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Acta Mathematica et Informatica Universitatis Ostraviensis, Vol. 2 (1994), No. 1, 45--59

Persistent URL: http://dml.cz/dmlcz/120483

Weak convergence in infinite dimensional spaces

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Abstract. The aim of the paper si to describe the role of the weak convergence in Banach spaces. After definitions the notions are studied in several cases: the finite dimensional space, the space of infinite sequences and the space of integrable functions. Some strange properties of weak convergence are introduced. Remarks on application in operator theory, homogenization and optimal design close the paper. The paper is intended to be admissible for non-specialist in functional analysis.

1991 Mathematics Subject Classification: 46-XX, 46B10, 46A50

(Dedicated to the memory of Svatopluk Fučík)

Introduction

These notes have their origin in a lecture held at Brušperk on the occasion of opening the educational and scientific *Svatopluk's Center* in Brušperk on October 26–27, 1991. The aim of the contribution is to attract the attention of readers to the role of weak convergence in the analysis of infinite dimensional spaces and its application.

I want to express my thanks to Dr. Milan Konečný who initiated origin of this contribution.

The author wish the paper will be admissible for non-specialist in mathematical analysis, therefore even the simplest notions are briefly introduced.

Prologue

The sequence and its convergence ranks to the most important notions of Mathematical Analysis. Most problems in applications are solved by methods consisting of sequences of approximative problems. Such a method makes sense if the sequence of corresponding approximate solutions converges in some sense to the solution of the solved problem. Although in practice we are able to compute only a finite number of members of the sequence, the convergence ensures us that we can obtain as precise solution as we need; or at least that computing higher member of the sequence we obtain a more precise solution.

Let us mention one basic difference between Algebra and Analysis: In Algebra there are elements and operations with them; but there are just two possibilities:

the elements are either equal (identical, etc.) or not. There is no possibility to express some closeness of the objects. In Analysis we can measure a distance between the objects if we have a metric. In more general cases we can at least express tending (drawing together) of elements in a sequence to a limit element. This is done by convergence, or at a higher level by topology.

Even the most important notion of Analysis i. e. continuity is based on convergence or topology.

1 Basic notions

We start with definitions that appear in almost all textbooks on functional analysis. We shall confine to Banach spaces, although the weak convergence can be introduced also in more general spaces.

Banach space and strong convergence

The space V is called *linear* iff the elements can be added (i. e. if $x_1, x_2 \in V$ then also $x_1 + x_2 \in V$) and multiplied by real numbers (i. e. if $x \in V$ and $\alpha \in \mathbb{R}$ then also $\alpha x \in V$).

The linear space is called *normed* iff it is endowed with a real function $\|\cdot\|$ called *norm* which measures the distance of any element x from the zero element 0. Thus due to linearity of the space we can measure distance of any two elements x_1 and x_2 of the space by $\|x_1 - x_2\|$.

The *norm* is supposed to satisfy natural conditions:

— it distinguishes the elements from zero element: the norm of an element $\|x\|$ is a positive real number except for the zero element 0, where $\|0\| = 0$,

- — it is "homogeneous": $\|\alpha x\| = |\alpha| \cdot \|x\|$ for $\alpha \in \mathbb{R}$
- — and it satisfies the triangle inequality $\|x_1 + x_2\| \le \|x_1\| + \|x_2\|$.

We shall deal with sequences $\{x_k\}_k \equiv \{x_1, x_2, x_3, \ldots\}$ of elements $x_k \in V$. The norm induces a convergence:

We say that a sequence $\{x_k\}_k$ *converges* to an element $x \in V$, iff the distance of x_k from x tends to 0 with $k \to \infty$. This convergence called *strong convergence* will be denoted by the arrow $\to$:

$$x_k \to x \qquad \text{iff} \qquad \|x_k - x\| \to 0\,. \tag{1.1}$$

Besides convergent sequences there are *fundamental* (or *Cauchy*) sequences. The distances between their members tend to zero, i. e. $\|x_k - x_j\| \to 0$ for $k, j \to \infty$, but the sequence need not have a limit, in place of the limit there may be a "hole" in the space.

The space without these "holes", i. e. each fundamental sequence has its limit in the space, are called *complete*. We close the subsection by a definition:

A complete normed linear space is called Banach *space.*

Dual space and weak convergence

To introduce the second convergence we need the notion of a dual space.

Let us consider continuous linear forms on V. The form on V i. e. a mapping $f : V \to \mathbb{R}$ with its value at $x \in V$ denoted by $\langle f, x \rangle$ is *linear* iff

$$\langle f, \alpha_1 x_1 + \alpha_2 x_2 \rangle \;=\; \alpha_1 \langle f, x_1 \rangle + \alpha_2 \langle f, x_2 \rangle \qquad \forall x_1, x_2 \in V\,, \quad \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

and *continuous* on V iff

$$x_k \to x \quad \Longrightarrow \quad \langle f, x_k \rangle \to \langle f, x \rangle \quad \forall\, x_k, x \in V\,.$$

Since the form f is linear, it is sufficient to check the continuity only for sequences $x_k \to 0$. But the continuity of a linear form is equivalent to its boundedness. Thus the form f is continuous iff there exists a constant $c \in \mathbb{R}$ such that

$$|\langle f, x \rangle| \le c\,||x|| \quad \forall x \in V\,.$$

Since the smallest of the constants c satisfy the conditions of the norm, it will be proclaimed to be the *norm* of the form.

The set of all continuous linear forms on V is called the *dual space* and denoted by V'. Its elements are given by defining its values on elements of V. The linear continuous forms can be added

$$\langle f_1 + f_2, x \rangle = \langle f_1, x \rangle + \langle f_2, x \rangle\,, \quad \forall x \in V$$

multiplied by real numbers

$$\langle \alpha f, x \rangle = \alpha \langle f, x \rangle \quad \forall \alpha \in \mathbb{R}\,, x \in V$$

and normed

$$||f|| = \max_{x \in V} \frac{|\langle f, x \rangle|}{||x||}\,. \tag{1.2}$$

Thus the dual space is a normed linear space, too. Moreover, it is complete with respect to its norm, since the set $\mathbb{R}$ is complete. Thus the dual space of a normed linear space is a Banach space.

Now we are able to introduce the weak convergence. Instead of measuring the distance of x_k and the limit x by the norm, we measure it by all continuous linear forms:

We say that a sequence $\{x_k\}_k$ in V is *weakly converging* to an element $x \in V$ iff it converges with respect to all forms f on V. The convergence is called the *weak convergence* and will be denoted by an half-arrow $\rightharpoonup$ in contrast to the strong convergence:

$$x_k \rightharpoonup x \qquad \text{iff} \qquad \langle f, x_k \rangle \to \langle f, x \rangle \quad \forall f \in V'\,. \tag{1.3}$$

Let us note that in order to introduce the weak convergence we need continuous linear forms on V i. e. we need a linear space with some continuity to define them.

We introduced the definition for the case of a real space, it can be rewritten also for the case of complex spaces.

2 The finite dimensional spaces

Let us start with the finite dimensional spaces. Everybody is able to imagine the elements and their distance in spaces of dimension one, two and three. In higher dimensional spaces this imagination has no counterpart with our everyday experiences.

An element x in the space of dimension 2 can be represented not only as a point in the plane, but also as a tuple (x^1, x^2) of its coordinates. Since we use the subscripts for sequences, the coordinates will be denoted by superscripts, i. e. $x \equiv (x^1, x^2)$. In the plane we measure the distance of two elements x_1, x_2 by the so-called Euclidean norm $\|x_1 - x_2\|$ given by

$$\|x_1 - x_2\| = \left[\sum_{i=1}^{2}(x_1^i - x_2^i)^2)\right]^{1/2} .$$

Besides of this norm one can introduce a family of norms by replacing the exponent 2 by a parameter p ($p \in [1, \infty)$), but it is not the subject of this paper.

In the case of the higher finite dimension N the space V can be identified with $\mathbb{R}^N$. Indeed, choosing a convenient base each element x can be uniquely represented by the family of its coordinates $(x^1, x^2, \ldots, x^N) \in \mathbb{R}^N$. Thus we can imagine the finite dimensional space as the space of finite sequences.

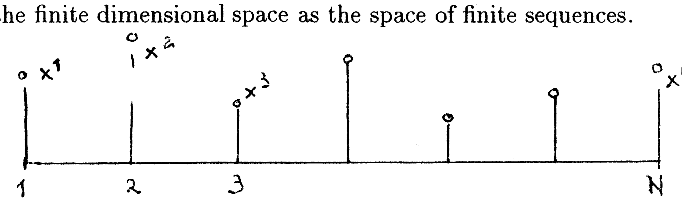

Fig. 1 Elements of $\mathbb{R}^N$ can be represented by the finite sequences of their coordinates.

The elements of $\mathbb{R}^N$ can be added by adding the corresponding coordinates and multiplied by a real number by multiplying the corresponding coordinates $(\alpha x = (\alpha x_1, \ldots, \alpha x_N))$.

The definition of the plane norm can be naturally extended to dimension N:

$$\|x\| = \left[\sum_{i=1}^{N} |x^i|^2\right]^{1/2}$$

The strong convergence is defined by the norm: $x_k \to x$ by definition iff $\|x_k - x\| = \left[\sum_i |x_k^i - x^i|^2\right]^{1/2} \to 0$. Since the sum contains the finite number of non-negative terms, each of them tends to zero. Thus strong convergence $x_k \to x$ implies the convergence in all coordinates $x_k^i \to x^i$, $\quad i = 1, 2, \ldots, N$. The converse implication is obvious.

Let us conclude: *In the finite dimensional space $\mathbb{R}^N$ the sequence x_k converges strongly iff it converges in all coordinates. Moreover, the space $\mathbb{R}^N$ is complete.*

Continuous linear forms

Let us deal with linear forms on $\mathbb{R}^N$. Let us denote the canonical base of the space $\mathbb{R}^N$ by $e_1, \dots, e_N$: $e_k = (0, \dots, 0, 1, 0, \dots, 0)$, where 1 stands at the k-th position, the other coordinates are zero. Due to linearity of the form we can write the value of the form f at x in the form

$$\langle f, x\rangle = \langle f, \sum_i x^i e_i\rangle = \sum_i x^i \langle f, e_i\rangle .$$

Let us denote $f^i = \langle f, e_i\rangle$. Thus the form f can be represented by the family $(f^1, \dots, f^N)$ from $\mathbb{R}^N$, i.e. by its values f^i at the elements e_i $i = 1, 2, \dots, N$ of the base of the space. On the other hand any family from $\mathbb{R}^N$ defines a linear form on R^N.

Since the finite family $f^1, \dots, f^N$ is bounded and x_k converges to 0 iff $x_k^i \to 0$ as $k \to \infty$ for all i, all linear forms are continuous. Indeed, $x_k \to 0$ yields

$$|\langle f, x_k\rangle| = \left|\sum_i x_k^i f^i\right| \leq \max_i |f^i| \cdot \sum_i |x_k^i| \to 0 \quad \text{as } k \to \infty .$$

Thus *all linear forms on $\mathbb{R}^N$ are continuous, they form a linear space which can be represented again by $\mathbb{R}^N$.*

Let us consider a weakly converging sequence $x_k \rightharpoonup 0$. Choosing a form f defined by $\langle f, x\rangle = x^i$, we find that any component a the weakly converging sequence is convergent, too.

We conclude the section by the statement that *in the finite dimensional spaces the weak and the strong convergence coincide. Moreover, the dual space to $\mathbb{R}^N$ can be represented by $\mathbb{R}^N$, it is complete normed space with coinciding the strong and the weak convergence characterized by the convergence in each component.*

3 The space of infinite sequences

The space of infinite sequences is the simplest example of the infinite dimensional space. This infinite dimensional space is generated by canonical base $\{e_1, e_2, e_3, \dots\}$ where e_k has 1 at the k-th position, the other terms are zero. In contrast to Algebra we take not only the finite linear combinations but also some (not all) infinite linear combinations $\sum_i x^i e_i$. Thus representing the element by the sequence of its coordinates we can imagine the element of the space as a infinite sequence:

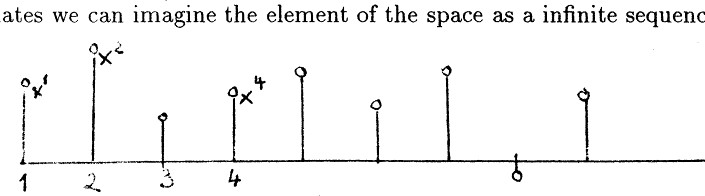

Fig. 2 Elements of the space can be represented by the infinite sequences of their coordinates.

The norm can be generalized without problems to the infinite dimensional space:

$$\|x\| = \left[\sum_{i=1}^{\infty} |x^i|^2\right]^{1/2} \tag{3.1}$$

Now the infinite dimension causes the first problems. Not all infinite sequences can be normed by the norms, even if we consider bounded sequences only. We can take only the elements which are can ne normed, i. e. such that $\|x\|$ is a finite number. We shall deal with the set of all sequences that can be normed by (3.1) and we denote the set by l^2:

$$l^2 = \left\{x = (x^1, x^2, x^3, \ldots) \;\middle|\; \|x\| < \infty\right\} .$$

The sequences can be added and multiplied by a real number in natural way. It is simple to check that the sum and multiple of sequences from the set l^2 remains in the same set, i. e. the sequences of l^2 form a *linear space.* The space contains all finite sequences extended by zero, thus each $\mathbb{R}^N$ can be supposed to be a subspace of l^2. Since the norm implies the convergence in components, the space is also complete with respect to this norm.

In addition the space l^2 is even Hilbert space if we endow it with the scalar product

$$(x, y) = \sum_i x^i y^i . \tag{3.2}$$

Continuous linear forms

How to represent linear forms on l^2 ? Let f be such a linear form. Taking the values of f at elements of the base $\{e_k\}_k$, we can represent it by a sequence $(f^1, f^2, \ldots)$, where $f^i = \langle f, e_i\rangle$. Indeed, by linearity

$$\langle f, x\rangle = \langle f, \sum_i^{\infty} x^i e_i\rangle = \sum_i^{\infty} \langle f, e_i\rangle x^i = \sum_i^{\infty} f^i\, x^i .$$

Thus each linear form can be represented by a sequence. But not every sequence defines a linear form on l^2 since the series need not converge. Using the Schwartz inequality yielding

$$|\langle f, x\rangle| = \left|\sum_i^{\infty} f^i\, x^i\right| \le \left[\sum_i^{\infty} |f^i|^2\right]^{1/2} \cdot \left[\sum_i^{\infty} |x^i|^2\right]^{1/2} = \|f\| \cdot \|x\|$$

we find that if the form f is bounded (which is for linear forms equivalent to continuity) then its representing sequence $(f^1, f^2, \ldots)$ is in l^2. On the other hand any sequence of l^2 defines a bounded form on l^2. Thus by representation

$$\langle f, x\rangle = \sum_{i=1}^{\infty} f^i x^i . \tag{3.3}$$

we can coincide the dual space $(l^2)'$ with l^2. Moreover, one can see that the norm of $f \in (l^2)'$ is equal to the norm of its representing sequence; the representation is isometric.

3.1 Weak convergence

Following the definition the sequence $\{x_k\}$ converges weakly to x iff $\langle x_k - x, f\rangle \to 0$. Since $f\, x \mapsto x^i$ is a continuous linear form on l^2 for any i, we see that the sequence converges in each component: $x^i_k \to x^i$ as $k \to \infty$. But the convergences in the components need not be uniform.

The sequence of the base elements e_k serves as the well known example of the sequence converging weakly but not strongly. Indeed, e_k cannot converge since each two different elements have distance $\|e_k - e_l\| = \sqrt{2}$. On the other hand considering any i-th component we obtain a sequence containing from $i+1$ member only zero. Thus

$$e_k \not\to 0 \qquad \text{but} \qquad e_k \rightharpoonup 0\,. \tag{3.4}$$

It is caused by the fact that there is no linear form on l^2 which would exclude such "bad" sequences; the form $(1, 1, 1, \ldots$ which can detect this non-convergence is absent in l^2.

4 Spaces of integrable functions

These spaces are also infinite dimensional, the dimension can be said to be even incountable, since the segment consists of incountable many points. But they form the bases for analysis of problems in most applications e. g. in mathematical physics, mathematical modelling, optimal design etc.

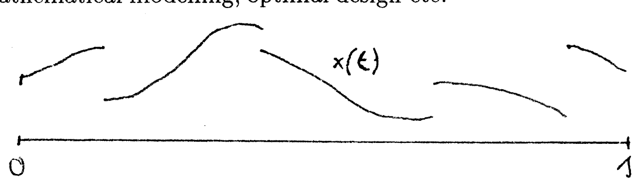

Fig. 3 Elements of the space are square integrable functions.

Let us fix a bounded interval, say $I = (0, 1)$. We shall consider functions $x(t)$ on this interval. The distance between functions can be measured using various norms, we shall deal with the integral norm and integral scalar product

$$\|x\| = \left[\int_I |x(t)|^2\,\mathrm{d}t\right]^{1/2} \qquad (x, y) = \int_I x(t)y(t)\,\mathrm{d}t\,. \tag{4.1}$$

We have to overcome several difficulties. We cannot accept all functions since the integrals should exist and be finite. Thus we admit only the functions which are measurable and square integrable i. e.

$$L^2(I) = \left\{ x\, I \to \mathbb{R} \;\middle|\; \int_I |x(t)|^2 \, \mathrm{d}t < \infty \right\} . \tag{4.2}$$

But there appears another problem: there are functions which have zero norm $\|x\| = 0$ but they are not identically zero, e. g. a function having value 1 at finite numbers of points t_i, otherwise zero. Thus the first property of the norm is violated. The solution consists in identifying functions which differ only on a zero measure subset — we say that they are equal "almost everywhere". This space of square integrable functions with "a.e. equality" will be denoted by $L^2(I)$.

Remark. There is another procedure of introducing the space. It consists in starting with less functions e. g. with functions being continuous on the compact interval $\bar{I}$ which are bounded an integrable. Then we make completion in the norm (4.1). By this way we obtain the same space.

Dual space

In general the characterization of continuous forms on spaces of functions is not an easy task. We shall confine to the integrable function, where the situation is simple. In case of square integrable functions following Riesz representation theorem each continuous form F on $L^2(I)$ can be represented by a function $f \in L^2(I)$ by

$$\langle F, x \rangle = (f, x) \equiv \int_I f(t)\, x(t)\, \mathrm{d}t \qquad \forall x \in L^2(I) . \tag{4.3}$$

That means that each continuous form F on $L^2(I)$ can be written in the form (4.3) with some $f \in L^2(I)$, and on the other side each function $f \in L^2(I)$ defines by (4.3) a continuous form F on $L^2(I)$. In addition, norm of the form F equals to the norm of the corresponding representing function f. Thus using this representation we can identify the dual space $(L^2(I))'$ with the space $L^2(I)$ itself.

Weak convergence

Let us consider a sequence of functions $x_k(\cdot) \in L^2(I)$ and a function $x(\cdot)$ of $L^2(I)$. The strong convergence $x_k \to x$ given by

$$\|x_k - x\| = \left[\int_I |x_k(t) - x(t)|^2 \, \mathrm{d}t \right]^{1/2} \to 0$$

implies convergence $x_k(t) \to x(t)$ for almost all $t \in I$.

The functions $x_k(\cdot)$ converge to $x(\cdot)$ weakly iff $\langle F, x_k - x \rangle \to 0 \quad \forall\, F \in (L^2(I))'$. Using the representation of linear forms it means

$$x_k \rightharpoonup x \qquad \text{iff} \qquad \int_I f\,(x_k - x)\, \mathrm{d}t \to 0 \quad \forall f \in L^2(I) . \tag{4.4}$$

According to the name the weak convergence is "weaker" i.e. each strongly converging sequence is converging weakly, but the converse is not true.

A simple counterexample is the following sequence of functions

$$x_k(t) = \sin k\pi t\,. \tag{4.5}$$

It is not converging strongly, since it is not converging in any point except for $t = 0, 1$. But it converges weakly to zero. For f constant the convergence is obvious. Let f be constant on a subinterval $(a, b) \subset [0, 1]$. Then

$$\int_a^b f\,\sin k\pi t\,\mathrm{d}t = \frac{f}{k\pi}[\sin k\pi t]_a^b \to 0 \quad \text{for } k \to \infty$$

since $\sin k\pi t$ is bounded. Using this argument we see that $(x_k, f) \to 0$ for any "step-like" function f. Since these functions are dense in $L^2(I)$ the assertion follows.

5 Compactness in finite and infinite dimensional spaces

The compactness of a set M can be formulated as follows:

Definition. A subset M of the space V is said to be compact iff each sequence $\{x_k\}$ of M contains a subsequence $\{x_{k'}\}$ converging to an element x in M.

Finite dimensional spaces

In the finite dimensional spaces each bounded closed set is compact. The compactness has several important consequences:

Theorem. *Let M be a compact set and T a continuous real function on M. Then:*

— *T is bounded on M,*

— *T attains its biggest and smallest value on M.*

The proof is simple. Let $\{x_k\}$ be a sequence such that $T(x_k)$ is increasing to the biggest value m. To be more precise, the values of T form a subset of $\mathbb{R}$. Each subset of $\mathbb{R}$ has its lowest upper bound m called *supremum* which can be written as the limit of a sequence $T(x_k) \to m$. Since M is compact, $\{x_k\}$ contains a subsequence $\{x_{k'}\}$ converging to an element $x \in M$. Due to continuity of T we have $Tx_{k'} \to Tx = m$. Thus m is a finite number and the result follows.

Infinite dimensional spaces

In the infinite dimensional spaces the closed bounded sets need not be compact. In the space of infinite sequences l^2 the closed unit ball B is not compact. Indeed, B contains the sequence $\{e_k\}_{k=1}^{\infty}$ which contains no converging subsequence since each two members have distance bounded from bellow by a positive constant: $\|e_j - e_k\| = \sqrt{2}$.

In the spaces of functions $L^2(I)$ the unit ball is not compact either since the sequence $\{x_k\}$ defined by (4.5) cannot contain a converging subsequence by the same argument.

Thus a continuous function on a bounded set B need not be bounded and need not admit its biggest and smallest value. Similarly, having a bounded sequence of approximate solutions x_k we cannot apply the usual procedure, since we have no limit element x.

Fortunately, the weak convergence can solve the problem. The infinite dimensional spaces $V = l^2$ and $V = L^2(I)$ have the following property:

Theorem. *Let M be a closed convex bounded subset of V. Then it is compact with respect to the weak convergence, i. e. each sequence $\{x_k\}$ contains a subsequence weakly converging to an element $x \in M$.*

The spaces with this property are called *reflexive.* Many infinite dimensional spaces commonly used in applications have this property, e. g. Lebesgue spaces $L^2(\Omega)$ for Ω bounded and Sobolev spaces $W^{1,2}(\Omega)$ etc.

With this weak convergence there appears another problem: the usual continuity of the function T: $x_k \to x \quad \Longrightarrow \quad Tx_k \to Tx$ is not sufficient: we need the continuity with respect to the weak convergence:

$$x_k \rightharpoonup x \quad \Longrightarrow \quad Tx_k \to Tx\,.$$

But this "weak continuity" is not ensured by the usual continuity. Moreover, it is very restrictive variational functionals. Thus instead of the continuity we assume only that the functional T is *weakly lower semi-continuous,* i. e.

$$x_k \rightharpoonup x \quad \Longrightarrow \quad \liminf_{k\to\infty} Tx_k \geq Tx \quad \forall x_k, x \in V\,.$$

6 Strange properties of weakly converging sequences

The "good" properties of the weak convergence are balanced by some "unpleasant" ones. One must be very careful when passing to limits in products of sequences converging only weakly.

Let $\{x_k\}_k$ $\{y_k\}_k$ be two sequences in $L^2(I)$. If they are converging strongly $x_k \to x$ and $y_k \to y$, then their scalar product converges $(x_k, y_k) \to (x, f)$. The same holds if there is only one weakly converging sequence and the other converge strongly. If there is the scalar product of sequences converging only weakly $x_k \rightharpoonup x$, $y_k \rightharpoonup y$, then it need not be true: their scalar product (x_k, y_k) do not converges to product of the weak limits (x, f).

We illustrate it by the following examples

$$\begin{aligned} x_k(t) &= \sin k\pi t \rightharpoonup 0\,, \quad & y_k(t) &= \sin k\pi t \rightharpoonup 0 \quad & &\text{but} \ (x_k, y_k) \to \tfrac{1}{2}\,, \\ x_k(t) &= \sin k\pi t \rightharpoonup 0\,, \quad & y_k(t) &= -\sin k\pi t \rightharpoonup 0 \quad & &\text{but} \ (x_k, y_k) \to -\tfrac{1}{2}\,, \\ x_k(t) &= \sin k\pi t \rightharpoonup 0\,, \quad & y_k(t) &= \cos k\pi t \rightharpoonup 0 \quad & &\text{but} \ (x_k, y_k) \to 0\,. \end{aligned} \tag{6.1}$$

Let us remark that Another strange property is the similar effect for inverse operation. Let $\{x_k\}_k$ be a sequences in $L^2(I)$ of functions bounded on I and satisfying $|x_k(t)| > c$, where c is a positive constant. Then their "inverse" i.e. functions $x_k^{-1} = 1/x_k(t)$ are also in $L^2(I)$. If x_k converge to x strongly, then also $(x_k)^{-1} \to x^{-1}$. If they are converging only weakly $x_k \rightharpoonup x$, then their inverses may converge but to other limit as can be seen in the example:

$$x_k(t) = 2 + \cos k\pi t \rightharpoonup 2 \qquad \text{but} \qquad \frac{1}{x_k} = \frac{1}{2+\cos k\pi t} \rightharpoonup \frac{1}{\sqrt{3}} \neq \frac{1}{2}\,.$$

7 Weakly continuous operators

Let us consider an operator $AV \to V'$ on a Hilbert space V. The usual assumption that the operator is continuous is not sufficient. The proof will be simple if we can adopt the assumption that the operator is weakly continuous. This continuity is defined by means of the weak convergence:

Definition. The operator A is said to be *weakly continuous* iff

$$x_k \rightharpoonup x \quad \Longrightarrow \quad A\,x_k \rightharpoonup A\,x \quad \forall x_k, x \in V\,.$$

Let us suppose that we have a sequence of problems on finite dimensional subspaces (e.g. a sequence of Galerkin approximations) and a sequence of corresponding solutions x_k. Let us assume that the sequence $\{x_k\}$ is bounded (this can be assured e.g. by means of coercivity of the operator). The bounded sequence contains a weakly converging subsequence $\{x_{k'}\}$ converging $x_{k'} \rightharpoonup x$, where x is an element of V. Then $Ax_k \rightharpoonup Ax$ due to weak continuity of A. To finish the proof it remains to verify that Ax satisfies the solved infinite dimensional problem. Moreover, the weak continuity yields continuity on finite dimensional subspaces which can be used for solving finite dimensional approximation. For the abstract equation $Ax = y$ the proof can be found in [3].

If the operator is not weakly continuous one can use the theorem on pseudomonotone operators or operators satisfying the property (M), both these notions are defined by means of weak convergence, see the survey [4].

8 Applications in operator theory

In the last section we shall mention (often without proof) some examples where the weak convergence appear. We have raised one level up; from weak convergence in the spaces of functions to weak convergence on spaces of operators $A\,V \to V'$.

We shall deal with second order linear differential operators of type

$$A\,x = -\frac{\mathrm{d}}{\mathrm{d}t}\left[a(\cdot)\,\frac{\mathrm{d}u}{\mathrm{d}t}(\cdot)\right] \tag{8.1}$$

where the coefficients $a(t)$ are measurable functions satisfying

$$\alpha < a(t) < M \tag{8.2}$$

with fixed constants $0 < \alpha < M < \infty$. It is not difficult to prove that the boundary value problem for differential equation with operator of this type

$$A\,x \equiv -\,[a(\cdot)x']' = f$$
$$x_k(0) = x_k(1) = 0$$

admits unique solution $x(t)$ in a generalized sense, see e. g.[2].

The problem describes e. g. the stationary case of heat flow in a thin bar. The unknown function $x(t)$ means the temperature in place t, the right-hand side f means the intensity of internal heat sources and the boundary condition prescribes the temperature at the ends of the bar. The coefficient $a(t)$ describes the local properties of the material, in this model it is the heat conductivity.

Homogenization

Let $a_k(t)$ be a sequence of periodic functions with diminishing period $1/k$ satisfying (8.2). As an example we shall consider

$$a_k(t) = \frac{1}{2 + \cos 2k\pi t}\,. \tag{8.3}$$

The sequence determines a sequence of differential operators A_k. The corresponding boundary value problems describe heat conduction in bars made of a composite material - the non-constant conductivity is described by the periodic functions $a_k(t)$.

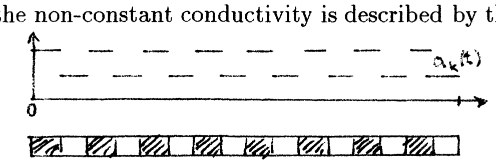

Fig. 4 Heat conduction in a thin bar made of a composite material.

In this way we obtained a sequence of materials having periodic structure with diminishing period and the corresponding sequence of solutions x_k. One can prove that the solutions x_k converge to some function x_∞ which is a solution to a problem of the same type

$$A_\infty x_\infty \equiv -\,[a_\infty x']' = f \qquad x_\infty(0) = x_\infty(1) = 0\,.$$

Let us remark that in spite of the functions a_k converge to $\bar{a}$ the coefficient a_∞ differs from the limit $\bar{a}$ of the coefficients a_k.

In the case of coefficients (8.3) with $f = 1$ we can even compute the solutions:

$$x_k(t) = t(1-t) + \frac{1-2t}{4k\pi}\sin 2k\pi t + \frac{1}{(2k\pi)^2}[1 - cos 2k\pi t]\,.$$

Clearly, the sequence x_k converges to a function $x_\infty(t) = t(1-t)$, which is the solution to the problem

$$A_\infty x_\infty = -a_\infty \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = 1 \qquad x_\infty(0) = x_\infty(1) = 0\,,$$

where $a_\infty = \frac{1}{2}$ but the limit of a_k is $\bar{a} = \frac{1}{\sqrt{3}}$.

The introduced example shows that the natural operator convergence $A_k \to A$ defined by $A_k x \to A\,x \quad \forall x$ which corresponds to the weak convergence of coefficients is not convenient for homogenization.

Since the problem (8.2) admits unique solution the inverse operator $A^{-1}f$ makes sense and we can define the so called G-convergence for operators of type (8.1) as follows

$$A_k G - \text{converge to } A_\infty \qquad \text{iff} \qquad (A_k)^{-1} f \to (A_\infty)^{-1} \quad \forall f\,. \tag{8.4}$$

In the one-dimensional case of operators of type (8.1) the G-convergence is equivalent to the weak convergence $1/a_k \rightharpoonup 1/a_\infty$¿ In more dimensional cases (heat conduction in a thin plate or in a body made of a composite material) the situation is more complicated. The G-convergence does not coincide with any coefficient convergence, studying G-convergence one must deal with differential equations.

8.1 Compactness and optimal design problems

Let us consider a class of operators of type (8.2) satisfying condition (8.3) with fixed constants α, M. This class of operators is compact in the following sense: each sequence of operators contains a subsequence which is G-convergent to an operator of the same class.

This result is important in some applications. We shall introduce an optimal design problem.

The problem consists in looking for the distribution of two or more materials (or a material and holes) with given volume ratio within the given domain such that the stresses or other quantities are in some sense smallest. The distribution of the material is given by the coefficient $a(t)$: which takes two values: one for materials, the second for the other material or zero for holes.

The optimal design is reached by the usual procedure: We find a sequence of designs tending to the optimum. Using the compactness we obtain the limit design. But this limit is not often a "two-value" function: in spite of the two-value coefficients the coefficients of the limit operator admits also the values between limits. Thus the optimal design often does not exist. We compute this generalized design minimizing the so-called *relaxed* functional (it exists, but does not correspond to real material), and then we construct an real almost optimal design.

Example

Let us consider the following simple problem. We are looking for the minimum of the integral functional

$$\Phi(x) = \int_I f(x(\cdot)')\,\mathrm{d}t\,, \tag{8.5}$$

where $I = (0,1)$ and f is a special discontinuous function

$$f(\xi) = \begin{cases} 0 & \text{for } \xi = 0 \\ 1 + |\xi|^2 & \text{for } \xi \neq 0 \end{cases}\,. \tag{8.6}$$

The minimum is looked for in the set of continuous functions $x(t)$ with square integrable derivatives $x(t)' \in L^2(I)$ satisfying $x(0) = 0$, $x(1) = c$, where c is a positive constant.

For big c ($c > 2$) it is clear that linear $x(t) = ct$ minimizes the functional yielding the minimum $1 + c^2$. For small c one obtain a smaller value if $x(t)$ is a functions with "steps" and "slopes", i. e. constant on some subintervals of I and with the slope 2 on the rest of I of total length $a/2$. Using such functions we obtain the value $\Phi(x) = 2a$ that is smaller then the value $1 + c^2$ reached by the linear $x(t)$. The effect is caused by the fact that the function f is not convex.

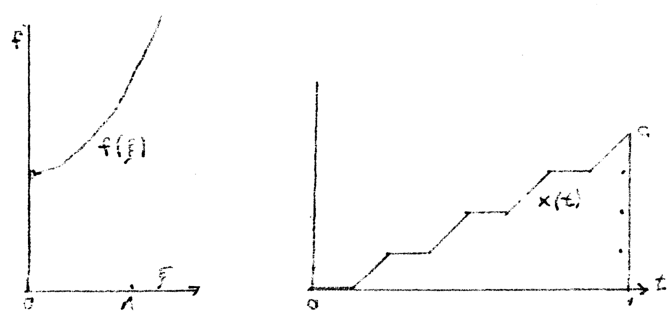

Fig. 5 Function f and the minimizing "step-slope" function.

In the introduced one dimensional example the smallest value of the functional Φ was attained, in the higher dimensional case it is not the case. Let us consider the integral functional

$$\Phi(x) = \iint_\Omega f(|\nabla x|)\mathrm{d}s\mathrm{d}t\,,$$

where $\Omega = (0,1) \times (0,1)$, $\nabla x = \left(\frac{\partial x}{\partial t}, \frac{\partial x}{\partial s}\right)$ and f is given by (8.6). We are looking for the minimum over the functions satisfying linear boundary conditions

$$x(s,t) = ct \quad \forall (t,s) \in \partial\Omega\,.$$

Again for small c the linear function does not yield the minimum. The minimum would be attained by the similar "stair and slope" function, but it does not satisfy the boundary conditions on the "sides" $s = 0, 1$.

Thus this minimizer must be corrected by slopes along the sides to satisfy the boundary conditions.

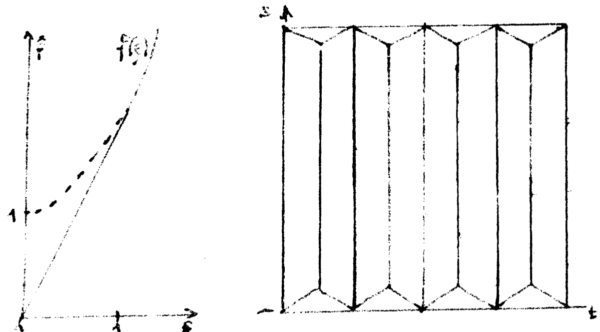

Fig. 6 Relaxed function f and "almost minimizer" for the two-dimensional problem.

Thus we obtained "almost minimum", the narrower the steps are the narrower are the side stripes and the value is closer to the minimum.

The solution of optimal design problem consist in replacing the functional by the relaxed one with the relaxed function f

$$f(\xi) = \min\{2\xi, 1 + |\xi|^2\} \quad \text{for } \xi > 0 .$$

The relaxed functional is convex and thus attains its minimum. We compute the minimum to the relaxed problem and then construct an "almost minimum" of the problem. Let us mentioned that the minimizing sequence of "almost minima" is converging weakly and represents a sequence of composite materials with refining structure.

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(*Received* June 8, 1994)