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Oscillation criterions

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OSCILLATION CRITERIONS.

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1. In what follows all functions are real-valued and continuous for large positive t, say for $t \geqq t_0$. Let $f(t)$ be such a function. Then from STURM'S separation theorem it follows that either all non-trivial solutions of the linear differential equation

$$x'' + f(t)\, x = 0 \tag{1}$$

have an infinity of zeros or all have only a final number of them. In the first case (1) is called oscillatory, in the second non-oscillatory. When namely a non-trivial solution $x(t)$ of (1) vanishes it changes its sign, since $x(t)$ and $x'(t)$ cannot vanish simultaneously.

It is important to know the conditions under which (1) is oscillatory. KNESER [1] already dealt with this problem in the case where $f(t)$ has a constant sign. Recently WINTNER [2] examined the same question but without the supposition that $f(t)$ has a constant sign for large positive t. If we denote

$$F(t) = \int^t f(s)\, \mathrm{d}s,$$

the sufficient conditions for (1) being oscillatory are: either

$$\lim_{t\to\infty} F(t) = \infty \tag{2}$$

or only

$$\lim_{t\to\infty} \frac{\int^t F(s)\, \mathrm{d}s}{t} = \infty. \tag{3}$$

In this note we shall deal with STURM'S differential equation

$$(Q(t)\,.\,x')' + f(t)\, x = 0 \tag{4}$$

where $Q(t)$ is positive, and we shall prove two oscillation criterions which are a generalization of (2). As a corollary of the first criterion we shall obtain a generalisation of a certain oscillation criterion of KNESER.[1])

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[1]) Added in proof. Two new papers containing similar results have got into my hands: J. G. MIKUSIŃSKI, On Fite's oscillation theorems, Colloquium math. **2** (1949), 34—39; W. LEIGHTON, The detection of the oscillation of solutions of a second order linear differential equation, Duke Math. J. **17** (1950) 57—61.

2. Theorem I. *Let the integral* $\int^{t} \frac{ds}{Q(s)}$ *be divergent and let exist a positive function* $\omega(t)$ *having a continuous first derivative such that*

$$\int^{\infty} \frac{Q(s)}{\omega(s)} \omega'^2(s)\, ds < \infty, \tag{5}$$

$$\int^{t} \omega(s) \, . \, f(s)\, ds \to \infty \quad \textit{for } t \to \infty. \tag{6}$$

Then (4) *is oscillatory.*

Proof. Assume (4) to be non-oscillatory. First of all we are going to prove that then there exists one unbounded solution $x_0(t)$.

If $x_1(t)$, $x_2(t)$ are two linearly independent solutions of (4), it is easy to see that their Wronskian is equal to $\frac{W_0}{Q(t)}$, where W_0 is a non-vanishing constant, i. e.

$$x_1(t) \, . \, x'_2(t) - x'_1(t) \, . \, x_2(t) = \frac{W_0}{Q(t)}.$$

$x_1(t)$ and $x_2(t)$ can be assumed such that $W_0 = 1$. As we suppose (4) to be non-oscillatory, the fractions $\frac{x_1(t)}{x_2(t)}$ and $\frac{x_2(t)}{x_1(t)}$ are continuous for large positive t and either $\frac{x_1(t)}{x_2(t)} > 0$ or $\frac{x_2(t)}{x_1(t)} < 0$. In the first case it is

$$\left(\frac{x_1(t)}{x_2(t)}\right)' = -\frac{1}{Q(t)\, x_2^2(t)}$$

so that $|x_2(t)|$ must be unbounded. For if $|x_2(t)| < K$, then

$$\frac{x_1(t)}{x_2(t)} = \frac{x_1(t_1)}{x_2(t_1)} + \int_{t_1}^{t} \left(\frac{x_1(s)}{x_2(s)}\right)' ds = \frac{x_1(t_1)}{x_2(t_1)} - \int_{t_1}^{t} \frac{ds}{Q(s)\, x_2^2(s)} < \frac{x_1(t_1)}{x_2(t_1)} -$$

$$- \frac{1}{K^2} \int_{t_1}^{t} \frac{ds}{Q(s)} \to -\infty$$

which is contradictory to $\frac{x_1(t)}{x_2(t)} > 0$. In the second case

$$\left(\frac{x_2(t)}{x_1(t)}\right)' = \frac{1}{Q(t)\, x_1^2(t)}$$

so that $|x_1(t)|$ must be unbounded. Therefore at least one of the solutions $x_1(t)$ and $x_2(t)$ is in absolute value unbounded. We denote it $x_0(t)$ and can assume $x_0(t) > 0$ for large positive t, say for $t \geq t_1$.

Put now $u(t) = Q(t) \dfrac{x_0'(t)}{x_0(t)}$, $t \geqq t_1$. In view of (4) $u(t)$ satisfies RICCATI'S differential equation

$$u' = -\frac{1}{Q(t)} u^2 - f(t). \tag{7}$$

Multiply (7) with $\omega(t)$, integrate the left side by parts and use SCHWARZ'S inequality and (5). One gets

$$\omega(t) . u(t) = a + \int^t \omega'(s)\, u(s)\, ds - \int^t \frac{\omega(s)}{Q(s)} u^2(s)\, ds - \int^t \omega(s) . f(s)\, ds$$

$$\leqq a + \left\{ \int^t \frac{\omega(s)}{Q(s)} u^2(s)\, ds . \int^t \frac{Q(s)}{\omega(s)} \omega'^2(s)\, ds \right\}^{\frac{1}{2}} -$$

$$- \int^t \frac{\omega(s)}{Q(s)} u^2(s)\, ds - \int^t \omega(s)\, f(x)s\, ds$$

$$\leqq a + I_0^{\frac{1}{2}} \left\{ \int^t \frac{\omega(s)}{Q(s)} u^2(s)\, ds \right\}^{\frac{1}{2}} - \int^t \frac{\omega(s)}{Q(s)} u^2(s)\, ds - \int^t \omega(s)\, f(s)\, ds$$

where $I_0 = \int^\infty \frac{Q(s)}{\omega(s)} \omega'^2(s)\, ds$, so that by (6) $\omega(t) . u(t) \to -\infty$ for $t \to \infty$. Therefore $u(t)$ and consequently $x'_0(t)$ are negative for large positive t. But this is a contradiction because $x_0(t)$ is positive and unbounded.

Corollary I. *Let ε be positive and*

$$\int^t s^{1-\varepsilon} f(s)\, ds \to \infty \textit{ for } t \to \infty.$$

Then (1) *is oscillatory.*

Proof: We use Theorem I with $Q(t) = 1$ and $\omega(t) = t^{1-\varepsilon}$.

As a further corollary we shall derive from Theorem I a generalization of an oscillation criterion of KNESER [1]. With a slight extension, we can formulate KNESER'S result as follows:

If

$$\liminf_{t \to \infty} t^2 . f(t) > \tfrac{1}{4}$$

then (1) is oscillatory.

Recently this criterion has been generalised simultaneously by HILLE (see [3], p. 249) and HARTMAN (see [4], p. 778). If we write

$$\log_1 t = \log t, \; \log_p t = \log\log_{p-1} t, \; p = 1, 2, 3, \ldots$$

$$L_0(t) = t, \; L_p(t) = L_{p-1}(t) \log_p t,$$

$$S_p(t) = \sum_{n=0}^{p} [L_n(t)]^{-2},$$

then their sufficient condition is

$$\liminf_{t\to\infty} [L_p(t)]^2\{f(t) - \tfrac{1}{4} S_{p-1}(t)\} > \tfrac{1}{4}.$$

Our corollary doesn't contain this result but it isn't also contained in it because it admits the possibility that the above mentioned lower limit is smaller or equal to $\frac{1}{4}$.

Corollary 2. *If*

$$\int^t L_p(s)\{f(s) - \tfrac{1}{4}S_p(s)\}\, ds \to \infty \quad \textit{for } t \to \infty,$$

then (1) *is oscillatory.*

Proof: By the assumption and Theorem I with $Q(t) = L_p(t)$ and $\omega(t) = 1$ the differential equation

$$(L_p(t) \,.\, z')' + L_p(t)\{f(t) - \tfrac{1}{4} S_p(t)\} \,.\, z = 0 \tag{8}$$

is oscillatory. If we put $z(t) = \dfrac{x(t)}{\sqrt{L_p(t)}}$, $x(t)$ is also oscillatory. $x(t)$ however satisfies (1) as we are going to prove.

From (8) it follows that $x(t)$ satisfies the differential equation

$$x'' + \left\{\frac{1}{4}\, \frac{L_p'^2(t) - 2\, L_p(t) \,.\, L_p''(t)}{L_p^2(t)} + f(t) - \tfrac{1}{4} S_p(t)\right\} x = 0.$$

Therefore it is sufficient to prove that

$$\frac{L_p'^2(t) - 2L_p(t)\, L_p''(t)}{L_p^2(t)} = S_p(t). \tag{9}$$

For $p = 0$ (9) is evident. Suppose that (9) holds for $p = m - 1$. It is

$$(\log_m t)' = \frac{1}{L_{m-1}(t)},\ L_m'(t) = (L_{m-1}(t) \,.\, \log_m t)' = L_{m-1}'(t) \,.\, \log_m t + 1,$$

$$L_m''(t) = L_{m-1}''(t) \log_m t + \frac{L_{m-1}'(t)}{L_{m-1}(t)}$$

so that

$$\frac{L_m'^2(t) - 2L_m(t)\, L_m''(t)}{L_m^2(t)} = \frac{[\log_m t]^2 \,.\, [L_m'(t) - 2L_{m-1}(t) \,.\, L_{m-1}''(t)] + 1}{L_{m-1}^2(t)[\log_m t]^2} =$$

$$= \frac{L_{m-1}'^2(t) - 2L_{m-1}(t)\, L_{m-1}''(t)}{L_{m-1}^2(t)} + \frac{1}{L_m^2(t)} = S_{m-1}(t) + \frac{1}{L_m^2(t)} = S_m(t).$$

3. $\limsup_{t\to\infty} \dfrac{\int^t F(s)\, ds}{t} = \infty$ is weaker than WINTNER'S supposition (3) and

$$\limsup_{t\to\infty} F(t) = \infty \tag{10}$$

is still weaker.

I have dealt with the question whether (10) suffices in order that (1) may be oscillatory. I have succeeded only in showing that (10) is a sufficient condition when $f(t)$ is bounded from below. More is contained in the following

Theorem II. *Let the integral* $\int^t \frac{ds}{Q(s)}$ *be divergent,* $Q(t) \cdot f(t)$ *bounded from below and let* (10) *hold. Then* (4) *is oscillatory.*

Proof: Let be $Q(t) \cdot f(t) > -M$, $M > 0$. Assume (4) to be non-oscillatory. Then by the proof of Theorem I there exists an unbounded solution $x_0(t)$ positive from a certain t_1.

If we put $u(t) = Q(t) \frac{x'_0(t)}{x_0(t)}$ again, we get from (7) $u'(t) \leqq -f(t)$. Hence a quadrature shows that

$$u(t) \leqq a - F(t). \tag{11}$$

By (10) and (11) there exists $t_2 > t_1$ such that $u(t_2) < -\sqrt{M}$. $u(t)$ decreases in a certain neighbourhood of the number t_2 for

$$u'(t_2) = \frac{-u^2(t_2) - Q(t_2) \cdot f(t_2)}{Q(t_2)} < \frac{-M + M}{Q(t_2)} = 0.$$

$u(t)$ however must decrease for all $t \geqq t_2$. Let namely t_3 be the first number greater than t_2 and such that $u'(t_3) = 0$. Within the interval (t_2, t_3), $u(t)$ decreases and as $u(t_2) < -\sqrt{M}$ it is also $u(t_3) < -\sqrt{M}$. Therefore

$$u'(t_3) = \frac{-u^2(t_3) - Q(t_3) \cdot f(t_3)}{Q(t_3)} < 0.$$

We have proved that $u(t)$ decreases for $t \geqq t_2$. Therefore

$$u(t) \leqq u(t_2) < -\sqrt{M} < 0$$

for $t \geqq t_2$ and consequently $x_0'(t) < 0$ for $t \geqq t_2$. But this is a contradiction because $x_0(t)$ is positive and unbounded.

Corollary. *Let $f(t)$ be bounded from below and let* (10) *hold. Then* (1) *is oscillatory.*

References.

[1] A. Kneser: Untersuchungen über die reellen Nullstellen der Integrale linearer Differentialgleichungen, Math. Annalen **42** (1893), pp. 409—435.

[2] A. Wintner: A criterion of oscillatory stability, Quart. Appl. Math. **5** (1949), pp. 115—117.

[3] E. Hille: Non-oscillation theorems, Trans. Amer. Math. Soc. **64** (1948), pp. 234—252.

[4] P. Hartman: On the linear logarithmico-exponential differential equation of the second order, Amer. J. Math. **70** (1948), pp. 764—779.

Oscilační kriteria.

(Obsah předcházejícího článku.)

Nechť $f(t)$ a $Q(t)$ jsou reálné funkce spojité pro velká kladná t a $Q(t) > 0$. Pak ze STURMOVA srovnávacího teorému plyne, že buďto všechna reálná netriviální řešení STURMOVY diferenciální rovnice (4) mají nekonečně mnoho nulových bodů nebo jen konečný jejich počet. V prvém případě nazýváme řešení oscilatorická, ve druhém případě neoscilatorická, poněvadž prochází-li řešení nulovým bodem, mění znaménko. V tomto článku jsou dokázány dvě kriteria:

I. *Nechť integrál* $\int^t \frac{ds}{Q(s)}$ *diverguje a nechť existuje taková kladná funkce* $\omega(t)$ *mající spojitou první derivaci, že*

$$\int^{\infty} \frac{Q(s)}{\omega(s)} \omega'^2(s) \, ds < \infty,$$

$$\int^t \omega(s) f(s) \, ds \to \infty \text{ pro } t \to \infty.$$

Pak řešení STURMOVY *diferenciální rovnice* (4) *jsou oscilatorická.*

II. *Nechť integrál* $\int^t \frac{ds}{Q(s)}$ *diverguje, součin* $f(t) \, . \, Q(t)$ *je zdola ohraničen a*

$$\limsup_{t \to \infty} \int^t f(s) \, ds = \infty.$$

Pak řešení Sturmovy diferenciální rovnice (4) *jsou oscilatorická.*

Z kriteria I plynou dva koroláry, z nichž druhý je zobecnění jednoho KNESEROVA oscilačního kriteria:

Korolár 1. *Nechť* $\varepsilon > 0$ *a*

$$\int^t s^{1-\varepsilon} f(s) \, ds \to \infty \text{ pro } t \to \infty.$$

Pak řešení diferenciální rovnice (1) *jsou oscilatorická.*

Korolár 2. *Nechť*

$$\int^t L_p(s) \, . \, \{f(s) - \tfrac{1}{4} S_p(s)\} \, ds \to \infty \text{ pro } t \to \infty.^{2})$$

Pak řešení diferenciální rovnice (1) *jsou oscilatorická.*

[2]) Definici funkcí $L_p(t)$ a $S_p(t)$ viz v článku.