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Existence Results on the Semiinfinite Interval for First and Second Order Integrodifferential Equations in Banach spaces with Nonlocal Conditions

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Abstract

In this paper we investigate the existence of mild solutions defined on the semiinfinite interval for initial value problems for a class of first and second order semilinear integrodifferential equations in Banach spaces. The results are based on the Schauder–Tychonoff fixed point theorem.

Key words: Initial value problems, semilinear integrodifferential equations, mild solution, semiinfinite interval, fixed point.

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1 Introduction

In the few past years several books and papers have been devoted to study the existence on compact intervals of strong, classical and mild solutions for differential equations in abstract spaces. We refer to the books of Goldstein [8], Heikkila and Lakshmikantham [9], Ladas and Lakshmikantham [11] and Pazy [13], to the papers of Ghisi [7], Heikkila and Lakshmikantham [10] and Lakshmikantham and Leela [12].

In Section 3 we study the existence of mild solutions, defined on a semiinfinite interval $J = [0, \infty)$, for the Initial Value Problems (IVP) for semilinear evolution integrodifferential equations of the form

$$y' - Ay = f\left(t, y, \int_0^t K(t,s,y)ds\right), \quad t \in J := [0, \infty), \tag{1}$$

$$y(0) + g(y) = y_0, \tag{2}$$

where $f : J \times E \times E \to E$, $K : D \times E \to E$, $D = \{(t,s) \in J \times J : t \geq s\}$, $g \in C(C(J,E),E)$, are given functions, $y_0 \in E$, A is the infinitesimal generator of a strongly continuous semigroup $T(t)$, $t \geq 0$ and E a real Banach space with norm $|\cdot|$.

Section 4 is devoted to the study of the second order semilinear integrodifferential equation of the form

$$y'' - Ay = f\left(t, y, \int_0^t K(t,s,y)ds\right), \quad t \in J := [0, \infty), \tag{3}$$

$$y(0) + g(y) = y_0, \quad y'(0) = y_1 \tag{4}$$

where K, y_0, g and f are as in problem (1)–(2), A is a linear infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in the Banach space E and $y_1 \in E$.

Recently, the authors studied existence results, for ordinary differential equations with nonlocal conditions, of first and second order, in [1] and [2] respectively. Here we extend these results to integrodifferential equations. Nonlocal evolution problems were initiated by Byszewski [3]. For the importance of nonlocal conditions in many areas of applied mathematics we refer to [3] and the references cited therein.

The method we are going to use is to reduce the existence of mild solutions to problems (1)–(2) and (3)–(4) to the search for fixed points of a suitable map on a Fréchet space $C(J,E)$. In order to prove the existence of fixed points, we shall rely on the theorem of Schauder–Tychonoff.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

$J_m = [0, t_m]$ where $t_1 < t_2 < \ldots < t_m \uparrow \infty$.

$C(J,E)$ is the linear metric Fréchet space of continuous functions from J into E with the metric (see Corduneanu [4])

$$d(y,z) = \sum_{m=0}^{\infty} \frac{2^{-m}\|y-z\|_m}{1+\|y-z\|_m} \quad \text{for each } y, z \in C(J,E),$$

where $\|y\|_m := \sup\{|y(t)| : t \in J_m\}$.

$B(E)$ denotes the Banach space of bounded linear operators from E into E.

A strongly measurable function $y : J \to E$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable. (For properties of the Bochner integral see Yosida [16]).

$L^1(J, E)$ denotes the linear space of equivalence classes of strongly measurable functions $y : J \to E$ which are Bochner integrable normed by

$$|y|_{L^1} = \int_0^\infty |y(t)|dt \quad \text{for all } y \in L^1(J, E).$$

The convergence in $C(J, E)$ is the uniform convergence on compact intervals, i.e. $y_j \to y$ in $C(J, E)$ if and only if for each $m \in \mathbb{N}$, $\|y_j - y\|_m \to 0$ in $C(J_m, E)$ as $j \to \infty$.

$M \subseteq C(J, E)$ is a bounded set if and only if there exists a positive function $\phi \in C(J, \mathbb{R}_+)$ such that

$$|y(t)| \leq \phi(t) \quad \text{for all } t \in J \text{ and all } y \in M.$$

From the definition of the metric defined on the Fréchet space $C(J, E)$ a set $M \subseteq C(J, E)$ is compact if and only if for each $m \in \mathbb{N}$, M is a compact set in the Banach space $(C(J_m, E), \|\cdot\|_m)$.

We say that a family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(E)$ is a strongly continuous cosine family if

(i) $C(0) = I$ (I is the identity operatorin E),

(ii) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $s, t \in \mathbb{R}$,

(iii) the map $t \mapsto C(t)y$ is strongly continuous for each $y \in E$;

The strongly continuous sine family $\{S(t) : t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, is defined by

$$S(t)y = \int_0^t C(s)y\, ds, \quad y \in E,\ t \in \mathbb{R}.$$

The infinitesimal generator $B : E \to E$ of a cosine family $\{C(t) : t \in \mathbb{R}\}$ is defined by

$$By = \frac{d^2}{dt^2} C(t)y\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [8] and Fattorini [6], and the papers of Travis and Webb [14], [15].

The operator $G : E \to E$ is said to be completely continuous if $G(D)$ is relatively compact in E for every bounded subset $D \subseteq E$.

Our considerations are based on the following theorem.

Theorem 1 (Schauder–Tychonoff [5], [4]) *Let Ω be a closed convex subset of a locally convex Hausdorff space E. Assume that $N : \Omega \to \Omega$ is continuous and that $N(\Omega)$ is relatively compact in E. Then N has at least one fixed point in Ω.*

3 First order integrodifferential equations

Definition 1 A continuous solution $y(t)$ of the integral equation

$$y(t) = T(t)y_0 - T(t)g(y) + \int_0^t T(t-s)f\Big(s, y(s), \int_0^s K(s,u,y(u))\,du\Big)\,ds, \quad t \in J,$$

is called a *mild solution* of (1)–(2) on J.

Now, we are able to state and prove our main theorem. We will need the following assumptions:

(H1) A is the infinitesimal generator of the linear bounded and compact semigroup $T(t)$, $t > 0$;

(H2) The function g is completely continuous and there exists a constant $G > 0$ such that $|g(y)| \le G$ for each $y \in C(J, E)$.

(H3) K is continuous in all its arguments and there exists $q \in L^2(J, \mathbb{R}_+)$ such that

$$\left|\int_0^t K(t,s,y)\,ds\right| \le q(t)\,\psi(|y|) \quad \text{for each } t \in J \text{ and } y \in E$$

where $\psi : \mathbb{R}_+ \to (0, \infty)$ is continuous and increasing with

$$\int_1^\infty \frac{du}{u + \psi(u)} = \infty.$$

(H4) f is continuous in all its arguments and

$$|f(t,u,v)| \le p(t)(|u| + |v|) \quad \text{for almost all } t \in J \text{ and all } u, v \in E,$$

where $p \in L^2(J, \mathbb{R}_+)$.

Theorem 2 *Assume that hypotheses (H1)–(H4) are satisfied. Then the problem (1)–(2) has at least one mild solution on J.*

Proof We transform the problem (1)–(2) into a fixed point problem. Consider the map, $N : C(J,E) \to C(J,E)$ defined by

$$N(y)(t) = T(t)y_0 - T(t)g(y) + \int_0^t T(t-s)f\Big(s, y(s), \int_0^s K(s,u,y(u))\,du\Big)ds, \quad t \in J.$$

Let $\Omega := \{y \in C(J,E) : |y(t)| \le a(t),\ t \in J\}$ where

$$a(t) = I^{-1}\Big(M\int_0^t p(s)\,\widehat{q}(s)\,ds\Big), \quad \widehat{q}(t) = \max\{1, q(t)\}, \quad M = \sup\{|T(t)| : t \ge 0\},$$

and

$$I(z) = \int_c^z \frac{du}{u + \psi(u)}, \quad c = M|y_0| + MG.$$

Clearly Ω is a convex subset of $C(J, E)$.

We shall show that Ω is closed and the operator N defined on Ω has values in Ω and it is compact. The proof will be given in several steps.

Step 1. *Ω is closed.*

Let $y_n \in \Omega$ with $\|y_n\|_m \to \|y\|_m$ (i.e. y_n converges uniformly to y on J_m) for each $m \in \{1, 2, \ldots\}$. Then for each fixed $t \in J_m$ we have $\|y_n(t)\|_m \leq a(t)$ which implies $\|y(t)\|_m \leq a(t)$. So $y \in \Omega$.

Step 2. $N(\Omega) \subseteq \Omega$.

Let $y \in \Omega$ and fix $t \in J$. We must show $N(y) \in \Omega$. Let $x < t$, then

$$\begin{aligned}
\|(Ny)(x)\|_m &\leq M|y_0| + MG + M \int_0^x p(s)[|y(s)| + q(s)\,\psi(|y(s)|)]\,ds \\
&\leq M|y_0| + MG + M \int_0^x p(s)\,\widehat{q}(s)[a(s) + \psi(a(s)))]\,ds \\
&= M|y_0| + MG + \int_0^x a'(s)\,ds = a(x),
\end{aligned}$$

since

$$\int_c^{a(s)} \frac{du}{u + \psi(u)} = M \int_0^s p(\tau)\,\widehat{q}(\tau)\,d\tau.$$

Thus $N(y) \in \Omega$, so $N : \Omega \to \Omega$.

Step 3. *N is continuous.*

Let $y_n \to y$ in $C(J, E)$. We will to show that $N(y_n) \to N(y)$ in $C(J, E)$. Now, $\|y_n\|_m \to \|y\|_m$ implies that there exists $r > 0$ such that $\|y_n\|_m \leq r$ and $\|y\|_m \leq r$. The Lebesque dominated convergence theorem implies that

$$\begin{aligned}
\|N(y_n) - N(y)\|_m = \sup_{t \in J_m} \Bigg| \int_0^t T(t-s) \Bigg[f\left(s, y_n(s), \int_0^s K(s, u, y_n(u))\,du\right) \\
- f\left(s, y(s), \int_0^s K(s, u, y(u))\,du\right)\Bigg]\,ds - T(t)g(y_n) + T(t)g(y)\Bigg| \to 0.
\end{aligned}$$

Thus N is continuous.

Step 4. *N maps bounded sets in $C(J, E)$ into uniformly bounded sets.*

Let $B_r = \{y \in C(J, E) : |y| \leq r\}$ be a bounded set in $C(J, E)$. Then

$$\begin{aligned}
\|N(y)(t)\|_m &\leq M|y_0| + MG + M \int_0^t \left| f\left(s, y(s), \int_0^s K(s, u, y(u))du\right)\right| ds \\
&\leq M|y_0| + MG + M \int_0^t p(s)[|y(s)| + q(s)\psi(|y(s)|)]\,ds \\
&\leq M|y_0| + MG + M \int_0^t p(s)\widehat{q}(s)[|y(s)| + \psi(|y(s)|)]\,ds \\
&\leq M|y_0| + MG + M(r + \psi(r)) \int_0^{t_m} p(s)\widehat{q}(s)\,ds.
\end{aligned}$$

Step 5. *N maps bounded sets in $C(J,E)$ into equicontinuous family.*

Let $\tau_1, \tau_2 \in J_m$, $\tau_1 < \tau_2$ and $B_r = \{y \in C(J,E) : |y| \le r\}$ be a bounded set in $C(J,E)$. Thus

$$\begin{aligned}|N(y)(\tau_2) - N(y)(\tau_1)| &\le |(T(\tau_2)y_0 - T(\tau_1))y_0| + |(T(\tau_2)g(y) - T(\tau_1))g(y)| \\ &+ \left|\int_{\tau_1}^{\tau_2} T(\tau_1 - s) f\left(s, y(s), \int_0^s K(s,u,y(u))du\right) ds\right| \\ &+ \left|\int_0^{\tau_2} [T(\tau_2 - s) - T(\tau_1 - s)] f\left(s, y(s), \int_0^s K(s,u,y(u))du\right) ds\right|.\end{aligned}$$

As $\tau_2 \to \tau_1$ the right-hand side of the above inequality tends to zero.

As a consequence of Step 3, Step 4, Step 5 and (H1) together with the metric of the Fréchet space $C(J,E)$ we can conclude that $N(B_r)$ is relatively compact in $C(J,E)$.

As a consequence of the Schauder–Tychonoff theorem we can conclude that N has a fixed point y in Ω, which is, by Theorem 1, a mild solution of (1)–(2). □

4 Second order integrodifferential equations

Definition 2 A continuous solution $y(t)$ of the integral equation

$$y(t) = C(t)(y_0 - g(y)) + S(t)y_1 + \int_0^t S(t-s) f\Big(s, y(s), \int_0^s K(s,u,y(u))\,du\Big) ds, \quad t \in J,$$

is called a mild solution of (3)–(4) on J.

In order to study the problem (3)–(4) we introduce the following hypotheses:

(H5) A is an infinitesimal generator of a given strongly continuous, bounded and compact cosine family $\{C(t) : t \in J\}$.

Now, we are able to state and prove our existence theorem for the problem (3)–(4).

Theorem 3 *Assume that hypotheses (H2)–(H4) and (H5) are satisfied. Then the problem (3)–(4) has at least one mild solution on J.*

Sketch of the proof We transform the problem into a fixed point problem. Consider the operator $\widetilde{N} : C(J,E) \to C(J,E)$ defined by

$$\widetilde{N}(y)(t) = C(t)(y_0 - g(y)) + S(t)y_1 + \int_0^t S(t-s) f\Big(s, y(s), \int_0^s K(s,u,y(u))du\Big) ds,$$

$t \in J$. Let $\Omega := \{y \in C(J,E) : |y(t)| \le a(t)$, for each $t \in J\}$ where

$$a(t) = I^{-1}\Big(\widetilde{M} \int_0^t p(s)\,\widehat{q}(s)\,ds\Big), \quad \widehat{q}(t) = \max\{1, q(t)\}, \quad \widetilde{M} = \sup\{|C(t)|;\ t \in J\},$$

and

$$I(z) = \int_{\widetilde{c}}^{z} \frac{du}{u + \psi(u)}, \quad \widetilde{c} = \widetilde{M}|y_0| + \widetilde{M}G.$$

We can also show as in Section 3 that Ω is closed, convex and the operator $\widetilde{N}$ defined on Ω has values in Ω and it is compact. The existence of the fixed point, which is a mild solution of (3)–(4), is then a consequence of Theorem 1.

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