)] \subset H$, and we let $\bar{H}^{r+1} = L^{r+1} - N^{r+1} + h^{r+1}$. The chain $K^{r+1} - \Sigma H^{r+1} + \Sigma \bar{H}^{r+1} \rightarrow \gamma^r$ does not meet Γ^{n-r-1}, contradicting the fact that γ^r and Γ^{n-r-1} are linked.

For the proof of the next theorem we need the following lemma:

Lemma H. *If certain sets M_m, finite in number, have a point P in common, and if for each m, M_m is locally r-avoidable at P and for some neighborhood U_m of P all r-cycles of $M_m . U_m - P$ bound on M_m, then for any $\varepsilon > 0$ there exist δ and η such that if γ^r is a cycle of $M_m . F(P, \delta)$, then γ^r bounds on $M_m - M_m . S(P, \eta)$.*

Proof. Let ε be small enough that all r-cycles of any $M_m . S(P, \varepsilon) - P$ bound on M_m. Since each M_m is r-avoidable at P, there exist, for each m, positive numbers δ_m and η_m such that any γ^r of $M_m . F(P, \delta_m)$ bounds on $M_m - M_m . S(P, \eta_m)$. Let δ be such that $\varepsilon > \delta > \delta_m$ for all m, and η such that $\eta_m > \eta > 0$ for all m. If γ^r is a cycle of $M_m . F(P, \delta)$, it bounds a chain K^{r+1} on M_m; if this chain meets $S(P, \eta)$, the portion of it in $S(P, \eta_m)$ (and hence the portion in $S(P, \eta)$) may be removed by methods similar to those used in proofs above.

Theorem 4. *Let M be a closed point set and r a non-negative integer $< n$ such that the complementary domains of M have (1) diameters that form a null sequence, (2) boundaries all but a finite number of which are simply r-connected, and (3) boundaries which at each* *point P satisfy the conditions placed on M_m in Lemma H. Then M is r-avoidable at all its points.*

The proof of Theorem 4 employs methods similar to those used in the proof of Theorem 3. We use Lemma H to provide δ and η such that any r-cycle of $B_m \,.\, F(P, \delta)$ bounds on $B_m \,.\, [E_n - S(P, \eta)]$, and δ_1 and η_1 such that any r-cycle of $F_m \,.\, F(P, \delta_1)$ bounds on $F_m \,.\, [E_n - S(P, \eta_1)]$.

Theorem 5. *Let M be a compact connected J^{n-2}. Then the diameters of the complementary domains of M form a null sequence.*[12]

Proof. Suppose M has infinitely many complementary domains of diameter greater than some $\varepsilon > 0$. Then there exists a point P of E_n and positive numbers δ and η, where $\delta > \eta$, such that infinitely many complementary domains of M, say $D_1, D_2, \ldots, D_m, \ldots$, contain points of both $F(P, \delta)$ and $F(P, \eta)$.

We may show that the set $M' = E_n - \sum_{m=1}^{\infty} D_m$ is a locally compact J^{n-2}, by methods used in the first part of the proof of Theorem 7 of L. C.

In each D_m there is an arc $x_m y_m$ such that x_m and y_m are points of $F(P, \delta)$ and $F(P, \eta)$, respectively, and $x_m y_m - x_m - y_m \subset S(P, \delta) - \overline{S(P, \eta)}$. Let Θ be such that $\delta > \Theta > \eta$, and let $S_1, S_2, \ldots, S_m, \ldots$ be a sequence of subdivisions of $F(P, \Theta)$ whose meshes form a null sequence. For a fixed integer h, only a finite number of the sets D_m can contain vertices of S_h, and consequently there exists for each h a domain $D_{m(h)}$ that contains no vertex of S_h.

Now by methods similar to those used in paragraphs seven to eleven of the proof of Theorem 7 of L. C., we can show the existence, for h great enough, of a cycle Γ_h^{n-1} which fails to meet the arc $x_{m(h)} y_{m(h)}$, and yet which approximates S_h as closely as we please (dependent on h). Since for h great enough such a cycle must meet the arc $x_{m(h)} y_{m(h)}$, a contradiction results.

Theorem 6. *Let M be a compact connected J^{n-2}. Then all but a finite number of the complementary domains of M are simply i-connected for $i = 1, 2, \ldots, n-2$.*

Proof. As M is compact, there exists an $\varepsilon > 0$ such that all i-cycles of diameter $< \varepsilon$ bound on M. If the complementary domains of M are infinite in number, then by Theorem 5 all but a finite number of them are of diameter less than ε, and we assert that those domains whose diameters are less than ε are simply i-connected. For if D is such a domain, and γ^i is a cycle of D which fails to bound in D, then γ^i is linked with a cycle Γ^{n-i-1} of the boundary

[12]) For the plane, this result was proved by Schoenflies. See Schoenflies, A., *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten*, Ergänzungsband, Jahresb. d. Deut. Math.-Ver., Leipzig, 1908, p. 237.

of D. But Γ^{n-i-1} is of diameter less than ε and must therefore bound on M, hence bound a chain which fails to meet γ^i.

We now state one of our principal theorems, the motive for which will be found in Principal Theorem D of G. C. M.

Principal Theorem A. *In order that a compact continuum M should have only complementary domains (1) whose boundaries are g. c. $(n-1)$-m's all but a finite number of which are simply i-connected for $i = 1, 2, \ldots, n-2$, and (2) whose diameters form a null sequence, it is necessary and sufficient that M be a J^{n-2} which is completely i-avoidable for $i = 0, 1, \ldots, n-3$, and locally $(n-2)$-avoidable.*

The necessity follows from the properties of g. c. $(n-1)$-m.'s and Theorems 2—6 above, and the sufficiency follows from Principal Theorem D of G. C. M.

As an important corollary of this theorem we have:

Corollary. *Among the compact connected J^{n-2}'s, those which have g. c. $(n-1)$-m.'s as boundaries of all their complementary domains are characterized by the fact that they are completely i-avoidable for $i = 0, 1, \ldots, n-3$ and locally $(n-2)$-avoidable.*

It should be noted here that by Lemma D, *Principal Theorem A and its Corollary remain true if the condition that the set be completely i-avoidable is replaced by the condition that all its points be local-non-i-cut-points* — a matter not without interest in view of the fact that (by Lemma A) these two conditions are in general independent.

Theorem 7. *Let M be a closed point set whose complementary domains have diameters that form a null sequence and whose boundaries are all simply i-connected $(i = 0, 1, \ldots, n-2)$ g. c. $(n-1)$-m.'s. Then all points of M are non-i-cut-points, i-avoidable and locally i-avoidable.*

Proof. By Theorem 1, M is a J^{n-2}. That M is simply i-connected follows from the duality theorem for closed sets. Hence by the Corollary above, M is completely i-avoidable for $i = 0, 1, \ldots, n-3$ and locally $(n-2)$-avoidable. By Lemma E, M has only non-i-cut-points, and its points are also i-avoidable and locally i-avoidable.

Theorem 8. *Let M be a compact continuum and D a domain complementary to M such that (1) D is u. l. i-c.*[13]) *for $i = 0, 1, \ldots, k$, where $k \leqq n-3$; (2) small i-cycles bound in D for $k < i \leqq n-2$, and (3) $\overline{M}$ is locally i-avoidable for $i = 0, 1, \ldots, n-k-3$. Then the boundary of D is a g. c. $(n-1)$-m.*

Proof. We show that D is u. l. i-c. for $k < i \leqq n-2$. Suppose D not u. l. i-c. Then there exist a point P of M and an $\varepsilon > 0$ such that for every $\eta > 0$ there exists in $D \, . \, S(P, \eta)$ a cycle ${}^{\eta}\gamma^i$ which fails to bound in $D \, . \, S(P, \varepsilon)$. By condition (3) there exist δ and η such that any j-cycle, γ^j, where $j = n-i-2$, of $M \, . \, F(P, \delta)$

[13]) U. l. i-c. = uniformly locally i-connected (defined in G. C. M.).

bounds on $M - M \,.\, S(P, \eta)$. We may take η so small that it not only satisfies this condition, but also the condition (2) that i-cycles of diameter $< \eta$ bound in D.

Consider a cycle ${}^{\eta}\gamma^i$. It bounds a chain K^{i+1} of D which a fortiori lies in $E_n - M \,.\, \overline{S(P, \delta)}$. Any chain L^{i+1} bounded by ${}^{\eta}\gamma^i$ in $S(P, \eta)$ also lies in $E_n - [F(P, \varepsilon) + M - M \,.\, S(P, \delta)]$. The cycle $K^{i+1} - L^{i+1}$ must fail to link $M \,.\, F(P, \delta)$, since j-cycles on the latter set bound on $M - M \,.\, S(P, \eta)$ and cannot meet the chain $K^{i+1} - L^{i+1}$. Thus by the Alexander Addition Theorem ${}^{\eta}\gamma^i$ bounds in $D \,.\, S(P, \varepsilon)$, D is u. l. i-c., and the boundary of D is a g. c. $(n - 1)$-m. by Principal Theorem C of G. C. M.

The proof of the following theorem is similar to the proof just given:

Theorem 8a. *Let M be a compact continuum and D a domain complementary to M such that (1) small i-cycles of D bound in D for $i = 1, 2, \ldots, n - 2$, and (2) M is locally i-avoidable for $i = 0, 1, \ldots, n - 2$. Then the boundary of D is a g. c. $(n - 1)$-m.*

The following corollaries are of interest.

Corollary. *In the plane, if M is a continuum all of whose points are locally 0-avoidable, then the boundaries of the complementary domains of M are simple closed curves.*

Corollary. *In 3-space, if M is a continuum all of whose points are locally 0- and 1-avoidable, and D is a complementary domain of M whose small 1-cycles bound in D, then the boundary of D is a closed 2-dimensional manifold.*

Theorem 9. *In order that the boundary, B, of a bounded, simply $(n - 1)$-connected domain D should be a g. c. $(n - 1)$-m., it is necessary and sufficient that (1) the small i-cycles of D bound in D for $i = 1, 2, \ldots, n - 2$ and that (2) B be locally i-avoidable for $i = 0, 1, \ldots, n - 2$.*

Proof. The necessity follows from the properties of a g. c. $(n - 1)$-m. and Lemma A (I $>$ II); the sufficiency from Theorem 8a.

As a Corollary of Theorems 6 and 8a we have:

Theorem 10. *If a compact continuum M is a J^{n-2} and locally i-avoidable for $i = 0, 1, \ldots, n - 2$, then all but a finite number of the complementary domains of M are bounded by simply i-connected g. c. $(n - 1)$-m.'s.*

Theorem 11. *In order that a simply i-connected ($i = 0, 1, \ldots, n - 2$) compact closed set should have only simply i-connected g. c. $(n - 1)$-m.'s as boundaries of its complementary domains, it is sufficient that M should be locally i-avoidable.*

Proof. Being simply 0-connected, M is a continuum. Condition (1) of Theorem 8a holds for any complementary domain of M, since M is simply i-connected for $i = 1, 2, \ldots, n - 2$, and condition (2) of Theorem 8a is part of our hypothesis. Consequently the boundaries of the domains complementary to M are g. c. $(n-1)$-m.'s.

Let D be a domain complementary to M, and B its boundary. By Principal Theorem A of G. C. M., $E_n - B$ is the sum of two domains D_1 and D_2, of which B is the common boundary. As $D \subset E_n - M \subset E_n - B$, we know that $D \subset D_1 + D_2$, and hence $D \subset D_1$, say. Then $D \equiv D_1$ and $D_2 \supset M - B$. Suppose $p^i(B) > 0$, where $1 \leqq i \leqq n-2$. Then $p^{n-i-1}(E_n - B) > 0$. It readily follows from the duality in Theorem 5 of G. C. M. that $p^{n-i-1}(D) > 0$, hence by the duality for closed sets that $p^i(M) > 0$, which is contrary to hypothesis.

Corollary. *In E_3, if M is a simply 1-connected continuum which is locally i-avoidable for $i = 0, 1$, then the complementary domains of M all have 2-spheres as boundaries.*

Principal Theorem B. *In order that a simply i-connected ($i = 0, 1, \ldots n-2$) compact J^{n-2} should have only simply i-connected g. c. $(n-1)$-m.'s as boundaries of its complementary domains, it is necessary and sufficient that it have only non-i-cut-points.*

Proof. The necessity follows from Theorems 7 and 5. As for the sufficiency: By Lemma D, M is locally i-avoidable at all points, and consequently by Theorem 11 the boundaries of the complementary domains of M are all g. c. $(n-1)$-m.'s.

We conclude with a theorem concerning the common boundary of two domains:

Theorem 12. *Let M be a compact, common boundary of two domains D_1 and D_2 such that (1) D_k is u. l. i-c. for $i = 0, 1, \ldots, n_k$ ($k = 1, 2$), where $n_1 + n_2 < n - 3$; (2) small i-cycles of D_1 bound in D_1 for $i = n_1 + 1, n_1 + 2, \ldots, n - n_2 - 2$; (3) M is locally i-avoidable for $i = n_2 + 1, n_2 + 2, \ldots, n - n_1 - 3$. Then M is a g. c. $(n-1)$-m.*

Proof. We first show that D_1 is u. l. i-c. for any i such that $n_1 + 1 \leqq i \leqq n - n_2 - 3$. If for some such i, D_1 is not u. l. i-c., there exist a point P of M and $\varepsilon > 0$ such that for any $\eta > 0$ there is a cycle ${}^{\eta}\gamma^i$ of $D_1 . S(P, \eta)$ that fails to bound in $D_1 . S(P, \varepsilon)$. However, let δ and η be selected so as to satisfy the local $(n-i-2)$-avoidability definition at P, as well as so that i-cycles of $D_1 . S(P, \eta)$ bound in D_1. By using the argument of the second paragraph of the proof of Theorem 8, we may now show that any ${}^{\eta}\gamma^i$ bounds in $D_1 . S(P, \varepsilon)$, thus obtaining a contradiction.

We conclude, then, that D_1 is u. l. i-c. for $i = 0, 1, \ldots, n - n_2 - 3$, and since D_2 is u. l. i-c. for $i = 0, 1, \ldots, n_2$ it follows[14]) that M is a g. c. $(n-1)$-m.

[14]) By Theorem 2 of my paper *A characterization of manifold boundaries* . . ., Bull. Amer. Math. Soc. 42 (1936), pp. 436—441.

Množiny, na nichž se lze vyhnouti danému bodu.

(Obsah předešlého článku.)

Je-li P bod topologického prostoru M, je-li Γ cyklus, jehož nosič neobsahuje bod P a je-li $\Gamma \sim 0$ v prostoru M, pak jedna z vyšetřovaných vlastností je, že Γ musí býti ~ 0 v prostoru $M - P$. Další vlastnosti (celkem je jich pět) vzniknou rozmanitými lokalisacemi. Některé z těchto vlastností se již dříve vyskytly (u autora i u jiných matematiků) při axiomatické definici variety pomocí homologie. V prvé části článku se studují vzájemné vztahy pěti vyšetřovaných vlastností. Ve druhé části jsou mimo jiné odvozeny podmínky, které stačí předpokládati o uzavřené množině M vnořené do euklidovského E_n, aby hranice každé komplementární oblasti byla $(n-1)$-rozměrnou varietou.
