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A Common Generalization of Permutability and 0-permutability

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Abstract

We present a congruence property which is a common generalization of congruence permutability and 0-permutability. We characterize varieties of algebras satisfying this property by a Mal'cev type condition as well as by a relational condition. Examples of such varieties are included.

Key words: Congruence permutability, 0-permutability, (f, g)-permutability, a compatible relation.

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Permutable algebras and varieties play an important role in universal algebra. Recall that an algebra $\mathcal{A}$ is *congruence permutable* (or simply *permutable*, in brief) if $\Theta \cdot \Phi = \Phi \cdot \Theta$ for every two congruences $\Theta, \Phi \in Con\,\mathcal{A}$. A variety $\mathcal{V}$ is *permutable* if each $\mathcal{A} \in \mathcal{V}$ has this property. It was shown by B. Jónsson that every permutable algebra $\mathcal{A}$ has modular its congruence lattice $Con\,\mathcal{A}$. In 1954, A. I. Mal'cev [3] derived a very useful condition characterizing permutable varieties:

A variety $\mathcal{V}$ is permutable if and only if there is a ternary term p of $\mathcal{V}$ such that the following identities are satisfied in $\mathcal{V}$:

$$p(x, z, z) = x \quad \text{and} \quad p(x, x, z) = z.$$

In 1984, H.-P. Gumm and A. Ursini [2] treated the so called ideals in universal algebras and described congruence classes containing a constant element 0. For this, it was suitable to restrict the investigation on the so called 0-*permutable varieties* (or *subtractive varieties* in the terminology of A. Ursini). Recall that an algebra $\mathcal{A}$ with 0 is 0-*permutable* if

$$[0]_{\Theta\cdot\Phi} = [0]_{\Phi\cdot\Theta}$$

holds for any congruences $\Theta, \Phi \in Con\,\mathcal{A}$. A variety $\mathcal{V}$ with a constant 0 is 0-*permutable* if each $\mathcal{A} \in \mathcal{V}$ has this property. Also this property can be characterized by a Mal'cev type condition:

> A variety $\mathcal{V}$ with 0 is 0-permutable if and only if there exists a binary term b of $\mathcal{V}$ such that the following identities hold in $\mathcal{V}$:
>
> $$b(x,x) = 0 \quad \text{and } b(x,0) = x.$$

Let us note that the latest identities can be easily derived from the foregoing ones by putting $b(x,y) = p(x,y,0)$ and hence the 0-permutability is a modification of permutability taken "in vicinity of 0".

The first attempt to unify both of above mentioned conditions was settled by the author and R. Bělohlávek in [1]. The aim of this paper is to show a different approach which enable us to find a number of applications in several well known varieties.

At first, we introduce the concept in question.

Definition Let $\mathcal{A} = (A, F)$ be an algebra of type τ. Let $f(x), g(x)$ be unary terms of this type τ. We say that $\mathcal{A}$ is *(f,g)-permutable* whenever for every $\Theta, \Phi \in Con\,\mathcal{A}$ and each $a, b \in A$ it holds

$$(P) \qquad \langle f(a), g(b)\rangle \in \Theta\cdot\Phi \text{ if and only if } \langle f(a), g(b)\rangle \in \Phi\cdot\Theta.$$

A variety $\mathcal{V}$ of type τ is *(f,g)-permutable* if each $\mathcal{A} \in \mathcal{V}$ has this property.

Remark The condition (P) can be expressed in the form:

$$f(a) \in [g(b)]_{\Theta\cdot\Phi} \quad \text{if and only if} \quad f(a) \in [g(b)]_{\Phi\cdot\Theta}.$$

Hence, an algebra $\mathcal{A}$ is *permutable at g* in the sense of [1] if and only if $\mathcal{A}$ is (f,g)-permutable by our definition with the trivial term $f(x) := x$.

Special cases:

(1) Having $f(x) = x = g(x)$, then (f,g)-permutability is just congruence permutability.

(2) Putting $f(x) = x$ and $g(x) = 0$, where 0 is a constant of $\mathcal{A}$ (i.e. it is a unary term with a constant value), our (f,g)-permutability is just 0-permutability mentioned above.
Hence, (f,g)-permutability is a common generalization of both the congruence properties.

Example 1 Let $\mathcal{L} = (L; \vee, \wedge, 0, 1)$ be a complemented lattice, where 0 or 1 is its least or greatest element, respectively. Set $f(x) = 1$, $g(x) = 0$. Then $\mathcal{L}$ is (1,0)-permutable.

Indeed, if $\langle 1, 0\rangle \in \Theta \cdot \Phi$ for $\Theta, \Phi \in Con\,\mathcal{L}$ then there exists an element $a \in L$ with $\langle 1, a\rangle \in \Theta$, $\langle a, 0\rangle \in \Phi$. Let b be a complement of a. Then

$$\langle b, 0\rangle = \langle b \wedge 1, b \wedge a\rangle \in \Theta$$
$$\langle 1, b\rangle = \langle a \vee b, 0 \vee b\rangle \in \Phi$$

which yield immediately $\langle 1, 0\rangle \in \Phi \cdot \Theta$.

Let us note that no chain with at least 3 elements is (1,0)-permutable. Namely, if $\mathcal{L}$ is a chain with 0 and 1 and $0 < x < 1$, then for $\Theta = \Theta(0, x)$, $\Phi = \Theta(x, 1)$ we have $\langle 0, 1\rangle \in \Theta \cdot \Phi$ but $\langle 0, 1\rangle \notin \Phi \cdot \Theta$.

We are able to characterize (f, g)-permutable varieties by means of a Mal'cev type condition:

Theorem 1 *Let $f(x), g(x)$ be unary terms of a variety $\mathcal{V}$. $\mathcal{V}$ is (f, g)-permutable if and only if there exists a ternary term $p(x, y, z)$ such that the following identities hold in $\mathcal{V}$:*

$$p(x, g(y), y) = f(x), \quad p(x, f(x), y) = g(y).$$

Proof Consider the free algebra $F_v(x, y, z)$ of $\mathcal{V}$ and the congruences $\Theta(f(x), z)$, $\Theta(z, g(y))$ on $F_v(x, y, z)$. Of course,

$$\langle f(x), g(y)\rangle \in \Theta(f(x), z) \cdot \Theta(z, g(y))$$

thus, due to (f, g)-permutability, also

$$\langle f(x), g(y)\rangle \in \Theta(z, g(y)) \cdot \Theta(f(x), z)).$$

Hence, there is an element $q \in F_v(x, y, z)$ with

$$\langle f(x), q\rangle \in \Theta(z, g(y))$$

and

$$\langle q, g(y)\rangle \in \Theta(f(x), z).$$

Of course, $q = p(x, z, y)$ for a suitable ternary term and from

$$\langle f(x), p(x, z, y)\rangle \in \Theta(z, g(y))$$

we conclude immediately

$$p(x, g(y), y) = f(x).$$

Analogously, $\langle p(x, z, y), g(y)\rangle \in \Theta(f(x), z)$ gives us $p(x, f(x), y) = g(y)$.

Conversely, if $\mathcal{V}$ satisfies the identities of Theorem 1 and $\mathcal{A} = (A, F) \in \mathcal{V}$, $a, b \in A$, $\Theta, \Phi \in Con\,\mathcal{A}$ and $\langle f(a), g(b)\rangle \in \Theta \cdot \Phi$, then $\langle f(a), c\rangle \in \Theta$ and $\langle c, g(b)\rangle \in \Phi$ for some elements $c \in A$. Applying the identities, we have

$$\langle g(b), p(a, c, b)\rangle = \langle p(a, f(a), b), p(a, c, b)\rangle \in \Theta$$
$$\langle p(a, c, b), f(a)\rangle = \langle p(a, c, b), p(a, g(b), b)\rangle \in \Phi$$

which yield $\langle g(b), f(a)\rangle \in \Theta \cdot \Phi$, i.e. also $\langle f(a), g(b)\rangle \in \Phi \cdot \Theta$. □

Example 2 The variety of ortholattices is (1,0)-permutable. For this, we can take $p(x, y, z) = y^{\perp}$. Then

$$p(x, g(y), y) = p(x, 0, y) = 0^{\perp} = 1 = f(x)$$
$$p(x, f(x), y) = p(x, 1, y) = 1^{\perp} = 0 = g(y).$$

It can be easily generalized for algebras with 0 and 1 having a unary term $v(x)$ such that $v(0) = 1$ and $v(1) = 0$.

Hence, the variety of pseudocomplemented semilattices (or lattices) is (1,0)-permutable.

Example 3 The variety of all pseudocomplemented lattices is (f, f)-permutable, where $f(x) = x^{**}$.

For this, put

$$x \oplus y := (x^* \wedge y^{**}) \vee (x^{**} \wedge y^*).$$

One can verify that the operation $\oplus$ is associative and for

$$p(x, y, z) = x^{**} \oplus y^{**} \oplus z^{**}$$

we have

$$p(x, f(x), z) = x^{**} \oplus x^{****} \oplus z^{**} = z^{**} = f(z)$$

$$p(x, f(z), z) = x^{**} \oplus z^{****} \oplus z^{**} = x^{**} = f(x).$$

Example 4 The variety of pseudocomplemented semilattices is (f, f)-permutable for $f(x) = x^{**}$. Analogously as in the previous case, we can set $p(x, y, z) = x^{**} \oplus y^{**} \oplus z^{**}$, where

$$x \oplus y := [(x^* \wedge y^{**})^* \wedge (x^{**} \wedge y^*)^*]^*.$$

It is necessary only to check that the operation $\oplus$ is associative. Other verifications are evident.

We can search for a relational characterization of (f, g)-permutable varieties in a way similar to that for permutable varieties done by H. Werner [4]. At first we are going to formulate some necessary conditions:

Theorem 2 *Let $\mathcal{A} = (A, F) \in \mathcal{V}$, $a, b, c \in A$ and R be a compatible relation on $\mathcal{R}$.*

(a) *If $\mathcal{V}$ is (f, g)-permutable and*

$$\langle a, b\rangle \in R,\ \langle b, c\rangle \in R \text{ and } \langle g(b), f(b)\rangle \in R \text{ then } \langle f(a), g(c)\rangle \in R;$$

(b) *If $\mathcal{V}$ is (f, f)-permutable and R is, moreover, reflexive then*

(*i*) $\langle a, b\rangle \in R,\ \langle b, c\rangle \in R \Rightarrow \langle f(a), f(c)\rangle \in R$

(*ii*) $\langle a, b\rangle \in R \Rightarrow \langle f(b), f(a)\rangle \in R.$

Proof

(*a*) Consider the ternary term p of Theorem 1. If $\langle a, b\rangle \in R$, $\langle b, a\rangle \in R$ and $\langle g(b), f(b)\rangle \in R$ then, due to identities valid in $\mathcal{V}$, we have

$$\langle f(a), g(c)\rangle = \langle p(a, g(b), b), p(b, f(b), c)\rangle \in R.$$

(*b*) If $f = g$ and R is also reflexive, then, of course, $\langle f(b), g(b)\rangle = \langle f(b), f(b)\rangle \in R$. Applying (*a*) in this case, we conclude (*i*) immediately. For (*ii*), we have

$$\langle f(b), f(a)\rangle = \langle p(a, f(a), b), p(a, f(b), b)\rangle \in R$$

whenever $\mathcal{V}$ is (f, f)-permutable. □

Now, we can give a full relational characterization of (f, g)-permutable varieties:

Theorem 3 *A variety $\mathcal{V}$ is (f, g)-permutable if and only if for each $\mathcal{A} \in \mathcal{V}$ and every reflexive and compatible relation R on $\mathcal{A}$*

$$(S) \qquad \langle f(a), g(b)\rangle \in R \text{ implies } \langle g(b), f(a)\rangle \in R$$

for any elements a, b of $\mathcal{A}$.

Proof (1) Let (S) be valid for every reflexive and compatible relation R on each $\mathcal{A} \in \mathcal{V}$. Choose $\mathcal{A} = F_v(x, y)$, the free algebra of $\mathcal{V}$ with two free generators x, y. Let R be a reflexive and compatible relation on $\mathcal{A}$ generated by a single pair $\langle f(x), g(y)\rangle$. By (S), we have $\langle g(y), f(x)\rangle \in R$, thus there exists a unary polynomial φ over $\mathcal{A}$ with $g(y) = \varphi(f(x))$ and $f(x) = \varphi(g(y))$. However, $\mathcal{A}$ is generated by $\{x, y\}$, i.e. there exists a ternary term p such that

$$\varphi(t) = p(x, t, y),$$

i.e. $g(y) = p(x, f(x), y)$ and $f(x) = p(x, g(y), y)$. By Theorem 1, $\mathcal{V}$ is (f, g)-permutable.

(2) Suppose that $\mathcal{V}$ is (f, g)-permutable. Let $\mathcal{A} \in \mathcal{V}$ and R be a reflexive and compatible relation on $\mathcal{A}$. Let a, b be elements of $\mathcal{A}$ and $\langle f(a), g(b)\rangle \in R$. Then also $\langle a, a\rangle \in R$, $\langle b, b\rangle \in R$ and, applying the term p and the identities of Theorem 1, we conclude

$$\langle g(b), f(a)\rangle = \langle p(a, f(a), b), p(a, g(b), b)\rangle \in R$$

proving (S). □

Lemma 1 *Let $\mathcal{V}$ be an (f,g)-permutable variety. If $\mathcal{A} \in \mathcal{V}$ and R be a reflexive and compatible relation on $\mathcal{A}$ then*

$$f(b) \in [g(a)]_{R^{-1}} \text{ implies } f(b) \in [g(a)]_R.$$

Proof Let $\mathcal{A} \in \mathcal{V}$, let R be a reflexive and compatible relation on $\mathcal{A}$ and for a, b of $\mathcal{A}$ we have $f(b) \in [g(a)]_{R^{-1}}$. Then $\langle f(b), g(a)\rangle \in R^{-1}$ thus also

$$\langle g(a), f(b)\rangle = \langle p(b, f(b), a), p(b, g(a), a)\rangle \in R^{-1}$$

which implies

$$\langle f(b), g(a)\rangle \in (R^{-1})^{-1} = R.$$

Hence $f(b) \in [g(a)]_R$. □

Lemma 2 *Let $\mathcal{V}$ be an (f,g)-permutable variety such that the identities*

$$f(f(x)) = f(x) \quad \text{and} \quad g(g(x)) = g(x)$$

hold in $\mathcal{V}$. Let $\mathcal{V} = (A, F) \in \mathcal{V}$ and R be a reflexive and compatible relation on $\mathcal{A}$. Then for each $a, b \in A$

$$f(b) \in [g(a)]_{R \cdot R} \quad \text{implies} \quad f(b) \in [g(a)]_R.$$

Proof Under the assumptions of Lemma 2, let $f(b) \in [g(a)]_{R \cdot R}$. Then $\langle f(b), g(a)\rangle \in R \cdot R$, i.e. there is $c \in A$ with $\langle f(b), c\rangle \in R$ and $\langle c, g(a)\rangle \in R$. With respect to the identities in assumption, we conclude

$$\langle f(b), g(a)\rangle = \langle f(f(b)), g(g(a))\rangle = \langle p(f(b), g(c), c), p(c, g(c), g(a)\rangle \in R$$

where p is the term satisfying the identities of Theorem 1.

Hence, $f(b) \in [g(a)]_R$. □

Let $R \subseteq A \times A$ for an algebra $\mathcal{A} = (A, F)$. We denote by $\Theta(R)$ the least congruence on $\mathcal{A}$ containing R. Of course, if R is a reflexive and compatible relation on $\mathcal{A}$, then $\Theta(R)$ is the transitive hull of $R \cup R^{-1}$. Applying of Lemma 1 and Lemma 2, we conclude:

Theorem 4 *Let $\mathcal{V}$ be an (f,g)-permutable variety satisfying $f(f(x)) = f(x)$ and $g(g(x)) = g(x)$. Let $\mathcal{A} = (A, F) \in \mathcal{V}$ and R be a reflexive and compatible relation on $\mathcal{A}$. Then for every $a, b \in A$ we have*

$$f(b) \in [g(a)]_R \quad \text{iff} \quad f(b) \in [g(a)]_{\Theta(R)}.$$

For every algebra $\mathcal{A} = (A, F)$ and $\Phi, \Psi \in Con\,\mathcal{A}$, we have

$$\Phi \vee \Psi = (\Phi \cdot \Psi) \cup (\Phi \cdot \Psi \cdot \Phi) \cup (\Phi \cdot \Psi \cdot \Phi \cdot \Psi) \cup \cdots$$

in $Con\,\mathcal{A}$. Hence, $\Phi \vee \Psi = \Theta(\Phi \cdot \Psi)$ and Theorem 4 implies the following

Corollary *Let $\mathcal{V}$ be an (f,g)-permutable variety satisfying $f(f(x)) = f(x)$ and $g(g(x)) = g(x)$, let $\mathcal{A} = (A,F) \in \mathcal{V}$ and $\Phi, \Psi \in Con\,\mathcal{A}$. Then for any $a, b \in A$ we have*

$$f(b) \in [g(a)]_{\Phi\vee\Psi} \quad \textit{if and only if} \quad f(b) \in [g(a)]_{\Phi\cdot\Psi}\ .$$

Let us note that the identities $f(f(x)) = f(x)$ and $g(g(x)) = g(x)$ are satisfied e.g. in the varieties of Examples 3 and 4.

It is a well-known statement that if an algebra $\mathcal{V}$ is permutable then the congruence lattice $Con\,\mathcal{A}$ is modular. Applying the foregoing results, we are able to show that in the case of (f,g)-permutable varieties, a weak form of modularity of $Con\,\mathcal{A}$ can be proven.

Theorem 5 *Let $\mathcal{V}$ be an (f,g)-permutable variety satisfying $f(f(x)) = f(x)$ and $g(g(x)) = g(x)$. Let $\Theta, \Phi, \Psi \in Con\,\mathcal{A}$ for $\mathcal{A} \in \mathcal{V}$ with $\Phi \subseteq \Psi$ and $a, b \in A$. Then*

$$f(b) \in [g(a)]_{(\Theta\vee\Phi)\wedge\Psi} \quad \textit{if and only if} \quad f(b) \in [g(a)]_{(\Theta\wedge\Psi)\vee\Phi}.$$

Proof Of course, $\Phi \subseteq \Psi$ yields $(\Theta \wedge \Psi) \vee \Phi \subseteq (\Theta \vee \Phi) \wedge \Psi$. Hence, to prove our assertion, we need only to show

$$f(b) \in [g(a)]_{(\Theta\cdot\Phi)\cap\Psi} \Rightarrow f(b) \in [g(a)]_{(\Theta\cap\Psi)\cdot\Phi}.$$

in account of the foregoing Corollary.

Suppose $f(b) \in [g(a)]_{(\Theta\cdot\Phi)\cap\Psi}$. Then $\langle f(b), g(a)\rangle \in \Psi$ and there is $c \in A$ with $\langle f(b), c\rangle \in \Theta$ and $\langle c, g(a)\rangle \in \Phi$. Hence, $\Phi \subseteq \Psi$ gives $\langle g(a), c\rangle \in \Psi$ and, due to transitivity of Ψ, also $\langle f(b), c\rangle \in \Psi$. Together, $\langle f(b), c\rangle \in \Theta \cap \Psi$, and $\langle f(b), g(a)\rangle \in (\Theta \cap \Psi)\cdot\Phi$ proving $f(b) \in [g(a)]_{(\Theta\cap\Psi)\cdot\Phi}$. □

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