Časopis pro pěstování matematiky a fysiky

Harry Levy

Conformal invariants in two dimensions. [II.]

Časopis pro pěstování matematiky a fysiky, Vol. 69 (1940), No. 3-4, 118--127

Persistent URL: http://dml.cz/dmlcz/123328

Conformal Invariants in Two Dimensions II.

Harry Levy, Urbana, Illinois.

(Received April 19, 1938.)

In a previous paper[1]) the author studied conformal properties of two or three curves on a surface. Here we continue that investigation obtaining a series of functions determined by a one parameter family of curves on an arbitrary surface. These functions are relative invariants under transformations of coordinates and absolute invariants under conformal transformations. In § 2 we obtain some relations between these functions; in § 3 we obtain the main theorems of this paper, necessary and sufficient conditions, expressed in terms of the invariants, that a given transformation be conformal, and that given one parameter families of curves (or given orthogonal nets) be conformally equivalent.

1. Let $\{C_1\}$ be a one parameter family of curves on a surface V and let $\{C_2\}$ be its orthogonal trajectories. We can orient the normal to any curve intrinsically by requiring that the positive normal lie on the same side of the tangent geodesic as does the curve itself.[2]) The tangent to a curve admits of no intrinsic orientation. But for an orthogonal net we are able to orient the tangents intrinsically by parametrizing the curves of each congruence so that the positive tangents of each congruence coincides with the positive normals to the other. With this convention, the Frenet equations become

$$\begin{array}{ll} D_1\lambda_1{}^i = k_1\lambda_2{}^i & D_1\lambda_2{}^i = -k_1\lambda_1{}^i \\ D_2\lambda_1{}^i = -k_2\lambda_2{}^i & D_2\lambda_2{}^i = k_2\lambda_1{}^i, \quad i = 1, 2. \end{array} \tag{1}$$

where D_α represents the covariant differential operator along $\{C_\alpha\}$, and k_α is the (geodesic) curvature of $\{C_\alpha\}$.

If V' is a second surface in conformal correspondence with V, and if the correspondence is established by pairing those points

[1]) Conformal Invariants in Two Dimensions I, Časopis. We shall refer to this paper as I.

[2]) Cf. I, § 4, and Hlavatý, Differenciální geometrie křivek a ploch a tensorový počet.

on the two surfaces which have equal coordinates, the fundamental tensors are related by the equations

$$g'_{ij} = \sigma g_{ij}. \tag{2}$$

Let λ'^{i}_{α} be the oriented unit components of the family on V' corresponding to $\{C_\alpha\}$, so that

$$\lambda'^{i}_{\alpha} = e_\alpha \sigma^{-\frac{1}{2}} \lambda^{i}_{\alpha} \tag{3}$$

where $\sigma^{\frac{1}{2}}$ is the positive square root and where e_α are each numerically equal to unity[3].)

$$e_\alpha = \pm 1. \tag{4}$$

From (3) (for $\alpha = 1$) we obtain by differentiation that

$$k'_1 \lambda'^{i}_{2} = \sigma^{-1}(k_1 \lambda_2^i + \sigma_j \lambda_1^j \lambda_1^i - \sigma^i), \tag{5}$$

where $\sigma_i = \dfrac{1}{2} \dfrac{\partial}{\partial x^i} \log \sigma$ and from (3) itself it follows that

$$k'_1 = e_2 \sigma^{-\frac{1}{2}}(k_1 - \lambda_2^j \sigma_j) \tag{6}$$

and the analogous relation obtained from $\{C_2\}$

$$k'_2 = e_1 \sigma^{-\frac{1}{2}}(k_2 - \lambda_1^j \sigma_j). \tag{7}$$

Let us designate directional differentiation along $\{C_1\}$ and $\{C_2\}$ by the subscripts S and N respectively, so that for example

$$f_S = \lambda_1^i \frac{\partial f}{\partial x^i}, \qquad f_N = \lambda_2^i \frac{\partial f}{\partial x^i}.$$

Then equations (6) and (7) may be written

$$\begin{aligned} \sigma_S &= k_2 - e_1 \sigma^{\frac{1}{2}} k'_2, \\ \sigma_N &= k_1 - e_2 \sigma^{\frac{1}{2}} k'_1. \end{aligned} \tag{8}$$

We shall have occasion to refer to the well known integrability conditions.[4])

$$f_{SN} - f_{NS} = k_1 f_S - k_2 f_N. \tag{9}$$

Finally we observe, if we indicate with S' and N' the corresponding differentiation in V', that

$$f_{S'} = e_1 \sigma^{-\frac{1}{2}} f_S, \qquad f_{N'} = e_2 \sigma^{-\frac{1}{2}} f_N. \tag{10}$$

If we differentiate equations (8) with respect to N and S respectively, eliminate σ_S and σ_N by means of (8) themselves

---

[3]) If we assign a (non-intrinsic) positive direction of rotation on each surface by defining the directed angle from C_1 to C_2 by $\sin \Theta = \sqrt{g} \begin{vmatrix} \lambda_1^1 & \lambda_1^2 \\ \lambda_2^1 & \lambda_2^2 \end{vmatrix}$, it follows that the given correspondence is directly or inversely conformal according as $e_1 e_2 = 1$ or -1.

[4]) Graustein, Invariant Methods in Classical Differential Geometry, Bulletin, Am. Math. Soc. 36 (1930), p. 497.

and apply the integrability conditions (9), we obtain by an immediate calculation (in which we must also use (10)) that

$$e_1e_2\{(k'_1)_{S'} - (k'_2)_{N'}\} = \sigma^{-1}\{(k_1)_S - (k_2)_N\}. \tag{11}$$

Since

$$g' = \sigma^2 g \tag{12}$$

there follows our previous result,[5]) that

$$\sqrt{g}\{(k_1)_S - (k_2)_N\}$$

is an absolute conformal invariant for directly conformal transformations while for inversely conformal transformations

$$\sqrt{g'}\{(k'_1)_{S'} - (k'_2)_{N'}\} = -\sqrt{g}\{(k_1)_S - (k_2)_N\}.$$

If we designate by $\Delta_2\lambda$ the invariant analogous to Beltrami's second differential parameter

$$\Delta_2\lambda = \lambda_{i,j}\, g^{ij}$$

it follows that

$$k_2 = -\Delta_2\lambda_1$$

and consequently $\sqrt{g}\{(k_1)_S - (k_2)_N\}$ is expressed explicitly in terms of the family $\{C_1\}$ alone. The above equations thus interpreted give us a conformal invariant of a single congruence (or semi-invariant, if $e_1e_2 = -1$).

Although we shall speak throughout the remainder of this paper of the conformal invariants of an orthogonal net we must bear in mind that the invariant is determined completely by a single one parameter family of curves.

2. From the invariant of the preceeding section we can develop a sequence of invariants in the following way. Suppose F and F', functions referred to V and V' respectively, satisfy the equations

$$F' = \sigma^{-n}F. \tag{13}$$

If we differentiate with respect to S or N, make use of (10), and eliminate the derivatives of σ by means of (8) we obtain that

$$e_1F'_1 = \sigma^{-n-1}F_1, \tag{14}$$

$$e_2F'_2 = \sigma^{-n-1}F_2, \tag{15}$$

where

$$F_1 = F_S - 2n\, Fk_2, \quad F_2 = F_N - 2n\, Fk_1$$

and F'_1 and F'_2 are the same functions of the primes.

Let us define a sequence of functions

$$f = (k_1)_S - (k_2)_N, \tag{16}$$

$$f_{\alpha_1\ldots\alpha_r 1} = (f_{\alpha_1\ldots\alpha_r})_S - (r+2)\, k_2 f_{\alpha_1\ldots\alpha_r}, \tag{17}$$

[5]) Cf. I, § 4 and the references there given to Kasner.

$$f_{\alpha_1\ldots\alpha_r 2} = (f_{\alpha_1\ldots\alpha_r})_N - (r+2)\, k_1 f_{\alpha_1\ldots\alpha_r} \tag{18}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_r = 1, 2;\ r = 0, 1, 2 \ldots$ and $f'_{\alpha\ldots\beta}$ is defined by (17) and (18) written with primes. Then

$$e_1^{p+1} e_2^{q+1} g'^{\frac{r+2}{4}} f'_{\alpha_1\ldots\alpha_r} = g^{\frac{r+2}{4}} f_{\alpha_1\ldots\alpha_r}, \tag{19}$$

where p is the number of subscripts 1, q of 2 in the set $\alpha_1 \ldots \alpha_r$, $p + q = r$.

From (19) we see that the functions $f_{\alpha\ldots\beta}$ as thus defined are conformal invariants of the net (except possibly for sign), and therefore of a single congruence. They are not all algebraically independent. For if we express $f_{\alpha\ldots\beta 12}$ and $f_{\alpha\ldots\beta 21}$ in terms of $f_{\alpha\ldots\beta}$ and its derivatives and make use of the integrability conditions (9) we obtain at once that

$$f_{\alpha\ldots\beta 12} - f_{\alpha\ldots\beta 21} = (r+2)\, f f_{\alpha\ldots\beta}, \tag{20}$$

where r is the number of indices $\alpha \ldots \beta$. Moreover from (20) itself we obtain by differentiation that

$$f_{\alpha\ldots\beta 12\gamma} - f_{\alpha\ldots\beta 21\gamma} = (r+2)\,(f f_{\alpha\ldots\beta\gamma} + f_\gamma f_{\alpha\ldots\beta}). \tag{21}$$

By induction it consequently follows

$$f_{\alpha\ldots\beta} = f_{1\ldots 1 2\ldots 2} + {*},$$

where * represents f's with fewer indices than appear in $f_{\alpha\ldots\beta}$ and $1\ldots1\ 2\ldots2$ is a permutation of $\alpha\ldots\beta$.

From (20) we observe that if all the functions $f_{\alpha\ldots\beta}$ with a given number of subscripts are equal, their common value must be zero and that finally f itself must be zero. Likewise if $f_{\alpha\ldots\beta 1} = 0$ and $f_{\alpha\ldots\beta 2} = 0$ we obtain by applying the integrability conditions (9) that $f_{\alpha\ldots\beta}$ must vanish.

3. In this section we shall seek sufficient conditions that a given point correspondence between two surfaces be conformal and that given orthogonal nets be conformally equivalent. Let us recall that in any point correspondence between two surfaces there necessarily exists on each surface an orthogonal net whose transform is also orthogonal. We shall call any such net a Tissot net of the correspondence. We can associate with a pair of corresponding Tissot nets two numbers e_1 and e_2 ($e_1^2 = e_2^2 = 1$) in the following way: let the Tissot nets be parametric on each surface and let the directions of increasing parameter on one surface be the intrinsic orientations of the tangent vectors of the net. Then on the second surface the intrinsic orientations determined by its net may differ in sign from the directions of increasing parameter. Let e_1 and e_2 indicate these differences in sign. With

this agreement the functions f and f' are completely determined; we shall proceed to prove the following theorem:

If, in a point correspondence between two surfaces, for a pair of corresponding Tissot nets

$\sqrt{g'}\, e_1 e_2 \{(k'_1)_{S'} - (k'_2)_{N'}\}$ is equal to $\sqrt{g}\{(k_1)_S - (k_2)_N\}$

the two nets are conformally equivalent.

The linear elements of the two surfaces may be written

$$\mathrm{d}_S{}^2 = E\,\mathrm{d}u^2 + G\,\mathrm{d}v^2 \tag{22}$$

and

$$\mathrm{d}_{S'}{}^2 = ET_1{}^2\,\mathrm{d}u^2 + GT_2{}^2\,\mathrm{d}v^2,\ T_1 > 0,\ T_2 > 0. \tag{23}$$

The curvatures of the parametric curves of (22) are given by[6])

$$k_1 = -\frac{1}{\sqrt{G}}\frac{\partial}{\partial v}\log\sqrt{E}, \quad k_2 = -\frac{1}{\sqrt{E}}\frac{\partial}{\partial u}\log\sqrt{G} \tag{24}$$

so that

$$\sqrt{g}\, f = \frac{\partial^2}{\partial u\,\partial v}\log\sqrt{\frac{G}{E}}. \tag{25}$$

On the surface with fundamental form (23) we will have

$$\lambda_1{}^1 = \frac{e_1}{T_1\sqrt{E}}, \qquad \lambda_2{}^2 = \frac{e_2}{T_2\sqrt{G}}$$

so that we obtain in place of (24) and (25)

$$\begin{aligned} k'_1 &= \frac{-e_2}{T_2\sqrt{G}}\frac{\partial}{\partial v}\log(T_1\sqrt{E}), \\ k'_2 &= \frac{-e_1}{T_1\sqrt{E}}\frac{\partial}{\partial u}\log(T_2\sqrt{G}) \end{aligned} \tag{26}$$

and consequently

$$e_1 e_2\sqrt{g'}\, f' = \frac{\partial^2}{\partial u\,\partial v}\log\left(\frac{T_2}{T_1}\sqrt{\frac{G}{E}}\right). \tag{27}$$

By virtue of our hypothesis the left hand sides of (25) and (27) are equal; comparing the right hand sides it follows that

$$\log T_1 - \log T_2 = \varphi_1(u) - \varphi_2(v).$$

Consequently

$$T_1 = T_2\frac{U}{V}$$

[6]) Cf. Bianchi, Lezioni di Geometria Differenziale, Pisa (1922), p. 267. When we take geodesic curvatures as necessarily positive the formulas of Bianchi are valid only if the directions of increasing direction coincide with the intrinsic directions of the net. Cf. Hlavatý, l. c.

where $U(V)$ is a function of $u(v)$ alone. Substituting this value of T_1 in (23) we obtain that

$$ds^2 = \frac{T_2^2}{V^2}(EU^2\,du^2 + GV^2\,dv^2). \tag{28}$$

Now the transformation

$$\bar{u} = \int U\,du,\ \bar{v} = \int V\,dv$$

which represents only a change in the parametrization of the curves of the net shows that the parametric net on (28) is conformally equivalent to the given net on (22).

The linear element (28) shows that a transformation which leaves $\sqrt{g}\,f$ invariant (except possibly for sign) is not necessarily conformal. We are able however to obtain sufficient conditions that a transformation be conformal. The theorem follows:

If in a correspondence between two surfaces, $g^{\frac{1}{2}} f$, $g^{\frac{1}{2}} f_1$, and $g^{\frac{1}{2}} f_2$ formed for a Tissot net on one surface are equal respectively to $e_1 e_2 g'^{\frac{1}{2}} f'$, $e_2 g'^{\frac{1}{2}} f'_1$, and $e_1 g'^{\frac{1}{2}} f'_2$ formed for the corresponding net and if $f \neq 0$ the correspondence is conformal.

To prove this theorem it is sufficient to prove it for the linear elements given by (22) and

$$ds^2 = EU^2\,du^2 + GV^2\,dv^2 \tag{29}$$

(where U and V are functions of u and v alone) since the correspondence from (28) to (29) is already conformal and corresponding invariants for (28) and (29) are equal.

From (24) and (25) it follows that for the parametric net of (22) we have

$$g^{\frac{1}{2}} f_1 = \left(\frac{G}{E}\right)^{\frac{1}{4}} \left\{\frac{\partial^3 M}{\partial u^2\,\partial v} + \frac{\partial^2 M}{\partial u\,\partial v}\frac{\partial M}{\partial u}\right\} \tag{30}$$

and

$$g^{\frac{1}{2}} f_2 = \left(\frac{E}{G}\right)^{\frac{1}{4}} \left\{\frac{\partial^3 M}{\partial u\,\partial v^2} - \frac{\partial^2 M}{\partial u\,\partial v}\frac{\partial M}{\partial v}\right\}, \tag{31}$$

where

$$M = \log\sqrt{\frac{G}{E}}. \tag{32}$$

If we compute f'_1 and f'_2 for the parametric net of (29) the equations corresponding to the hypotheses of the theorem become the following

$$\frac{dU}{du} = -\sqrt{E}\,\frac{f_1}{f}\,\frac{U}{\sqrt{V}}\,(\sqrt{U} - \sqrt{V}) \tag{33}$$

and

$$\frac{dV}{dv} = \sqrt{G}\,\frac{f_2 V}{f\sqrt{U}}(\sqrt{U} - \sqrt{V}). \tag{34}$$

We shall show that functions U and V satisfying these equations are necessarily equal (and therefore constant), so that (29) is conformal to (22). If we differentiate (33) and (34) with respect to v and u respectively and eliminate the derivatives of U and V by means of (33) and (34) themselves, we obtain

$$\begin{aligned} (\sqrt{U} - \sqrt{V})\left(\frac{\partial}{\partial v}\frac{\sqrt{E}\,f_1}{f} - \frac{1}{2}\frac{\sqrt{EG}\,f_1 f_2}{f^2}\right) &= 0, \\ (\sqrt{U} - \sqrt{V})\left(\frac{\partial}{\partial u}\frac{\sqrt{G}\,f_2}{f} - \frac{1}{2}\frac{\sqrt{EG}\,f_1 f_2}{f^2}\right) &= 0. \end{aligned} \tag{35}$$

By expanding the second factor in each of these equations and making use of the defining equations (17) and (18) we find they are reducible to

$$2ff_{12} - 3f_1 f_2,\quad 2ff_{21} - 3f_1 f_2$$

respectively. If these quantities were zero it would follow from (20) that f would necessarily vanish. But this is contrary to our hypotheses and therefore the first factor in (35), namely $\sqrt{U} - \sqrt{V}$ must be zero.

4. It is well known that the vanishing of f is a necessary and sufficient condition that a net be isothermal.[7] In this section we propose to give some examples of nets for which $f_1 = 0$, $f \neq 0$. Let us take a linear element in the form

$$ds^2 = du^2 + G\,dv^2 \tag{36}$$

and then it follows from (25) and (30) that

$$\sqrt{G}\,f_1 = R_{u\,uv} + R_{uv} R_u, \tag{37}$$

where

$$R = \log \sqrt{G} \tag{38}$$

and where the subscripts indicate partial differentiation. If (36) were Euclidean, $\frac{\partial^2}{\partial u^2}\sqrt{G}$ would be zero and it would easily follow that the vanishing of f_1 inplies the vanishing of f, so that there exists no family of curves in the plane with rectilinear orthogonal trajectories and such that $f_1 = 0$, $f \neq 0$. We can however find other examples of curves for which $f_1 = 0$. We obtain from (37) that a necessary and sufficient condition is that

$$R_{u\,uv} + R_{uv} R_u = 0. \tag{39}$$

[7] Cf. Hlavatý, l. c.

Multiply by e^R and integrate with respect to u, obtaining

$$R_{uv} = Ve^{-R}, \quad V = V(v). \tag{40}$$

If we multiply (39) in turn by R_u and $\frac{1}{V} Rv$ (which is possible since the vanishing of V implies that f also is zero) we obtain

$$\frac{\partial}{\partial v}(R_u)^2 = -2\frac{\partial}{\partial u}(e^{-R}V), \tag{41}$$

$$\frac{\partial}{\partial u}\frac{1}{V}(R_v)^2 = -2\frac{\partial}{\partial v}e^{-R}. \tag{42}$$

By making use of (40) we can integrate (41) with respect to v and (42) with respect to u obtaining

$$\begin{aligned} R_{uu} &= \frac{1}{2}U - R_u^2, && U = U(u), \\ R_{vv} &= -\frac{1}{2}R_v^2 + R_v\frac{V'}{V} + \frac{1}{2}VV_1, && (V_1 = V_1(v). \end{aligned} \tag{43}$$

We can show that for any choice of the arbitrary functions U, V, V_1, (40) and (43) are completely integrable and consequently the system (40) and (42) is equivalent to (39).

Let us now define a function $a(u)$ as a solution of

$$a'' = \tfrac{1}{2}(U - (a')^2) \tag{44}$$

and let us denote by $\bar{R}$

$$\bar{R} = R - a. \tag{45}$$

The first of (43) then becomes

$$\bar{R}_{uu} = -\tfrac{1}{2}\bar{R}_u^2 - \bar{R}_u a'. \tag{46}$$

Since $\bar{R}_u$ cannot be zero we obtain by one integration that

$$e^{\frac{1}{2}\bar{R}}R_u = 2e^{\frac{1}{2}\alpha - a}$$

and integrating a second time it follows that

$$\bar{R} = \alpha + 2\log\{\textstyle\int e^{-a}\,du + \beta\}, \tag{47}$$

where α and β are arbitrary functions of v. Returning to (40) and the second of (43) we find that for R to be a solution the arbitrary functions already introduced must satisfy the following conditions

$$V = -2e^{\alpha}\beta', \tag{48}$$

$$\alpha'' + \frac{1}{2}(\alpha')^2 - \frac{V'}{V}\alpha' - \frac{1}{2}VV_1 = 0. \tag{49}$$

Finally since $G = e^{2R}$, it follows that

$$G = e^{2(\alpha+a)} (\int e^{-a} \, du + \beta)^4.$$

Conversely if we select arbitrary functions $\alpha(v)$, $\beta(v)$, and $a(u)$ (subject to the restriction that $\beta' \neq 0$) and if we define V by (48) it follows that $f_1 = 0$, $f \neq 0$. If we introduce new parameters along the net by the transformation $\bar{u} = \int e^{-a} \, du$, $\bar{v} = \int e^{\alpha} \, dv$ we obtain the canonical form

$$ds^2 = du^2 + (u + \beta)^4 \, dv^2.$$

The invariant f_1 consequently vanishes for the parametric curves of the linear element

$$ds^2 = \frac{1}{1 + (u + \beta)^4} (du^2 + (u + \beta)^4 \, dv^2). \tag{50}$$

But here the parametric curves are the bisectors of a net of Tchebychef in which the angle ω of the net is given by[8])

$$\tan \tfrac{1}{2}\omega = (u + \beta)^2.$$

We can find an example of such a net in the plane by requiring (50) to have zero Gaussian curvature. One solution is $\beta = v$; then one family of the net of Tchebychef consists of parallel straight lines and the other family is generated by the curves whose parametric equations referred to Cartesian coordinates is[9])

$$x = \int \frac{1 + 16\alpha^4}{1 - 16a^4} \, d\alpha \qquad y = \int \frac{8a^2}{1 + 16\alpha^4} \, d\alpha.$$

More generally we can show by direct computation, the following: let a net of Tchebychef in the plane be generated by a straight line l and a curve C; its bisectors form an orthogonal net for which $f_1 = 0$ if and only if the angle ω between C and the lines parallel to l satisfies the equation

$$\left(\frac{d\omega}{ds}\right)^2 = \sin \omega \, (a + a \cos \omega + b \sin \omega),$$

where a, b are arbitrary constants and s is the arc of C.

As a last example we consider a net for which $f_{1 \ldots 1} = 0$, where the number of indices is m. If the linear element referred to the net is $E \, du^2 + G \, dv^2$ then $f_{1 \ldots 1}$ will still be zero for the parametric net on a surface with linear element $\frac{E}{G} du^2 + dv^2$.

[8]) Cf. Bianchi, l. c., p. 153.
[9]) Cf. Bianchi, l. c., p. 161.

But here the curves $du = 0$ are geodesics, and therefore $k_2 = 0$; then $f_{1\ldots1} = 0$ becomes $(k_1)_{ss\ldots s} = 0$ so that a net for which $f_{1\ldots1}$ (m indices) vanishes is equivalent to a net of geodesics and their geodesic parallels in which the latter have curvature whose $(m+1)^{st}$ arc derivate is zero.

Institute for Advanced Study, Princeton, N. J., and University of Illinois, Urbana, Illinois.

*

Konformní invarianty ve dvou dimensích II.

(Obsah předešlého článku.)

Autor rozšiřuje výsledky předešlé práce (stejně pojmenované) na studium konformních invariantů kongruencí křivek na plochách. Nalézá systém hustot, jež jsou absolutními konformními invarianty, udává jejich vzájemné vztahy a používá jich k řešení problému konformní ekvivalence kongruencí.

*

Konforminvarianten in zwei Dimensionen II.

(Auszug aus dem vorstehenden Artikel.)

Der Verfasser verallgemeinert die Resultate seiner vorigen gleich benannten Arbeit auf die Konforminvarianten der Kongruenzkurven auf einer Fläche. Er findet ein System von Dichten auf, welche absolute Konforminvarianten sind, untersucht ihre gegenseitige Beziehungen und benützt die erhaltenen Resultate, um das Problem der Equivalenz von zwei Kongruenzen zu lösen.