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Acta Universitatis Palackianae Olomucensis. Facultas Rerum Naturalium. Mathematica, Vol. 40 (2001), No. 1, 179--184

Persistent URL: http://dml.cz/dmlcz/120429

Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **40** (2001) 179–184

The Natural Affinors on $(J^rT^{*,a})^*$

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(Received May 10, 2000)

Abstract

Let $J^rT^{*,a}M$ be the r-th jet prolongation of the cotangent bundle with weight a of an n-dimensional manifold M. If $n \geq 2$ and $a < 0$, then all natural affinors on $(J^rT^{*,a}M)^*$ are the constant multiples of the identity affinor only.

Key words: Bundle functors, natural transformations, natural affinors.

2000 Mathematics Subject Classification: 58A20, 53A55

0. Let n and r be natural numbers and a be a real number. We consider a linear action $\alpha^a : GL(n,r) \times \mathbf{R}^n \to \mathbf{R}^n$ by $\alpha^a(B,x) = |det(B)|^a(B^{-1})^*x$ and let $T^{*,a}$ be the corresponding vector natural bundle over n-manifolds. We recall that $T^{*,a}M = LM \times_{\alpha^a} \mathbf{R}^n$ for any n-manifold M, and $T^{*,a}\varphi = L\varphi \times_{\alpha^a} id_{\mathbf{R}^n} : T^{*,a}M \to T^{*,a}N$ for any embedding $\varphi : M \to N$ between n-manifolds, where LM is the principal fibre bundle over M of linear frames. $T^{*,a}$ is called the cotangent bundle of weight a over n-manifolds. Let $J^rT^{*,a}$ be the r-jet prolongation of $T^{*,a}$. We recall that $J^rT^{*,a}$ is a vector natural bundle over n-manifolds such that $J^rT^{*,a}M = \{j^r_x\sigma \mid \sigma \text{ is a section of } T^{*,a}M,\ x \in M\}$ and $J^rT^{*,a}\varphi : J^rT^{*,a}M \to J^rT^{*,a}N$, $J^rT^{*,a}\varphi(j^r_x\sigma) = j^r_{\varphi(x)}(T^{*,a}\varphi \circ \sigma \circ \varphi^{-1})$, $j^r_x\sigma \in J^rT^{*,a}M$, where M and φ are as above. Let $(J^rT^{*,a})^*$ be the dual (to $J^rT^{*,a}$) vector natural bundle over n-manifolds, i.e. $(J^rT^{*,a})^*M = (J^rT^{*,a}M)^*$ and $(J^rT^{*,a})^*\varphi = (J^rT^{*,a}\varphi^{-1})^*$ for any M and φ as above.

In general, a natural affinor A on a natural bundle F over n-manifolds is a system of affinors $A : TFM \to TFM$ (i.e. tensor fields of type (1,1) on FM) for any n-manifold M which is invariant with respect to local embeddings between n-manifolds. For example, the family $id = id_{TFM} : TFM \to TFM$ for any n-manifold M is a natural affinor on F.

The main result of this short note is the following classification theorem.

Theorem 1 *If $n \geq 2$ and r are natural numbers and $a < 0$ is a negative real number, then all natural affinors on $(J^rT^{*,a})^*$ over n-manifolds are the constant multiples of the identity natural affinor id only.*

For $a = 0$ the classification is different. In [9], we proved that if r and $n \geq 2$ are natural numbers, then the vector space of natural affinors A on $(J^rT^*)^*$ is 2-dimensional.

In Item 1, for natural numbers $n \geq 2$ and r and a negative real number a we present a classification of all natural transformations $(J^rT^{*,a})^* \to (J^rT^{*,a})^*$ over n-manifolds. In Item 2, using similar arguments as in Item 1, for n, r and a as above we present a classification of all linear natural transformations $T(J^rT^{*,a})^* \to (J^rT^{*,a})^*$ over n-manifolds. In Item 3, as a corollary of the result from Item 2, we present a classification of natural affinors of vertical type on $(J^rT^{*,a})^*$ for n,r and a as above. In Item 4, for n, r and a as above we present a classification of all natural transformations $T(J^rT^{*,a})^* \to T$ over n-manifolds. In Item 5, using the results of Items 3 and 4, we prove Theorem 1. In Item 6, we remark the same results for $(J^r\tilde{T}^{*,a})^*$ instead of $(J^rT^{*,a})^*$, where $\tilde{T}^{*,a}$ is given by a linear action $GL(n,r)\times\mathbf{R}^n \to \mathbf{R}^n$, $(B,x) \to sgn(det(B))|det(B)|^a(B^{-1})^*x$.

Natural affinors on F play a very improtrant role in the differential geometry. For example, they can be used to define torsions of a connection on F, see [5]. That is why classifications of natural affinors on some natural bundles have been studied in many papers, see e.g. [1]–[3] and [6]–[9].

Throughout this note the usual coordinates on $\mathbf{R}^n$ are denoted by $x^1, \dots, x^n$ and $\partial_i = \frac{\partial}{\partial x^i}$, $i = 1, \dots, n$.

All manifolds and maps are assumed to be of class C^∞.

1. In this item we prove the following proposition.

Proposition 1 *If $n \geq 2$ and r are natural numbers and a is a negative real number, then every natural transformation $B : (J^rT^{*,a})^* \to (J^rT^{*,a})^*$ over n-manifolds is proportional (by a real number) to the identity natural transformation.*

Proof From now on the set of all pairs (α, i), where $\alpha \in (\mathbf{N} \cup \{0\})^n$ is such that $|\alpha| \leq r$ and $i = 1, \dots, n$, will be denoted by $P(r, n)$.

Clearly, sections of $T^{*,a}\mathbf{R}^n$ are 1-forms on $\mathbf{R}^n$ satisfying respective new transformation rules. Then any element v from the fibre $(J^rT^{*,a})^*_0\mathbf{R}^n$ is a linear combination of the $(j^r_0(x^\alpha dx^i))^*$ for all $(\alpha, i) \in P(r, n)$, where the $(j^r_0(x^\alpha dx^i))^*$ form the basis dual to the basis $j^r_0(x^\alpha dx^i) \in (J^rT^{*,a})_0\mathbf{R}^n$. From now on we denote the coefficient of v corresponding to $(j^r_0(x^\alpha dx^i))^*$ by $[v]_{\alpha,i}$

Of course, any natural transformation B as in the proposition is uniquely determined by the values $<B(u), j^r_0(x^\alpha dx^i)> \in \mathbf{R}$ for $u \in (J^rT^{*,a})^*_0\mathbf{R}^n$ and $(\alpha, i) \in P(r, n)$, where $j^r_0(x^\alpha dx^i) \in (J^rT^{*,a})_0\mathbf{R}^n$.

Since B is invariant with respect to the coordinate permutations, it is determined by the $<B(u), j_0^r(x^\alpha dx^1)>$. We are going to prove that B is determined by the values $<B(u), j_0^r(dx^1)>$ for $u \in (J^rT^{*,a})_0^*\mathbf{R}^n$, where $j_0^r(dx^1) \in (J^rT^{*,a})_0\mathbf{R}^n$.

For any $\tau \in \mathbf{R}$ and any $\alpha \in (\mathbf{N} \cup \{0\})^n$ with $|\alpha| \leq r$ the local diffeomorphism $\psi_{\tau,\alpha} = (x^1, \ldots, x^{n-1}, x^n + \frac{1}{\alpha_n+1}\tau x^{\alpha+1_n})$ sends the section dx^1 of $T^{*,a}\mathbf{R}^n$ into the section $(1 + \tau x^\alpha)^a dx^1$ near $0 \in \mathbf{R}^n$, i.e. it sends $j_0^r(dx^1) \in (J^rT^{*,a})_0\mathbf{R}^n$ into $j_0^r(dx^1) + \tau a j_0^r(x^\alpha dx^1) + \tau^2(\ldots)$ (we consider the Taylor expansion at $\tau = 0$ of $(1 + \tau x^\alpha)^a$ for any x), where the dots is the element from $(J^rT^{*,a})_0\mathbf{R}^n$ depending polynomially on τ. By the naturality of B with respect to $\psi_{\tau,\alpha}$, the values $<B(u), j_0^r(dx^1) + \tau a j_0^r(x^\alpha dx^1) + \tau^2(\ldots)>$ for $u \in (J^rT^{*,a})_0^*\mathbf{R}^n$ and $\tau \in \mathbf{R}$ are determined by the values $<B(u), j_0^r(dx^1)>$ for $u \in (J^rT^{*,a})_0^*\mathbf{R}^n$. Clearly, $<B(u), j_0^r(dx^1) + \tau a j_0^r(x^\alpha dx^1) + \tau^2(\ldots)>$ depends polynomially on τ for any u. The coefficient on τ of the above polynomial is $a<B(u), j_0^r(x^\alpha dx^1)>$. Hence (since $a \neq 0$) the values $<B(u), j_0^r(x^\alpha dx^1)>$ for $u \in (J^rT^{*,a})_0^*\mathbf{R}^n$ are fully determined by the values $<B(u), j_0^r(dx^1)>$ for $u \in (J^rT^{*,a})_0^*\mathbf{R}^n$. That is why B is fully determined by the values $<B(u), j_0^r(dx^1)> \in \mathbf{R}$ for $u \in (J^rT^{*,a})_0^*\mathbf{R}^n$.

We continue the proof of the proposition. For any $t \in \mathbf{R}_+$ and any $(\alpha, i) \in P(r,n)$ the homothety $a_t = (tx^1, \ldots, tx^n)$ sends $j_0^r(x^\alpha dx^i) \in (J^rT^{*,a})_0\mathbf{R}^n$ into $t^{na-|\alpha|-1} j_0^r(x^\alpha dx^i)$, i.e. $(j_0^r(x^\alpha dx^i))^*$ into $t^{|\alpha|+1-na}(j_0^r(x^\alpha dx^i))^*$. Then (since $a < 0$) by the naturality of B with respect to a_t and by the homogeneous function theorem, [4], we deduce that given $u \in (J^rT^{*,a})_0^*\mathbf{R}^n$ we have $<B(u), j_0^r(dx^1)> = \sum_{i=1}^n \mu_i [u]_{(0),i}$. Similarly, for any $t \in \mathbf{R}_+$ the homothety $b_t = (x^1, tx^2, \ldots, tx^n)$ sends $(j_0^r(dx^i))^*$ into $t^{1-(n-1)a}(j_0^r(dx^i))^*$ for $i = 2, \ldots, n$, and it sends $(j_0^r(dx^1))^*$ into $t^{-(n-1)a}(j_0^r(dx^1))^*$. Then $<B(u), j_0^r(dx^1)>$ is proportional to $[u]_{(0),1}$.

Hence the vector space of natural transformations $B : (J^rT^{*,a})^* \to (J^rT^{*,a})^*$ over n-manifolds has dimension ≤ 1. This ends the proof of Proposition 1. □

2. The crucial point in the proof of Theorem 1 is the following proposition.

Proposition 2 *If $n \geq 2$ and r are natural numbers and a is a negative real number, then every linear natural transformation $C : T(J^rT^{*,a})^* \to (J^rT^{*,a})^*$ over n-manifolds is 0.*

Proof The linearity of C means that C gives a linear map $T_y(J^rT^{*,a})^*M \to (J^rT^{*,a})_x^*M$ for any $y \in (J^rT^{*,a})_x^*M$, $x \in M$. We will use the notation as in the proof of Proposition 1.

Similarly as in the proof of Proposition 1 we deduce that C is fully determined by the values $<C(u), j_0^r(dx^1)> \in \mathbf{R}$ for $u \in (T(J^rT^{*,a})^*\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times (V(J^rT^{*,a})^*\mathbf{R}^n)_0 \tilde{=} \mathbf{R}^n \times (J^rT^{*,a})_0^*\mathbf{R}^n \times (J^rT^{*,a})_0^*\mathbf{R}^n$, where $\tilde{=}$ is the standard trivialization and the canonical identification and where $j_0^r(dx^1) \in (J^rT^{*,a})_0\mathbf{R}^n$.

We continue the proof of the proposition. Similarly as in the proof of Proposition 1, by the naturality of C with respect to a_t and the homogeneous function theorem, we deduce that given $u = (u_1, u_2, u_3) \in (T(J^rT^{*,a})^*\mathbf{R}^n)_0 = \mathbf{R}^n \times (J^rT^{*,a})_0^*\mathbf{R}^n \times (J^rT^{*,a})_0^*\mathbf{R}^n$, $u_1 = (u_1^1, \ldots, u_1^n) \in \mathbf{R}^n$, $u_2, u_3 \in (J^rT^{*,a})_0^*\mathbf{R}^n$ we have $<C(u), j_0^r(dx^1)> = \sum_{i=1}^n \lambda_i [u_2]_{(0),i} + \sum_{i=1}^n \mu_i [u_3]_{(0),i} + \ldots$, where λ_i,

μ_i are the reals and the dots denote the linear combination of monomials in $u_1^1, \dots, u_1^n$ of degree ≥ 2. Since C is linear, $<C(u), j_0^r(dx^1)>$ depends linearly on (u_1, u_3) for any u_2. Then $<C(u), j_0^r(dx^1)> = \sum_{i=1}^n \mu_i [u_3]_{(0),i}$ for the reals μ_i. Then, by the naturality of C with respect to b_t (see the proof of Proposition 1) and $a < 0$,

$$(*) \qquad <C(u), j_0^r(dx^1)> = \mu [u_3]_{(0),1}$$

for the real number $\mu = \mu_1$. In particular, if $n \geq 2$

$$(**) \qquad <C(\partial_1^C{}_{|\omega}), j_0^r(dx^1)> = <C(e_1, \omega, 0), j_0^r(dx^1)> = 0$$

for any $\omega \in (J^rT^{*,a})_0^*\mathbf{R}^n$, where $(\ )^C$ is the complete lift to $(J^rT^{*,a})^*$.

Clearly, the proof of the proposition will be complete after proving that $\mu = 0$, i.e. $<C(0, 0, (j_0^r(dx^1))^*), j_0^r(dx^1)> = 0$. But (if $n \geq 2$) we have

$$(***) \qquad \begin{aligned} 0 &= <C(((x^2)^{r+1}\partial_1)^C{}_{|\omega}), j_0^r(dx^1)> \\ &= <C(0, \omega, (j_0^r(dx^1))^* + \dots), j_0^r(dx^1)> \\ &= <C(0, 0, (j_0^r(dx^1))^*), j_0^r(dx^1)>, \end{aligned}$$

where $\omega = \frac{1}{r+1}(j_0^r((x^2)^r dx^2))^*$ and where the dots denote the linear combination with real coefficients of the $(j_0^r(x^\alpha dx^i))^*$ with $(\alpha, i) \in P(r, n) \setminus \{((0), 1)\}$.

The last equality of (***) is an immediate consequence of the formula (*).

We prove the first equality of (***). The vector fields ∂_1 and $\partial_1 + (x^2)^{r+1}\partial_1$ have the same r-jets at $0 \in \mathbf{R}^n$. Hence there exists a diffeomorphism ψ with $j_0^{r+1}(\psi) = id$ sending ∂_1 into $\partial_1 + (x^2)^{r+1}\partial_1$ near 0. Clearly, ψ preserves $j_0^r(dx^1) \in (J^rT^{*,a})_0\mathbf{R}^n$ because of the order argument. Then using the naturality of C with respect to ψ from (**) it follows that $<C((\partial_1 + (x^2)^{r+1}\partial_1)^C{}_{|\omega}), j_0^r(dx^1)> = 0$ for any $\omega \in (J^rT^{*,a})_0^*\mathbf{R}^n$. Next we apply the linearity of C and (**).

It remains to prove the second equality of (***). The flow of $(x^2)^{r+1}\partial_1$ is $\varphi_t = (x^1 + t(x^2)^{r+1}, x^2, \dots, x^n)$. Clearly, $det(d_0(\tau_{-\varphi_t(y)} \circ \varphi_t \circ \tau_y)) = 1$ for any $y \in \mathbf{R}^n$, where $\tau_y : \mathbf{R}^n \to \mathbf{R}^n$ is the translation by y. Then φ_{-t} sends dx^1 into $d(x^1 \circ \varphi_t)$ because of the Jacobian argument. Then

$$\begin{aligned} &<((x^2)^{r+1}\partial_1)^C{}_{|\omega}, j_0^r(dx^1)> = <\frac{d}{dt}_{|t=0}(J^rT^{*,a})^*(\varphi_t)(\omega), j_0^r(dx^1)> \\ &= \frac{d}{dt}_{|t=0}<(J^rT^{*,a})^*(\varphi_t)(\omega), j_0^r(dx^1)> = \frac{d}{dt}_{|t=0}<\omega, j_0^r(d(x^1 \circ \varphi_t))> \\ &= <\omega, j_0^r(d(\frac{d}{dt}_{|t=0}(x^1 \circ \varphi_t)))> = <\omega, j_0^r(d((x^2)^{r+1}))> = 1 \end{aligned}$$

because of the definition of ω. Then $((x^2)^{r+1}\partial_1)^C{}_{|\omega} = (j_0^r(dx^1))^* + \dots$ under the isomorphism $V_\omega(J^rT^{*,a})^*\mathbf{R}^n = (J^rT^{*,a})_0^*\mathbf{R}^n$. It implies the second equality. □

3. From Proposition 2 we obtain the following corollary

Corollary 1 *If r and $n \geq 2$ are natural numbers and a is a negative real number, then every natural affinor $A : T(J^rT^{*,a})^*M \to V(J^rT^{*,a})^*M$ on $(J^rT^{*,a})^*$ over n-manifolds is* 0.

Proof Define a linear natural transformation $\tilde{A} = pr_2 \circ A : T(J^rT^{*,a})^*M \to V(J^rT^{*,a})^*M \tilde{=} (J^rT^{*,a})^*M \times_M (J^rT^{*,a})^*M \to (J^rT^{*,a})^*M$, where pr_2 is the projection onto second factor. By Proposition 2, $\tilde{A} = 0$. Then $A = (\pi^T, \tilde{A}) = (\pi^T, 0) = 0$. □

4. The tangent map $T\pi : T(J^rT^{*,a})^*M \to TM$ of the bundle projection $\pi : (J^rT^{*,a})^*M \to M$ defines a natural transformation $T\pi : T(J^rT^{*,a})^* \to T$ over n-manifolds.

Proposition 3 *If r and n are natural numbers and a is a negative real number, then every natural transformation $D : T(J^rT^{*,a})^* \to T$ over n-manifolds is proportional (by a real number) to $T\pi$.*

Proof Clearly, any natural transformation D as in the proposition is determined by the contractions $\langle D(u), d_0x^1 \rangle$ for

$$u = (u_1, u_2, u_3) \in (T(J^rT^{*,a})^*\mathbf{R}^n)_0 = \mathbf{R}^n \times (J^rT^{*,a})^*_0\mathbf{R}^n \times (J^rT^{*,a})^*_0\mathbf{R}^n.$$

Using the invariancy of D with respect to the homotheties $a_t = (tx^1, ., tx^n)$ for $t \in \mathbf{R}_+$ and the homogeneous function theorem we deduce (similarly as in the proof of Proposition 1) that $\langle D(u), d_0x^1 \rangle$ for $u = (u_1, u_2, u_3)$ is the linear combination (with real coefficients) of the $u_1^1, \dots, u_1^n$ and it is independent of u_2 and u_3, where $u_1 = (u_1^1, \dots, u_1^n) \in \mathbf{R}^n$. Next, using the invariance of D with respect to the homotheties $b_t = (x^1, tx^2, \dots, tx^n)$ we see that $\langle D(u), d_0x^1 \rangle$ is proportional (by a real number) to $u_1^1 = \langle T\pi(u), d_0x^1 \rangle$. □

5. We are now in position to prove Theorem 1. Let $A : T(J^rT^{*,a})^*M \to T(J^rT^{*,a})^*M$ be a natural affinor on $(J^rT^{*,a})^*$ over n-manifolds. Then $T\pi \circ A : T(J^rT^{*,a})^* \to T$ is a natural transformation. By Proposition 3, $T\pi \circ A = \lambda T\pi$ for some λ. Clearly, $T\pi \circ id = T\pi$. Then $A - \lambda id$ is an affinor on $(J^rT^{*,a})^*$ of vertical type. Now, applying Corollary 1 of Proposition 2 we end the proof. □

6. Remark Starting from a linear action $GL(n, r) \times \mathbf{R}^n \to \mathbf{R}^n$, $(B, x) \to sgn(det(B))|det(B)|^a(B^{-1})^*x$ instead of the one from Item 0, we get natural vector bundle $\tilde{T}^{*,a}$. Clearly, all results presented in this note are true for $(J^r\tilde{T}^{*,a})^*$ instead of $(J^rT^{*,a})^*$. We use the same proofs with $(J^r\tilde{T}^{*,a})^*$ instead of $(J^rT^{*,a})^*$.

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