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Positivity Theorem

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Abstract. In this paper we show the positivity of the solution of continuity equation of Navier-Stokes system, with boundary conditions considered in [1]. From this result it follows the uniqueness of these solution. This enables us to simplify the solving of whole nonlinear Navier-Stokes system of equations, also in weak formulation (cf. [2]).

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We suppose that Ω is a domain in $\mathbf{R}^N$, whose boundary fulfils the Lipschitz condition, where N is a positive integer and $\partial\Omega = \Gamma^+ + \Gamma^-$. In this paper, we will use the terms "integral", "measurable" and "measure" instead of "Lebesque integral", "Lebesque measurable" and "complete measure".

Let T is a positive real number. We define function γ by conditions

$$\gamma = \begin{cases} \gamma^+ & (t,x) \in [0;T] \times \Gamma^- \\ \gamma^- & (t,x) \in [0;T] \times \Gamma^+ \end{cases}$$

where

$$\vec{u} \cdot \vec{n} = \gamma \qquad \text{in} \qquad [0;T] \times \partial\Omega$$

The function γ meets the condition:

$$\begin{cases} \gamma^+ & \text{is a nonnegative function} \\ \gamma^- & \text{is a nonpositive function} \end{cases}$$

We consider an equation

$$\frac{d\varrho(t)(x)}{dt} + div\varrho\vec{u}(t)(x) = 0 \qquad \text{in} \qquad [0;t] \times \Omega \tag{B3}$$

with the initial condition

$$\varrho(0) = \varrho^0 \qquad \text{in} \qquad \Omega \tag{B3}$$

and with the boundary condition

$$\varrho(t)(x) = \varrho^1(t)(x) \qquad \text{in} \qquad [0;t] \times \Gamma^+. \tag{B3}$$

We suppose, that the solution of the problem (1), (2), (3) is a continuous function on the set $[0;T] \times \Omega^*$.

In the following we define the solution on the corresponding space:

*Sufficient condition of continuity is: $\varrho \in C([0;T], W_p^1(\Omega))$, for $p > N$.

Definition. An abstract function g £ C([0;T],X) x C^n((0; T% X^1) we call the solution of the problém (1), (2), (3) if and only if, when the following conditions hold:

1. The condition (1) holds in X^1 for all t G (0;T).
2. The equation (2) holds in X.
3. For all t G [0;T] and a. e. z G T+ (3) holds.

For all íixed t G [0; T] we deíine the set

$$0"(í) = \{x \; £tt; g(t)\{x) < 0\}.$$

Lemma 1. *Let g be a solution of the problém* (1), (2), (3). *Then*

$$\int_{Ja-(t)} \sim g(t)(x)dx > Q \qquad \textbf{(B3)}$$

for all t G [0;T].

PROOF: After integrating equation (1) on the set Q~(č), we get

$$/ \quad -g\{t)\{x)dx + \int (g\{Ú-n))\{t)\{x)dx=:0. \qquad (B3)$$

But an"(t') = (r+) n (0íT(*)) + (r~) n (0íT(f)) + (íi) n (3íT°(*)). From conditions for 7 a £>x the integrál over the hrst and the second domain is nonpositive and the integrál over the third domain is a zero.

The proof is complete. •

Lemma 2. *For sufficiently small h and for all t* G (0; *T*) *it holds:*

$$h\{ \int_{o-(t)} g(t + h)(x)dx - \int_{a-(t)} g(t)(x)dx\} > 0$$

PROOF: The proof follows from the inequality

$$0 < \lim i\{ \int_{n-(t)} [\wedge + A)(x)-\wedge)W]dx]$$

We define function m as follows

$$m (*) = y_{O-(t)} *?(*)(*)<**$$

for $t \in [0;T]$.

Lemma 3. *The function m^- is non-decreasing on the interval $[0;T]$.*

PROOF: Let h is negative and sufficiently small. We have

$$\begin{aligned} m^-(t+h) - m^-(t) &= \int\limits_{\Omega^-(t+h)} \varrho(t+h)(x)dx - \int\limits_{\Omega^-(t)} \varrho(t+h)(x)dx + \\ &+ \int\limits_{\Omega^-(t)} \{\varrho(t+h)(x)dx - \varrho(t)(x)\}dx. \end{aligned} \tag{B3}$$

From the previous lemma we have

$$m^-(t+h) - m^-(t) \le \int\limits_{\Omega^-(t+h)} \varrho(t+h)(x)dx - \int\limits_{\Omega^-(t)} \varrho(t+h)(x)dx.$$

Thus

$$m^-(t+h) - m^-(t) \le \int\limits_{\Omega^-(t+h)\setminus\Omega^-(t)} \varrho(t+h)(x)dx - \int\limits_{\Omega^-(t)\setminus\Omega^-(t+h)} \varrho(t+h)(x)dx.$$

With respect to the definition of $\Omega^-(t)$ we have

$$\int\limits_{\Omega^-(t)\setminus\Omega^-(t+h)} \varrho(t+h)(x)dx \ge 0,$$

$$\int\limits_{\Omega^-(t+h)\setminus\Omega^-(t)} \varrho(t+h)(x)dx \le 0,$$

i.e. $m^-(t+h) - m^-(t) \le 0$.

The proof is complete. □

Theorem. *Let for every $t \in [0;T]$ and a. e. $x \in \Omega$, $\varrho^1(t)(x) \ge 0$ and $\varrho^0(x) \ge 0$ holds. If ϱ is a solution of the problem (1), (2), (3), then for every $t \in [0;T]$ and a. e. $x \in \Omega$ is $\varrho(t)(x) \ge 0$*

PROOF: From the previous lemma m^- is non-decreasing. It is a contradiction to $m^-(0) = 0$ and to the fact, that m^- is negative. □

Remark 1. *It is easy to see from the proof, that the same results we can obtain for the next two cases:*

1. *If we consider the inequality $\ge$ instead of equality in (1).*
2. *If we assume that the right side of (1) is nonnegative instead of zero.*

Remark 2. *If we assume everywhere the opposite inequality, then in the proposition of the theorem we obtain the opposite inequalities.*

Corollary (Uniqueness theorem). *If the problem* (1), (2), (3) *has a continuous solution, then it is uniquely defined.*

PROOF: By the contradiction. Let ϱ_1, ϱ_2 be two different solutions of the considered problem and let $\varrho = \varrho_1 - \varrho_2 \neq 0$. Then ϱ is the solution of the homogeneous problem. According to the remarks, ϱ is simultaneous nonpositive and nonnegative. Therefore it is zero. This is the contradiction. □

References

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[2] Fučík, S., Kufner, A., *Nonlinear Differential Equations*, Elsevier Scientific Publishing Company, Amsterdam–Oxford–New York, 1980.

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