Acta Univ. Palacki. Olomuc., Fac. rer. nat.,
Mathematica **39** (2000) 73–94

Optimal Quadratic Interpolatory Splines on General Knotset *

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(Received January 21, 2000)

Abstract

Free parameters of quadratic interpolatory spline can be used not only for some boundary conditions, but also to minimize some functionals with geometrical or physical meaning (curvature, energy). We shall consider some cases of functionals which can be expressed as quadratic form in some local spline parameters. Then the problem to find optimal values of free parameters under continuity constraints can be stated as quadratic programming problem with equality constrains and solved with standard QP algorithms or with some more simple algorithms (using pseudoinverse or least squares techniques) in special cases.

Key words: Quadratic interpolatory spline, norm optimization.

1991 Mathematics Subject Classification: 41A15, 65D05

1 Introduction

The linear space of quadratic interpolatory splines with given monotone sequence of spline knots x_i, $i = 0(1)n + 1$ and different sequence of points of interpolation t_i, $i = 0(1)n$ has two free parameters, used usually for boundary conditions prescribed. We can use these free parameters to find such interpolatory spline which minimizes some proper norm or another functional with

*Supported by the Council of Czech Government, J 14/98:153100011.

geometrical or physical meaning, which can be more easily recognized by the user than proper boundary conditions. Such extremal property of natural cubic splines with respect to the L_2-norm of second derivative (on some more general class of functions) was recognized many years ago and used in variational theory of splines and for construction of smoothing splines. Quadratic splines interpolating function values do not have such more general extremal property (but similar property was recognized in the problems of mean values or derivative values interpolation), so the optimization is done here in the class of interpolatory splines only. Quadratic splines on simple knotset (which have only one free parameter) with minimal curvature were studied in [2], more generally in [3]. Here we shall study the more general case of points of interpolation distributed between knots and problems of interpolation of function values, mean values and derivative values from this point of view.

2 Notation—norms and functionals used

2.1 Notation used for quadratic splines

Let us have given the spline knotset $\{x_i\}$, resp. points of interpolation $\{t_i\}$,

$$(\mathbf{x}, \mathbf{t}): \qquad x_0 < x_1 < \cdots < x_n < x_{n+1}, \quad x_i < t_i < x_{i+1}, \ i = 0(1)n. \tag{1}$$

We shall use the following notation:
$S_{21}(\mathbf{x})$—the linear space of interpolatory quadratic splines with defect one on the given knotset $\mathbf{x}$;
$s(x) \in C^1$—quadratic interpolatory spline, consisting from quadratic segments on intervals $[x_i, x_{i+1}]$, $i = 0(1)n$, with local parameters

$$s_i = s(x_i), \quad m_i = s'(x_i), \quad M_i = s''(t_i), \quad h_i = x_{i+1} - x_i\,.$$

The notation g_i will be used for complementary local parameter—function value (FV) or derivative value (DV) in point of interpolation t_i, local mean value (MV) over interval $[x_i, x_{i+1}]$ in considered interpolation problems.

When we choose some local representation of a quadratic spline, then the continuity of the first derivative (or even of function values, when they are not used in the local representation) can be expressed in the form of three term recursions, which we will call *continuity conditions* (CC). Different local representations and corresponding continuity conditions will be used in the following sections (see e.g. [4]) for the solution of the problems mentioned.

2.2 Functionals minimized

We can try to minimize similar functional as with natural cubic spline—for quadratic spline with piecewise constant second derivative we obtain

$$J_2(s) = \int_{x_0}^{x_{n+1}} [s''(x)]^2 dx = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} [s''(x)]^2 dx = \sum_{i=0}^{n} h_i M_i^2. \tag{2}$$

We can also consider the functionals corresponding to the discrete l_2-norm or its weighted form

$$J_{2d}(s) = \|\mathbf{M}\|_2^2 = \sum_{i=0}^{n} M_i^2\,, \quad J_{2d}^w(s) = \sum_{i=0}^{n} w_i M_i^2\,. \tag{3}$$

This last functional contains also the case of the approximation of the norm of the curvature of the spline curve, when we use for weighting coefficients the values $w_i = (1 + p_i^2)^{-3}$, with slopes $p_i = (s_{i+1} - s_i)/h_i$. In all these cases the functional minimized can be expressed as positive definite quadratic form in local second derivatives with diagonal matrix of weighting coefficients. In case of function values and mean values interpolation we can express the spline continuity conditions as recurrences for the values M_{i-1}, M_i, M_{i+1} and use then for minimization some techniques of quadratic programming (see Section 3). *In case of derivative values interpolation the continuity conditions could not be expressed in terms of second derivatives and prescribed values only* and we have to express the functional in terms of corresponding local parameters—the matrix of the quadratic form will then (and also in another cases discussed) have more general structure as will be shown in the following sections.

Natural quadratic spline interpolating local mean values (with boundary conditions $m_0 = m_{n+1} = 0$) gives minimum to the L_2-norm of the spline first derivative (see [4], [5]). We can try to minimize this norm with quadratic splines interpolating function or derivative values. With direct computation or Simpson's rule (using piecewise linearity of the first derivative $s'(x)$) we obtain for $J_1(s)$ the following expression in parameters m_i

$$J_1(s) = \int_{x_0}^{x_{n+1}} [s'(x)]^2 dx = \frac{1}{3} \sum_{i=0}^{n+1} h_i(m_i^2 + m_i m_{i+1} + m_{i+1}^2) = \frac{1}{6}\mathbf{m}^{\mathbf{T}}\mathbf{R}\mathbf{m} \tag{4}$$

with tridiagonal symmetric positive definite matrix $\mathbf{R}$, which has the main diagonal

$$\begin{aligned} \mathit{diag}(\mathbf{R}) &= [2h_0, 2(h_0 + h_1), \cdots, 2(h_{n-1} + h_n), 2h_n] \\ \text{and subdiagonal} \quad & [h_0, h_1, \cdots, h_{n-1}, h_n]. \end{aligned} \tag{5}$$

In case of the derivative values interpolation we can express this functional in terms of derivatives and function values—see Section 5.

We shall consider also minimization of the discrete l_2-norm of the vector $\mathbf{m}$ or its variation with weighting coefficients w_i—with functionals denoted here as

$$J_{1d}(s) = \sum_{i=0}^{n+1} m_i^2\,, \quad J_{1d}^w(s) = \sum_{i=0}^{n+1} w_i m_i^2 \tag{6}$$

in cases of the function and mean values interpolation. In case of derivative values interpolation the norm of $\mathbf{g} = [g_i]$ is given from the data; the norm of $\mathbf{m}$ depends on one free parameter—its optimal value we shall consider in Section 5.

We can consider in some cases to minimize the functionals (spline norms)

$$J_0(s) = \int_{x_0}^{x_{n+1}} [s(x)]^2 dx, \qquad \text{or} \qquad J_{0d}^w(s) = \sum_{i=0}^{n+1} w_i s_i^2 \tag{7}$$

under continuity and interpolation conditions expressed in proper local parameters.

We can choose different local representations of a quadratic spline and we obtain then the spline continuity conditions and mentioned functionals expressed in local parameters used. In some cases we have to use two different kinds of unknown parameters and we obtain then the continuity conditions expressed in terms of such local parameters. Then we have possibility to try to minimize some norm of the compound vectors as $[\mathbf{s}, \mathbf{m}]$ or $[\mathbf{m}, \mathbf{M}]$. The different possibilities in the choice of local parameters and corresponding algorithms will be presented in the following sections.

2.3 Optimization algorithms used

2.3.1 Pseudoinverse, explicit conditions for extremum

The spline continuity conditions $s^{(j)}(x_i - 0) = s^{(j)}(x_i + 0)$, $i = 1(1)n$, $j = 0, 1$ can be expressed in some proper spline local representation as an underdetermined system of linear equations $\mathbf{Ap} = \mathbf{b}$ with some spline local parameters $\mathbf{p}$. Its solution with minimal discrete l_2-norm of $\mathbf{p}$ can be computed simply with pseudoinverse matrix as $\mathbf{p} = \text{pinv}(\mathbf{A}) * \mathbf{b}$ (Matlab notation for pseudoinverse is used; for proof see e.g. [8, p. 15]).

In some simple cases we can express by induction directly the form of the dependence of some local parameters on its initial value and we can obtain then the necessary conditions of minima of $J_k(s)$ as another recurrence between local parameters used. We can complete with such relation the system of continuity conditions and solve such regular system for optimal values of parameters used. The examples of such approach will be given in the following sections.

2.3.2 Least squares technique

The system of linear constrains in our optimization problem has special structure—we can consider the continuity conditions (three-term recurrences between local parameters of a quadratic spline) as a linear second order nonhomogeneous difference equation on the finite set of knots. We can compute some proper fundamental system of such equation and use it in algorithm for computing corresponding parameters of optimal solution. From stability and accuracy reasons we shall use the boundary value method (BVM) technique described in [9], [6] for stable computing of particular solutions of such equation. We describe here only basic idea of such approach in our case.

Let us have the difference equation

$$a_0^i y_i + a_1^i y_{i+1} + a_2^i y_{i+2} = f_i \ , \ i = 0(1)n - 1, \tag{8}$$

and denote the solutions of homogeneous (HE) and nonhomogeneous (NE) equation determined as solutions of boundary (or initial) value problems (BVP, IVP) under conditions:

— $u^j \in (HE),\ u^j_j = 1,\ u^j_i = 0,\ i \neq j;$

— $u^k \in (HE),\ u^k_k = 1,\ u^k_i = 0,\ |k-j| \geq 1,\ i = 0(1)n+1;$

— $v \in (NE),\ v_j = v_k = 0.$

Generally we can use initial conditions and forward or backward reccurences for computing particular solutions determined by two initial values (with $[j,k] = [0,1]$ for forward recurrences). But there is the danger of instability in case that there are roots of the characteristic polynomial outside (inside) the unit circle. The boundary values problem can be stated and solved as corresponding linear tridiagonal system of equations which we obtain from difference equation (8) and BV given. So the indexes j, k have to be chosen according to the positions of roots of the characteristic polynomial of the difference equation in case of constant coefficients. We can expect similar behaviour also in the case of slightly changing cofficients.

In cases of constant coefficients of the characteristic polynomial of the (HE) with $k_1(k_2)$ roots inside (outside) the unit circle the stable computation of particular solutions has to be done with $k_1(k_2)$ prescribed conditions on the left (right) hand side ($k_1 + k_2 = 2$)—see [9] (it gives the rule for choise of indices i, j above). Each solution of equation (8) determined by prescribed values y_j, y_k can be then written as $y = y_j u^j + y_k u^k + v$.

In our problem of optimal spline interpolation we search for the solution which minimizes some proper functional. If the functional minimized can be expressed as scalar product

$$J(y) = J(y_j, y_k) = (y, y)_R = y^T \mathbf{R} y$$

with some positive definite matrix $\mathbf{R}$, then the necessary conditions for optimal values y_j, y_k giving the minimum to $J(y) = J(y_j, y_k)$ we can write as normal system of equations

$$\begin{bmatrix} (u^j,u^j)_R & (u^j,u^k)_R \\ (u^j,u^k)_R & (u^k,u^k)_R \end{bmatrix} \begin{bmatrix} y_j \\ y_k \end{bmatrix} = -- \begin{bmatrix} (u^j,v)_R \\ (u^k,v)_R \end{bmatrix}. \tag{9}$$

Remark 1 In the frequently used case of splines on the equidistant knotset the corresponding difference equation has constant coefficients on the left hand side—the matrix is symmetric, tridiagonal, diagonally dominant. The roots of the characteristic polynomial of (HE) are located one in the unit circle, the second outside the unit circle—see the following examples taken from splines on equidistant mesh and mentioned in the following (roots rounded):

coefficients	a_i	b_i	c_i	z_1	z_2
special quadratic	1	3	1	-0.38	-2.618
cubic, quadratic - MVI	1	4	1	-0.3	-3.7
quadratic - FVI,DVI	1	6	1	-0.17	-5.828
special geometric knotsets	1	6	2	-0.35	-5.65
	2	4	1	-0.29	-1.71

According to the above mentioned rule from [9], we set now $j = 0$, $k = n + 1$ and compute v as the particular solution of (NE) with boundary conditions $v_0 = v_{n+1} = 0$ in a stable manner from the corresponding system of linear equations with diagonally dominant matrix. Any solution of (NE) determined by BC y_0, y_{n+1} can be then written as $y_j = c_1 z_1^j + c_2 y_2^j$. For j satisfactory big ve then obtain from BC

$y_j \approx y_0 z_1^j + v_j$ —for small indices j on the left hand side,
$y_j \approx y_{n+1} z_2^{j-n-1} + v_j$ —near the right hand side.

We can see that in such stably computed solutions the roots z_1, z_2 influence only corresponding part of the "stationary" solution v and there is not the danger of instability, caused by forward or backward reccurences (see [9]). When we compute the particular solutions u^0, u^{n+1} of (HE) as mentioned above (with $j = 0$, $k = n + 1$), then for the coefficients c_1, c_2 computed from the normal system we obtain for large n approximately

$$c_1 \approx -(u^0, v)_R/(u^0, u^0)_R, \quad c_2 \approx -(u^{n+1}, v)_R/(u^{n+1}, u^{n+1})_R \,. \tag{10}$$

2.3.3 Quadratic programming techniques

We have considered the nonnegative minimized functionals till now and obtained special cases of diagonal and tridiagonal matrices of the quadratic form. In case that the functional $J(s)$ has more complete structure of the quadratic form (linear terms, no definitness), we can use some known techniques of quadratic programming with equality constrains to find optimal values of optimal local parameters (see e.g. [1]).

We have used the MATLAB function **qp** in some of our examples to compare the results with above mentioned more simple algorithms.

We can use the classical technique of Lagrange's multipliers, too. Our typical problem can be stated as to find for given matrices **R, A** and vectors **c, b**

$$\min\{\mathbf{m}^{\mathbf{T}}\mathbf{R}\mathbf{m} - \mathbf{c}^{\mathbf{T}}\mathbf{m};\ \mathbf{A}\mathbf{m} = \mathbf{b}\}.$$

The corresponding Lagrange's function with multipliers $\boldsymbol{\lambda}$ is

$$F(\mathbf{m}, \boldsymbol{\lambda}) = \mathbf{m}^{\mathbf{T}}\mathbf{R}\mathbf{m} - \mathbf{c}^{\mathbf{T}}\mathbf{m} + \boldsymbol{\lambda}^{\mathbf{T}}(\mathbf{A}\mathbf{m} - \mathbf{b}).$$

The components of optimal vectors $\boldsymbol{\lambda}, \mathbf{m}$ can be computed from the system of linear equations

$$\begin{bmatrix} \mathbf{2R} & \mathbf{A}^{\mathbf{T}} \\ \mathbf{A} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \mathbf{m} \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{b} \end{bmatrix}. \tag{LM}$$

3 Function values interpolation

In this section we describe some local representations, corresponding continuity conditions and coefficients of the quadratic form for mentioned functionals $J_k(s)$ in the FVI problem.

3.1 Norms of the second derivative minimized

Given the knotset $(\mathbf{x},\mathbf{t})$ with $h_i = x_{i+1} - x_i$, $k_i = t_{i+1} - t_i$ and prescribed values $g_i = s(t_i)$, the local representation with local parameters $s_j, M_j = s''(t_j)$ of the spline can be written as Taylor's expansion in $x = t_i$

$$s(x) = g_i + s'(t_i)(x - t_i) + \frac{1}{2}M_i(x - t_i)^2 \ . \tag{11}$$

The value $s'(t_i)$ can be computed from parameters $g_i, g_{i+1}, M_i, M_{i+1}$ (see [4]) using CC—e.g. in special case with $x_{i+1} = \frac{1}{2}(t_i + t_{i+1})$ for $i = 1(1)n-1$ we have

$$s'(t_i) = \frac{1}{k_i}(g_{i+1} - g_i) - \frac{1}{8}k_i(3M_i + M_{i+1}). \tag{12}$$

The first derivative continuity conditions we can then write generally as three term recurrences (see [3])

$$a_iM_{i-1} + b_iM_i + c_iM_{i+1} = f_i \ , \ i = 1(1)n-1 \tag{13}$$

with coefficients depending on the geometry of the knotset

$$\begin{aligned} a_i &= [(x_i - t_{i-1})/k_{i-1}]^2 k_{i-1}/(k_{i-1} + k_i) > 0, \\ b_i &= \{(t_i - x_i)[1 + (x_i - t_{i-1})/k_{i-1}] \\ &\quad + (x_{i+1} - t_i)[1 + (t_{i+1} - x_{i+1})/k_i]\}/(k_{i-1} + k_i) > 0 \ , \\ c_i &= [(t_{i+1} - x_{i+1})/k_i]^2 k_i/(k_{i-1} + k_i) > 0, \\ f_i &= 2[(g_{i+1} - g_i)/k_i - (g_i - g_{i-1})/k_{i-1}]/(k_{i-1} + k_i) = 2[t_{i-1}, t_i, t_{i+1}]g. \end{aligned} \tag{14}$$

In the special case of $x_{i+1} = \frac{1}{2}(t_i + t_{i+1})$ these expressions for coefficients can be simplified to

$$a_i = k_{i-1}/(k_{i-1} + k_i) \ , \ b_i = 3, \ c_i = 1 - a_i, \ f_i = 8[t_{i-1}, t_i, t_{i+1}]g. \tag{15}$$

Let us denote as $\mathbf{A}$ the tridiagonal $(n-1, n+1)$ - matrix and $\mathbf{f}$ the vector of the right hand side of the system (13) with components given in (14), $\mathbf{M} = [M_i]$. Then the problem to find quadratic interpolatory spline with minimal L_2-norm can be stated now as the problem to find parameters M_i, for which

$$J_2(s) = \sum_{i=0}^{n} h_i M_i^2 \to min \quad \textit{under conditions} \quad \mathbf{AM} = \mathbf{f}\,. \tag{16}$$

This formulation contains formally also the problems of minimizing functionals $J_{2d}(s), J_{2d}^w(s)$. On the equidistant knotset with $h_i = h$ (or in case of $J_{2d}(s)$) we can find the optimal values M_i simply using pseudoinverse solution approach to the system of CC (13). When we have $k_i = k$, $i = 0(1)n$, then the coefficients on the left side of the recursion (13) are constant and equal to [1, 6, 1]. Then we can use LSQ algorithm with particular solutions computed from linear systems (CC recurrence relations) stemming from corresponding BVP. On the general knotset the sufficient condition for the full rank of the matrix $\mathbf{A}$ can be written with $r_j = (x_j - t_{j-1})/k_{j-1}$ as the *knotset regularity condition* (not satisfied in cases that $t_i \to x_i, x_{i+1}$)

$$r_i^2 + r_{i+1}^2 < h_i + (t_i - x_i)r_i + (t_{i+1} - x_{i+1})r_{i+1}. \tag{RC}$$

Generally the quadratic programming algorithm have to be used.

Theorem 1 *There exists just one solution of the optimization problem (16) under condition (RC). Specially, there are unique quadratic interpolatory splines minimizing functionals $J_{2d}(s)$, $J_{2d}^w(s)$. The optimal values of corresponding local parameters M_i we can find using quadratic programming algorithm to the problem (16), special LSQ approach (constant coefficients in CC) or Moore–Penrose pseudoinverse in case of $h_i = h$.*

Proof follows from positive definitness of $J_2(s)$ and structure (diagonal dominancy) of the matrix $\mathbf{A}$ in (16).

Remark 2 The case mentioned in (15) contains an interesting case of so called *geometric knot sets* with k_i forming geometric sequence—we obtain in such nonequidistant case the constant coefficients on the left side of (13); see the examples in last two rows in the table of roots.

Example 1 For the monotone testing data on the equidistant knotset we can see the similar results of interpolation with natural cubic spline and optimal quadratic spline (with knots in midpoints) on Fig. 1. The values of norms of the vectors of the second derivatives are [32.62, 32.60].

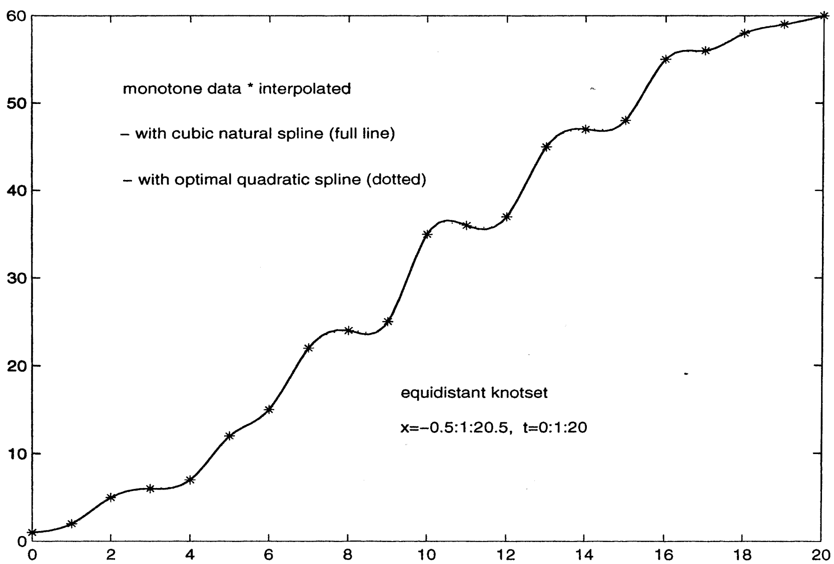

Fig. 1

3.2 The first derivative used and J_1, J_2 minimized

We can also use the spline local representation with parameters g_j, m_j and with local variable $u = (x - x_i)/h_i$ and parameter $d_i = (t_i - x_i)/h_i$

$$s(x) = g_i + h_i(u - d_i)[(2 - u - d_i)m_i + (u + d_i)m_{i+1}]/2\,. \tag{17}$$

The continuity conditions we can write now as recurrences

$$a_i m_{i-1} + b_i m_i + c_i m_{i+1} = f_i\,, \quad i = 1(1)n \tag{18}$$

with coefficients depending on the geometry of the $(\mathbf{x}, \mathbf{t})$-set and given as

$$\begin{aligned} a_i &= h_{i-1}(1 - d_{i-1})^2 > 0, \quad c_i = d_i^2 h_i > 0, \quad f_i = 2(g_i - g_{i-1}), \\ b_i &= h_i d_i(2 - d_i) + h_{i-1}(1 - d_{i-1}^2) > 0\,. \end{aligned} \tag{19}$$

The dominance of the middle coefficient in recursions (18) can be easily recognized for all $d_j \in (0,1)$. In equidistant case with $h_i = h$, $d_i = \frac{1}{2}$ we obtain again on the left hand side of CC the coefficients $[1, 6, 1]$. In case of the geometric knotset with $h_i = ch_{i-1}$, $d_i = 1/2$ these coefficients are $[1, 3(1 + c), c]$. The functional $J_2(s)$ (the norm of the second derivative) we can write now in parameters m_i as

$$\begin{aligned} J_2(s) &= \sum_{i=0}^{n} h_i M_i^2 = \sum_{i=0}^{n} \frac{1}{h_i}(m_{i+1} - m_i)^2 \\ &= \sum_{i=0}^{n} \frac{1}{h_i}(m_i^2 - 2m_i m_{i+1} + m_{i+1}^2) = \mathbf{m}^{\mathbf{T}}\mathbf{Q}\mathbf{m} \end{aligned} \tag{20}$$

where (singular) symmetric positive semidefinite tridiagonal matrix $\mathbf{Q}$ has

$$\begin{aligned} diag(\mathbf{Q}) &= [h_0^{-1},\ h_0^{-1} + h_1^{-1}, \cdots, h_{n-1}^{-1} + h_n^{-1},\ h_n^{-1}] \\ \text{and subdiagonal} &\quad [-h_0^{-1},\ \cdots,\ -h_n^{-1}]. \end{aligned} \tag{21}$$

We can try to use the LSQ algorithm to find optimal values of local parameters m_i minimizing $J_2(s)$, or to use (LM) approach (the existence and uniqueness of such minima were proved in Theorem 1).

The minimization of the L_2-norm of the spline first derivative leads us to the functional

$$J_1(s) = \frac{1}{6}\,\mathbf{m}^{\mathbf{T}}\mathbf{R}\mathbf{m}$$

with the positive definite symmetric tridiagonal matrix $\mathbf{R}$ described in (5) and with equality constrains (18). We can use for minimization of $J_1(s)$ the mentioned LSQ approach to find optimal values of parameters m_i. This approach we can use also for minimization of weighted discrete l_2-norm with J_{1d}^w. For minimization of the functional $J_{1d}(s)$ we can use simply pseudoinverse approach with the matrix described in (18)–(19). The results obtained we can summarize in the following Theorem.

Theorem 2 *There exist unique FVI quadratic splines which minimize functionals $J_1(s)$, $J_{1d}(s)$, $J_{1d}^w(s)$ under corresponding continuity conditions (18).*

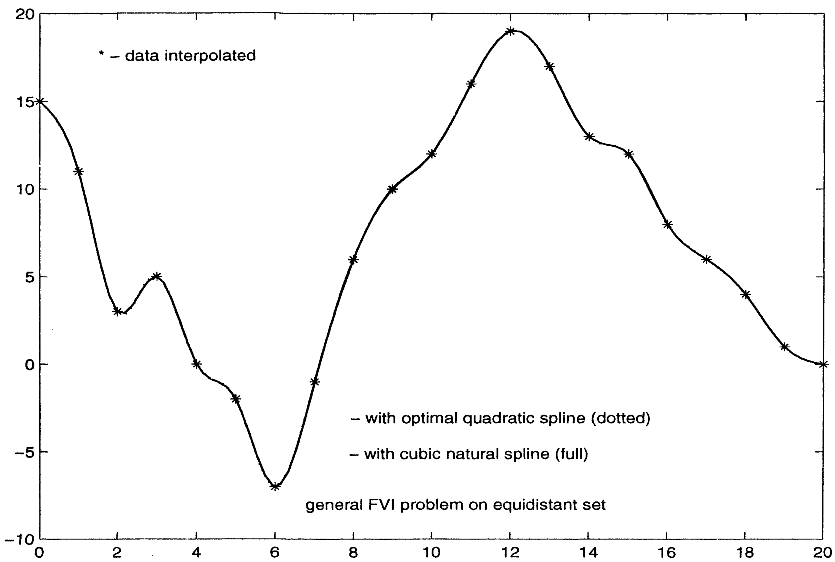

Fig. 2a

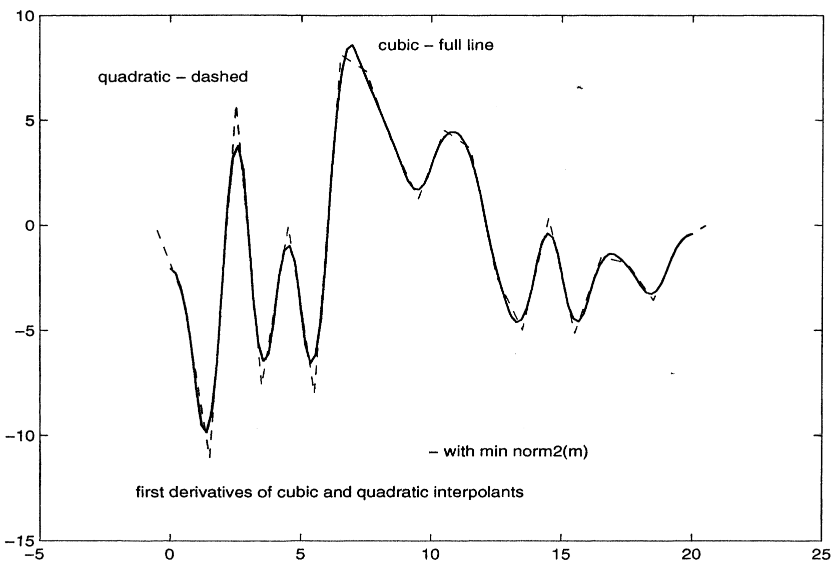

Fig. 2b

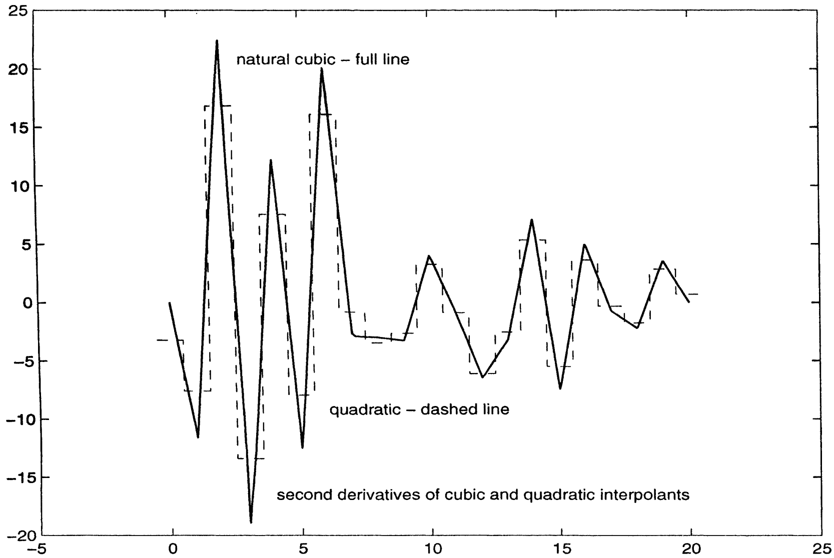

Fig. 2c

Example 2 The results of discrete data interpolation with natural cubic spline and quadratic spline with minimal l_2-norm of the vector $\mathbf{m}$ of the first derivatives and spline knots in midpoints are plotted in Fig. 2a. The first and second derivatives of that interpolatory splines are plotted on Fig. 2b, Fig. 2c and demonstrate closeness of these interpolants. Similar result we obtain with quadratic spline with minimal l_2-norm of the vector of the second derivatives.

3.3 Function values used and minimized

The another local parameters s_j, g_j are used in the local spline representation

$$s(x) = (1-u)(1-\frac{u}{d_i})s_i + u\frac{u-d_i}{1-d_i}s_{i+1} + \frac{u(1-u)}{d_i(1-d_i)}g_i. \tag{22}$$

The corresponding continuity conditions for the first derivatives are

$$\frac{1-d_{i-1}}{h_{i-1}d_{i-1}}s_{i-1} + \left[\frac{d_{i-1}}{h_{i-1}(1-d_{i-1})} + 2\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right) + \frac{1-d_i}{h_i d_i}\right]s_i$$

$$+ \frac{d_i}{h_i(1-d_i)}s_{i+1} = \frac{g_{i-1}}{h_{i-1}d_{i-1}(1-d_{i-1})} + \frac{g_i}{h_i d_i(1-d_i)}. \tag{23}$$

The middle coefficient is dominating for $d_j \in ((-1+\sqrt{2})/2, \sqrt{2}/2)$. In case of all $d_i = \frac{1}{2}$ it simplifies to recurrences

$$\frac{1}{h_{i-1}}s_{i-1} + 3\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)s_i + \frac{1}{h_i}s_{i+1} = 4\left(\frac{g_{i-1}}{h_{i-1}} + \frac{g_i}{h_i}\right) \tag{24}$$

with symmetric entries, in equidistant case again with coefficients [1, 6, 1] on the left side. (For knots x_i forming the geometric set we obtain the coefficients [c, 3(1+c), 1] here.) Given values g_j, we can find the local spline parameters s_j giving minimum to the functional $J_{0d}(s)$ described in (7) using pseudoinverse approach for CC given in (23) or (24); for the functional $J^w_{0d}(s)$ we can use here the LSQ approach.

For the value of the functional $J_0(s)$ we obtain from (22) after some computation the formula

$$J_0(s) = \int_{x_0}^{x_{n+1}} [s(x)]^2\, dx$$

$$= \frac{1}{30}\sum_{i=0}^{n} \frac{h_i}{d_i^2(1-d_i)^2}[g_i^2 + c_1 g_i s_i + c_2 g_i s_{i+1} + c_3 s_i^2 + c_4 s_i s_{i+1} + c_5 s_{i+1}^2] \quad (25)$$

with coefficients c_i depending on the knotset parameters d_i as

$$\begin{aligned} c_1 &= -2 + 7d_i - 5d_i^2, \\ c_2 &= (3 - 5d_i)d_i, \\ c_3 &= 1 - 7d_i + 21d_i^2 - 25d_i^3 + 10d_i^4, \\ c_4 &= (-3 + 13d_i - 20d_i^2 + 10d_i^3)d_i \\ c_5 &= d_i^2(6 - 15d_i + 10d_i^2) \end{aligned} \quad (26)$$

(for $d_i \in (0,1)$ there is $c_3 > 0$, $c_5 > 0$, $c_4 < 0$, the coefficients c_1, c_2 change sign here). In special case of $d_i = \frac{1}{2}$ we obtain more simple formula

$$J_0(s) = \frac{1}{15}\sum_{i=0}^{n} h_i[8g_i^2 + 2g_i(s_i + s_{i+1}) + 2s_i^2 + 2s_{i+1}^2 - s_i s_{i+1}]. \quad (27)$$

with symmetric positive definite matrix of the quadratic form. Using the local parameters g_j, m_j we obtain for $d_i = \frac{1}{2}$ and this local representation quadratic form with symmetric positive definite matrix

$$\begin{aligned} J_0(s) \;=\; & \sum_{i=0}^{n} h_i \Big[g_i^2 + 12h_i g_i(m_{i+1} - m_i) \\ & + \frac{17}{480}h_i^2 m_i m_{i+1} + \frac{23}{960}h_i^2(m_i^2 + m_{i+1}^2)\Big]. \end{aligned} \quad (28)$$

Remark 3 The matrix of the quadratic form in (27) belongs to M-matrices; in (28) we can recognize tridiagonal symmetric diagonally dominant matrix there. In the general case (25) we can find no simple quadratic form and the quadratic programming approach shall be used.

Theorem 3 *There exist local parameters* s$= [s_i]$ *of the quadratic FV interpolatory splines which realize the minimum of functionals* $J_0(s)$, $J_{0d}(s)$, $J^w_{0d}(s)$ *under equality constrains (23). We can find them using corresponding techniques of QP, LSQ or pseudoinverse. For* $d_i = 1/2$ *and in some its neighborhood the optimal spline is unique.*

Remark 4 The solution of Example 2 with minimal l_2-norm of **s** is close to that on Fig. 2a for all but last two points where it takes values about [68, 10].

3.4 Another cases

We can use also the local representation with parameters g_i, m_i, M_i which with local variable $u = (x - t_i)/h_i$, $d_i = (t_i - x_i)/h_i$ reads

$$s(x) = g_i + h_i(u - d_i)m_i + \frac{1}{2}h_i^2(u^2 - d_i^2)M_i\,. \tag{29}$$

The corresponding spline continuity conditions we can write now as

$$m_i - m_{i+1} + h_i M_i = 0, \quad i = 0(1)n \tag{30}$$

$$h_i(1 - d_i)m_i + h_{i+1}d_{i+1}m_{i+1} + \frac{1}{2}h_i^2(1 - d_i^2)M_i + \frac{1}{2}h_{i+1}^2 d_{i+1}^2 M_{i+1} = g_{i+1} - g_i\,.$$

We have obtained block system of linear equations for two kinds of unknown local parameters m_i, M_i. We can use pseudoinverse approach to find optimal values of m_i, M_i which minimize the l_2-norm of the vector $[\mathbf{m}, \mathbf{M}]$, or the LSQ approach with stable manner of computing of particular solutions of the the first order systems of linear difference equations described in [9] or [6]. We can use also the general algorithms of quadratic programming for another functionals considered. We can even simplify the system (30) when we eliminate some of parameters from the more simple first relation. We obtain then the reduced system of constrains in one of the forms

$$\begin{aligned}[h_i(1 - d_i) + h_{i+1}d_{i+1}]\, m_i + h_i\left[\frac{1}{2}h_i(1 - d_i^2) + h_{i+1}d_{i+1}\right] M_i \\ + \frac{1}{2}h_{i+1}^2 d_{i+1}^2 M_{i+1} = g_{i+1} - g_i\,,\end{aligned} \tag{31}$$

$$\begin{aligned}h_i(1 - d_i)m_i + h_{i+1}d_{i+1}(1 - \frac{1}{2}d_{i+1})m_{i+1} + \frac{1}{2}h_{i+1}d_{i+1}^2 m_{i+2} \\ + \frac{1}{2}h_i^2(1 - d_i^2)M_i = g_{i+1} - g_i\,.\end{aligned} \tag{32}$$

Now we apply the algorithms to one of these reduced system and we can compute the solution with smaller computational costs. The eliminated parameter we can compute then from (30).

When we eliminate parameter M_i from (30), we obtain the CC written in (18), (19). Similarly we can work with local parameters $[\mathbf{g}, \mathbf{m}]$, $[\mathbf{g}, \mathbf{M}]$.

4 Mean values interpolation

4.1 Second and first derivative used and minimized

In the mean value interpolation (MVI) problem we are given the mean values

$$g_i = \frac{1}{h_i}\int_{x_i}^{x_{i+1}} s(x)\,dx\,, \quad i = 0(1)n \tag{33}$$

to be interpolated by quadratic spline on the knotset $\mathbf{x}$. We can use the spline local representation

$$s(x) = g_i + h_i(u - \frac{1}{2})m_i + \frac{1}{2}h_i^2(u^2 - \frac{1}{3})M_i \tag{34}$$

and obtain the spline continuity conditions as the system of recurrences

$$\begin{aligned} h_i m_i + h_{i+1}m_{i+1} + \frac{2}{3}h_i^2 M_i + \frac{1}{3}h_{i+1}^2 M_{i+1} &= 2(g_{i+1} - g_i), \quad i = 0(1)n-1 \\ -m_i + m_{i+1} &= h_i M_i\,, \quad i = 0(1)n. \end{aligned} \tag{35}$$

We can use pseudoinverse approach to this four block matrix to find optimal values of parameters m_i, M_i giving minimum to the l_2-norm of the vector $[\mathbf{m},\mathbf{M}]$. We can again eliminate some of parameters to obtain smaller underdetermined system for these local parameters—e.g. one of the following

$$\begin{aligned} h_i m_i + \frac{2}{3}h_{i+1}m_{i+1} + \frac{1}{3}h_{i+1}m_{i+2} + \frac{2}{3}h_i^2 M_i = 2(g_{i+1} - g_i), \\ h_i m_i + 5h_{i+1}m_{i+1} + h_{i+1}^2 M_{i+1} = 6(g_{i+1} - g_i),\ i = 0(1)n-1. \end{aligned} \tag{36}$$

The computations with pseudoinverse will be more efective now. The undetermined parameter we can compute from (35).

When we eliminate the parameters M_i from (35), we obtain recursion in one parameter only, which will be given in (42) more easily with the use of proper local representation.

With the local representation

$$s_2(x) = (1 - 2u)s_i + 2ug_i + \frac{1}{6}h_i^2 u(-2 + 3u)M_i \tag{37}$$

we can obtain the system of continuity conditions

$$\begin{aligned} s_i + s_{i+1} - \frac{1}{6}h_i^2 M_i &= \frac{2}{h_i}g_i\,, \\ -\frac{1}{h_i}s_i + \frac{1}{h_{i+1}}s_{i+1} + \frac{1}{3}h_i M_i + \frac{1}{6}h_{i+1}M_{i+1} &= \left(\frac{g_{i+1}}{h_{i+1}} - \frac{g_i}{h_i}\right). \end{aligned} \tag{38}$$

In equidistant case we can eliminate one parameter and simplify it to

$$s_{i+1} + \frac{1}{12}h^2(M_i + M_{i+1}) = \frac{1}{2}(g_i + g_{i+1}), \tag{39}$$

$$\left(s_i = \frac{1}{2}(3g_i - g_{i+1}) + \frac{1}{12}h^2(3M_i + M_{i+1})\right).$$

We can eliminate parameters M_i and obtain recurrence (45), which will be obtained by more simple approach. When we eliminate parameters s_i, we obtain the CC expressed in parameters M_i only; with notation $c_j = (h_j + h_{j+1})^{-1}$ we can write the result as

$$c_{i-1}h_{i-1}^2 M_{i-1} + h_i\left[c_{i-1}h_{i-1} + 1 + c_i h_{i+1}\right] M_i + c_i h_{i+1}^2 M_{i+1} \tag{40}$$
$$= 6\left[c_{i-1}g_{i-1} + h_i^{-1}(-2 + c_{i-1}h_{i-1} + c_i h_{i+1})g_i + c_i g_{i+1}\right].$$

In equidistant case we obtain simple recursion

$$\frac{1}{6}(M_{i-1} + 4M_i + M_{i+1}) = \frac{1}{h^2}(g_{i-1} - 2g_i + g_{i+1}).$$

The sufficient condition for the dominance of the central coefficient in the general case and full rank of that matrix is

$$c_{i-1}h_{i-1}(h_i - h_{i-1}) + h_i + c_i h_{i+1}(h_i - h_{i+1}) > 0, \tag{DC}$$

which is not satisfied e.g. in case of $h_{i+1}/h_i = h_{i-1}/h_i = 2$. The unique optimal parameters $\mathbf{M}$ for $J_{2d}(s)$, $J_2(s)$ we can compute with the pseudoinverse or LSQ approach for full rank matrix in (40), generally with QP algorithm.

To find parameters m_i which minimize the functional $J_2(s)$, we can use the quadratic form (20) and local representation

$$s(x) = g_i + \frac{1}{6}h_i(-2 + 6u - 3u^2)m_i + \frac{1}{6}h_i(-1 + 3u^2)m_{i+1}. \tag{41}$$

The spline continuity conditions can be written now simply as

$$h_{i-1}m_{i-1} + 2(h_{i-1} + h_i)m_i + h_i m_{i+1} = 6(g_i - g_{i-1}), \quad i = 1(1)n \tag{42}$$

and LSQ approach for minimization of $J_2(s), J_{2d}^w(s), J_{2d}^w(s)$, or pseudoinverse approach for $J_{1d}(s)$ can be used. The constant coefficients [1, 4, 1] or [1, 2(1+c), c] we obtain on the left side of (42) in case of constant step size $h_i = h$ or steps h_i forming geometric sequence.

Using local representation (41) we obtain for functional $J_1(s)$ to be minimized the identical result as in (4)—

$$J_1(s) = \int_{x_0}^{x_{n+1}} \left[s'(x)\right]^2 dx = \frac{1}{3}\sum_{i=0}^{n} h_i(m_i^2 + m_i m_{i+1} + m_{i+1}^2) = \frac{1}{6}\mathbf{m}^{\mathrm{T}}\mathbf{R}\mathbf{m} \tag{43}$$

with the tridiagonal symmetric matrix $\mathbf{R}$ given in (5). For minimization of $J_1(s)$ again the LSQ approach and continuity conditions (42) have now to be used.

For the functional $J_2(s)$ we obtain result given in (20) for FVI.

The local spline representation

$$s(x) = (1 - 3u)(1 - u)s_i + u(3u - 2)s_{i+1} + 6u(1 - u)g_i \tag{44}$$

results in the following spline continuity conditions (see (24))

$$\frac{1}{h_{i-1}}s_{i-1} + 2\left(\frac{1}{h_{i-1}} + \frac{1}{h_i}\right)s_i + \frac{1}{h_i}s_{i+1} = 3\left(\frac{g_{i-1}}{h_{i-1}} + \frac{g_i}{h_i}\right), \; i = 1(1)n. \tag{45}$$

The functionals $J_p(s)$ can be expressed in local parameters s_i, g_i as

$$J_2(s) = 36 \sum_{i=0}^{n} \frac{1}{h_i^3} (s_i + s_{i+1} - 2g_i)^2, \tag{46}$$

$$J_1(s) = \sum_{i=0}^{n} \frac{1}{h_i} \left[3g_i^2 - 3g_i(s_i + s_{i+1}) + s_i^2 + s_i s_{i+1} + s_{i+1}^2\right], \tag{47}$$

$$J_0(s) = \frac{1}{15} \sum_{i=0}^{n} h_i \left[18g_i^2 - 3g_i(s_i + s_{i+1}) + 2s_i^2 + 2s_{i+1}^2 - s_i s_{i+1}\right]. \tag{48}$$

We can recognize the diagonal dominance in the corresponding SPD matrices of this forms for $p = 0, 1$ (singularity for $p = 2$ can cause nonuniqueness of the minimizer). So we can use pseudoinverse approach to minimize $J_{0d}(s)$, LSQ approach for $J_{2d}^w(s)$ and QP algorithms for $J_p(s)$ under constrains (45).

Theorem 4 *There exist unique MVI quadratic splines which minimize functionals $J_0(s)$, $J_{0d}(s)$, $J_1(s)$, $J_{1d}(s)$, $J_{2d}(s)$, $J_2(s)$ under corresponding continuity conditions (resp. (DC) for $J_2(s)$).*

Proof follows from positive definitness of corresponding matrices (see (2), (5), (48)) and full rank of matrices in CC (40), (42), (45).

Example 3 The results of the interpolation of the histogram on the nonequidistant knotset with quadratic spline with minimal l_2-norm of the vector $\mathbf{m}$ and minimal L_2-norms of the first and second derivatives are plotted on Fig. 3. We can see small differences in the middle intervals and diffferent behavior at the boundaries (according to the norms minimized).

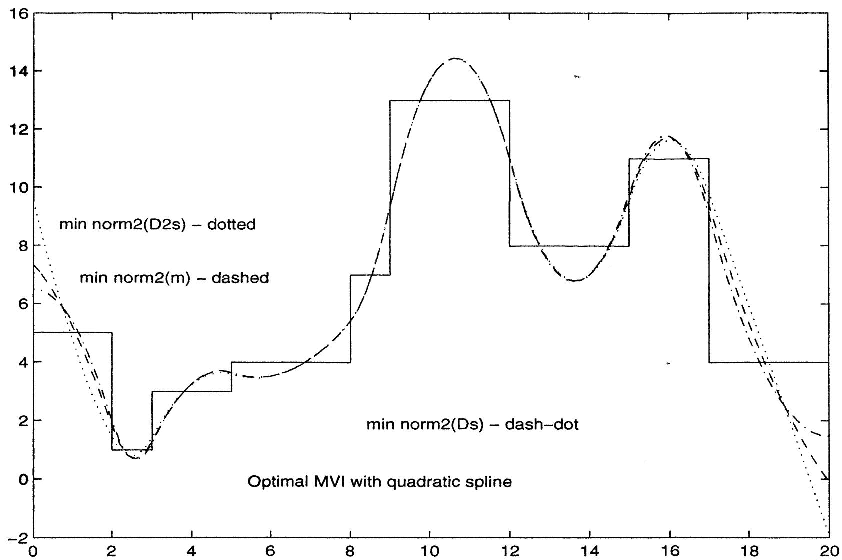

Fig. 3

5 Derivative values interpolation

In the problem of the derivative values interpolation (DVI) on the knotset **x**,**t** the values $g_i = s'(t_i)$, $i = 0(1)n$ are prescribed. There are some special features in case of $d_i = \frac{1}{2}$ (see [7]) which we have to consider. The quadratic DVI spline depends generally on two parameters and so we can again search for optimal spline minimizing given norm or another functional.

5.1 Second derivative used, minimized

We can try to use the Taylor's local representation similar to (11) with known values $s'(t_i) = g_i$, $d_i = (t_i - x_i)/h_i$ and unknown local parameters $s(t_i)$, M_i; $u = (x - x_i)/h_i$ (mention some differences in notation)

$$s(x) = s(t_i) + h_i(u - d_i)g_i + \frac{1}{2}h_i^2(u - d_i)^2 M_i \tag{49}$$

and we obtain the spline continuity condition as the recursions

$$\begin{aligned} -s(t_i) + s(t_{i+1}) - \frac{1}{2}h_i^2(1-d_i)^2 M_i + \frac{1}{2}d_{i+1}^2 h_{i+1}^2 M_{i+1} \\ = h_i(1-d_i)g_i + d_{i+1}h_{i+1}g_{i+1}\,, \quad i = 0(1)n-1, \\ h_i(1-d_i)M_i + d_{i+1}h_{i+1}M_{i+1} = g_{i+1} - g_i\,. \end{aligned} \tag{50}$$

We can use the pseudoinverse approach to find vector $[\mathbf{s},\ \mathbf{M}]$ with minimal l_2-norm. We can eliminate one parameter and obtain the reduced system of constraints—e.g.

$$\begin{aligned} -s(t_i) + s(t_{i+1}) - \frac{1}{2}h_i(1-d_i)\left[h_i(1-d_i) + d_{i+1}h_{i+1}\right] M_i \\ = \left[h_i(1-d_i) - \frac{1}{2}d_{i+1}h_{i+1}\right] g_i + \frac{1}{2}d_{i+1}h_{i+1}g_{i+1} \end{aligned} \tag{51}$$

and to use pseudoinverse approach to that reduced matrix.

More simple CC system we obtain using the local representation

$$s(x) = s_i + h_i u g_i + \frac{1}{2}h_i^2 u(u - 2d_i)M_i\,. \tag{52}$$

We can again search for vector $[\mathbf{s},\ \mathbf{M}]$ with minimal norm under spline continuity constrains

$$s_i - s_{i+1} + \frac{1}{2}h_i^2(1-2d_i)M_i = -h_i g_i\,, \tag{53}$$

$$h_i(1-d_i)M_i + d_{i+1}h_{i+1}M_{i+1} = g_{i+1} - g_i\,,\ i = 0(1)n-1. \tag{54}$$

In both representations the first derivative CC in (50),(54) is identical and does not contain parameters s_j. It follows in both representations from the fact, that the spline we search has two free parameters: one value of the second derivative M_j (which influences the value of our functional) and one function value s_k, which causes some shift of the graph of the spline and don't influence the functional. So for minimization of $J_{2d}(s)$ we can use some separation e.g. in the following *algorithm*:

1) *Compute optimal values of parameters* $\mathbf{M}$ *with pseudoinverse solution of the system (54);*
2) *Compute values* $\mathbf{s}$ *from CC (50) or (53) with known parameters* $\mathbf{M}$.

In the special case of $d_i = \frac{1}{2}$ the CC (53), (54) have simple form

$$s_{i+1} - s_i = h_i g_i\,, \quad h_i M_i + h_{i+1} M_{i+1} = 2(g_{i+1} - g_i) \tag{55}$$

and by induction we obtain

$$h_j M_j = (-1)^j h_0 M_0 + \sum_{i=0}^{j} c_i g_i; \quad dM_j/dM_0 = (-1)^j h_0/h_j$$

and the necessary condition of minima of the functional $J_2(s) = \sum h_i M_i^2$, $dJ_2(s)/dM_0 = 0$ completes the system of second part of CC to the regular system

$$h_i M_i + h_{i+1} M_{i+1} = 2(g_{i+1} - g_i), \;\; i = 0(1)n-1, \;\; \sum_{i=0}^{n} (-1)^i M_i = 0. \tag{56}$$

When we compute optimal values M_i and choose the free parameter s_0, the remaining values s_i we can compute from the first recursion in CC (55).

Similarly we can treat the case with the functional $J_{2d}(s)$.

Theorem 5 *There exist the unique values of* $\mathbf{M}$ *for the DVI quadratic spline which minimize the functionals* $J_2(s), J_{2d}(s)$ *and unique vectors* $\mathbf{m}$, $\mathbf{M}$ *with the minimal value of the* l_2*-norm of the vector* $[\mathbf{m}, \mathbf{M}]$*; the optimal splines are determined till the additive constant.*

5.2 First derivative used and minimized

The first derivative $m_i = s'(x_i)$ is used in local representation

$$s(x) = s_i + h_i u \left(1 - \frac{1}{2d_i} u\right) m_i + \frac{h_i}{2d_i} u^2 g_i\,. \tag{57}$$

The system of spline continuity conditions can be written now as

$$\frac{s_{i+1} - s_i}{h_i} + \left(\frac{1}{2d_i} - 1\right) m_i = \frac{1}{2d_i} g_i\,, \tag{58}$$

$$(1 - d_i) m_i + d_i m_{i+1} = g_i\,, \quad i = 0(1)n. \tag{59}$$

We can see again the appearance of parameters m_i only in the first derivative CC. Eliminating the parameter g_i we can reduce the system to

$$m_i + m_{i+1} - \frac{2}{h_i}(s_{i+1} - s_i) = 0, i = 0(1)n. \qquad (60)$$

We can use now the pseudoinverse approach to search for the vector [**s**,**m**] with minimal l_2-norm.

The relation (59) demonstrates the dependence of values m_i and the functional J_{1d} on one such a parameter only. We can choose m_0, to follow analytically the dependence of m_i on m_0 with the result

$$m_{k+1} = m_0 \prod_{i=0}^{k}(1 - d_i^{-1}) + \sum_{i=0}^{k} c_i g_i$$

and write down the necessary condition of minima of the functional $J_{1d}(s)$ as

$$\sum_{i=0}^{n+1} m_i \prod_{j=0}^{i-1}(1 - d_j^{-1}) = 0 \qquad \left(\sum_{i=0}^{n+1}(-1)^i m_i = 0 \quad \text{in case } d_i = \frac{1}{2}\right). \qquad (61)$$

This equation completes the system of CC (59) for computation of parameters m_i to the regular system of equations. The parameters s_i then we can compute recursivelly from given value s_0 and (58). This approach can be easily extended to minimize $J_{1d}^w(s)$.

More simply, we can use pseudoinverse approach to compute optimal values m_i from (59) , choose s_0 (the second free parameter) and then to compute s_i, $i = 1(1)n + 1$ recursivelly from (58).

For the spline second derivative we obtain from (57) $s''(x) = h_i(g_i - m_i)/d_i$ and then

$$J_2(s) = \int_{x_0}^{x_{n+1}} [s''(x)]^2 dx = \sum_{i=0}^{n} \left(\frac{h_i}{d_i}\right)^2 (g_i - m_i)^2, \qquad (62)$$

which we can minimize again under conditions (59) (unique solution) and then compute values s_i from (58).

The functional $J_1(s) = \frac{1}{6}\mathbf{m}^T\mathbf{R}\mathbf{m}$ we can minimize under conditions (59) using LSQ technique and then compute **s** from (58). For functional $J_1(s)$ we can from (57) also obtain

$$J_1(s) = \sum_{i=0}^{n} h_i \left[\left(1 - \frac{1}{d_i} + \frac{1}{3d_i^2}\right) m_i^2 - \frac{1}{d_i}\left(1 + \frac{2}{3d_i}\right) g_i m_i + \frac{1}{3d_i^2} g_i^2\right] \qquad (63)$$

with positive coefficients at m_i^2 for $d_i \in (0, 1)$. So we can use QP technique with respect to conditions (59) to find optimal value of m_i minimizing $J_1(s)$.

For the values of $J_0(s)$ we obtain with (57) and (58)

$$J_0(s) = \frac{1}{60}\sum_{i=0}^{n} h_i^3(8m_i^2 + 9m_i m_{i+1} + 3m_{i+1}^2) + \sum_{i=0}^{n} h_i \left[s_i^2 + \frac{1}{3}h_i s_i(2m_i + m_{i+1})\right] \qquad (64)$$

For its minimization under conditions (60) (both parameters s_i, m_i unknown) we can use QP approach.

Theorem 6 *There exist the unique values of* $\mathbf{m}$ *for the DVI quadratic spline which minimize the functionals* $J_1(s)$, $J_{1d}(s)$. *The optimal splines are determined till the additive constant.*

5.3 Function values used and minimized

In case of $d_i \neq \frac{1}{2}$ we can use the local representation of DVI spline

$$s(x) = (1-b(u))s_i + b(u)s_{i+1} + \frac{u(u-1)}{2d_i - 1} g_i\,, \quad b(u) = \frac{u(u-2d_i)}{1-2d_i}. \tag{65}$$

The spline continuity conditions can be written (see [7]) as the system of recurrences

$$-a_i s_{i-1} + (a_i - b_i)s_i + b_i s_{i+1} = \frac{1}{2}\left(\frac{g_{i-1}}{1-2d_{i-1}} + \frac{g_i}{1-2d_i}\right) \tag{66}$$

with coefficients

$$a_i = \frac{1-d_{i-1}}{h_{i-1}(1-2d_{i-1})}, \qquad b_i = \frac{d_i}{h_i(1-2d_i)}. \tag{67}$$

The functionals considered we can write now as

$$J_2(s) = 4\sum_{i=0}^{n} \frac{1}{h_i(1-2d_i)^2}\left(s_{i+1} - s_i - g_i\right)^2\,, \tag{68}$$

$$J_1(s) = \sum_{i=0}^{n} \frac{1}{h_i}\left[(s_i - s_{i+1})^2 + \frac{1}{3}(g_i - s_i + s_{i+1})^2/(1-2d_i)^2\right], \tag{69}$$

$$J_0(s) = \sum_{i=0}^{n} \frac{h_i}{15}\left[12s_i^2 + 2s_{i+1}^2 + s_i s_{i+1} + (9s_i + s_{i+1})g_i - g_i^2\right] \tag{70}$$

for $d_i = \dfrac{1}{4}$ (too lenghty in general).

The approximation of the norm of the spline curvature ($s'(x) \approx g_i$) is given as

$$J_c(s) = 4\sum_{i=0}^{n} w_i(s_{i+1} - s_i - g_i)^2, \quad w_i^{-1} = h_i^3(1+g_i^2)^3(1-2d_i)^2. \tag{71}$$

In the special case of $d_i = 1/2$ we can find from the recurrence $s_{i+1} = s_i + h_i g_i$ by induction

$$s_{j+1} = \sum_{i=0}^{j}(s_j + h_j g_j), \quad ds_i/ds_0 = 1.$$

For the functional $J_{0d}(s) = J_{0d}(s_0)$ we can write the necessary condition of minima $dJ_{0d}/ds_0 = 0$ and to complete thus the system of CC to the regular system of equations for computing optimal function values s_i

$$-s_i + s_{i+1} = h_i g_i, \quad i = 0(1)n, \quad \sum_{i=0}^{n+1} s_i = 0.$$

Theorem 7 *There exist quadratic DVI splines which minimize functionals $J_0(s)$, $J_{0d}(s)$. For $J_0(s)$ and $d_i = \frac{1}{4}$, $J_{0d}(s)$ with $d_i = \frac{1}{2}$ and problem with minimal norm of* $[\mathbf{s}, \mathbf{m}]$ *we have the unique solution.*

Example 4 For the function $f(x) = x * sin(x)$ and

$$\mathbf{x} = 0 : 0.5 : 10, \qquad \mathbf{t} = 0.25 : 0.5 : 9.75,$$

and values g_i computed exactly from $f(x)$, the DVI quadratic interpolatory spline with minimal value of $J_1(s)$ and $s_0 = f(x_0)$ is plotted in Fig. 4. The plots of $f(x), s(x)$ are nearly identical, with differences in the function values of the order 10^{-2}. The results for splines with minimal values of $J_1(s)$, $J_{2d}(s)$ and $s(0) = 0$ are identical to four decimals, spline **s0** with minimal value of $J_{0d}(s)$ is shifted down at about 0.6 (corresponding norms of **f, s0,** s are 17.46, 17.42, 17.66).

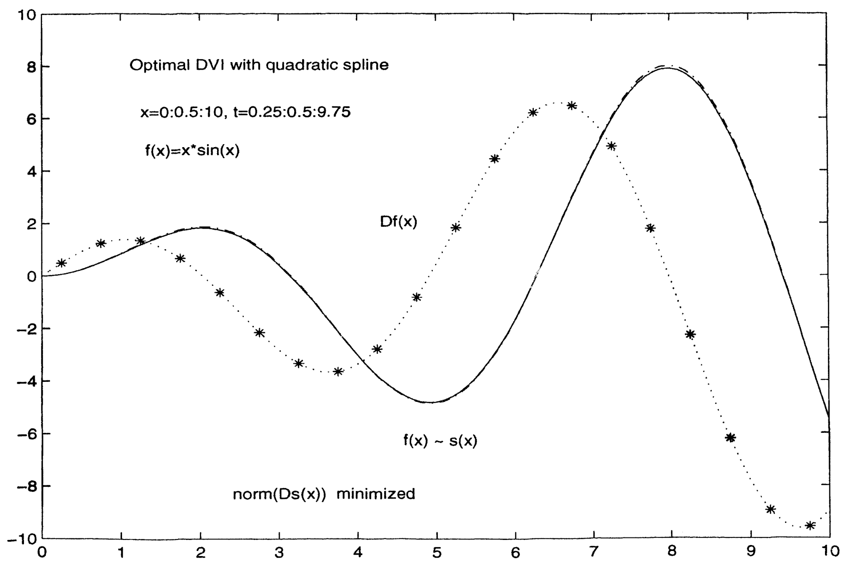

Fig. 4

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